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Random quantum graphs are asymmetric

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0 Outline

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1. Quantum graphs
2. Random quantum graphs

1 Origin story

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Point of view for today: quantum graphs are interesting mathematical objects in their own right.

1 Quantum confusability graphs

We start from a quantum channel $\Phi : M_n \to M_m$ with a **Kraus decomposition** $\Phi(x) := \sum_i S_i x S_i^*$, where $\sum S_i^* S_i = \mathbb{1}$, because Φ is trace preserving.

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V is an **operator system** independent of the Kraus decomposition of Φ. Moreover, any operator subsystem of M_n arises in this way.

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A quantum relation on M_n is an operator subspace $V \subset M_n$. It is **symmetric** if $V = V^*$ and **reflexive** if $\mathbb{1} \in V$.

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A **quantum adjacency matrix** $A : M_n \to M_n$ is a completely positive map such that $m(A \otimes A)m^* = A$, A is self-adjoint and $m(A \otimes \mathrm{Id})m^*(\mathbb{1}) = \mathbb{1}$.

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The inner product on M_n is $\langle A, B \rangle := n \, \mathrm{Tr}(A^* B)$.

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Are these definitions equivalent?

1 The equivalences

Operator systems vs projections

Let $p : M_n \to V$ be the orthogonal projection wrt the trace. As $\mathrm{B}(\mathrm{HS}_n) \simeq M_n \otimes M_n^{\mathrm{op}}$, we get a corresponding $P \in M_n \otimes M_n^{\mathrm{op}}$.

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If $P \in M_n \otimes M_n^{\mathrm{op}}$ is a projection then $A_P : M_n \to M_n$ given by $A_P(x) := (\mathrm{Id} \otimes n \, \mathrm{Tr})(P(\mathbb{1} \otimes x))$ is cp and satisfies $m(A_P \otimes A_P)m^* = A_P$.

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If $A : M_n \to M_n$ is cp and such that $m(A \otimes A)m^* = A$ then $P_A := (A \otimes \mathrm{Id})m^*(\mathbb{1})$, its **Choi matrix**, is a projection in $M_n \otimes M_n^{\mathrm{op}}$.

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$(A \otimes \mathrm{Id})m^*(\mathbb{1}) = \frac{1}{n} \sum_{i,j=1}^{n} A(e_{ij}) \otimes e_{ji}$

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1 Intermission – graphs as operator systems

Classical graphs

Classical graphs also fit into this framework. To a graph we associate the following space $V := \mathrm{span}\{e_{ij} : i \sim j \text{ or } i = j\}$.

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Quantum invariants

It turns out that two graphs are isomorphic if and only if the corresponding operator systems are isomorphic (Ortiz-Paulsen). This allows to associate "quantum" invariants to graphs, using quantities appearing in the theory of operator systems.

1 Automorphisms of quantum graphs

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The degree matrix

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If the spectrum of D is simple then the automorphism group is automatically abelian.

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2 Random models

The easiest thing to do is the following. We fix $0 \leqslant d \leqslant n^2 - 1$. Then we take d independent random Hermitian matrices X_1,...,X_d and consider $V_d := \mathrm{span}\{\mathbb{1}, X_1, \ldots, X_d\}$.

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$G(n, M)$

The model above corresponds to the Erdős-Rényi random graph $G(n, M)$ with a fixed number of edges. We can also build a version of the $G(n, p)$, where we fix the number of vertices and the probability that a given pair of vertices is connected by an edge.

2 The results

Theorem (Chirvasitu-W.)

If $1 \leqslant d \leqslant n^2 - 2$ *then the degree matrix* D *has almost surely simple spectrum.*

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If $1 \leqslant d \leqslant n^2 - 2$ then the automorphism group is almost surely abelian.

Theorem (Chirvasitu-W.)

If $2 \leqslant d \leqslant n^2 - 3$ then the automorphism group is almost surely trivial.

2 Diagonal degree matrices

The degree matrix

Let $V \subset M_n$ be an operator system with an orthonormal basis $(A_1, \ldots, A_d)$ consisting of Hermitian matrices. Then the degree matrix is equal to $D = \sum_{i=1}^{d} A_i^2$.

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Define $f_{ij} := \frac{e_{ij}+e_{ji}}{\sqrt{2}}$ for $i < j$ and $f_{ij} := \frac{i(e_{ij}-e_{ji})}{\sqrt{2}}$ for $i > j$. Then the set (f_{ij}) is orthonormal and f_{ij}^2 is a *diagonal* matrix.

2 Degree matrices with a simple spectrum

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If $d > n-2$ then we take $d-n+2$ matrices from the set (f_{ij}) and $n-2$ random mutually orthogonal diagonal matrices. The degree matrix is diagonal and for almost all choices it has distinct entries.

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The idea is to construct an operator system with a diagonal degree matrix such that the only diagonal unitaries preserving it upon conjugation are trivial.
Then we add some random diagonal matrices so that the degree matrix has distinct entries and conclude, because the only possible automorphisms had to be diagonal to begin with.

2 The construction

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We start with two matrices $X_1 := \sum_{i=1}^{[(n-1)/2]} f_{2i,2i+1}$ and $X_2 := \sum_{i=1}^{[n/2]} f_{2i-1,2i}$. Their span is preserved by a diagonal unitary matrix U iff all its entries are ± 1. Moreover these entries have to be 4-periodic.

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We have $u_1 = 1$ and $u_4 = u_2 u_3$, so only u_2 and u_3 have to be determined. To this end one has to add another matrix, for example $Y := f_{14} + f_{25} + f_{37}$ (and something slightly different for $n = 6$).

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What about quantum symmetries?

Thank you for your attention!