Well-posedness for some LWR models on a junction

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in collaboration with
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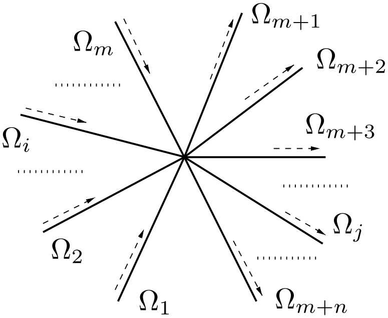
Statement of the problem

We consider a junction where *m* incoming and *n* outgoing roads meet.



- Incoming roads: $x \in \Omega_i = \mathbb{R}_-$, $i = 1, \ldots, m$;
- Outgoing roads: $x \in \Omega_j = \mathbb{R}_+$, $j = m+1, \ldots, m+n$;
- The junction is located at $x = 0$.

Statement of the problem

On each road the evolution of traffic is described by

$$\partial_t \rho_h + \partial_x f_h(\rho_h) = 0, \qquad h = 1, \dots, m+n,$$

- ρ_h density of vehicles, $[0, R]$-valued for all h
- f_h bell-shaped, non linearly non degenerate, Lipschitz fluxes

$$\forall h \quad f_h(0) = 0 = f_h(R)$$

Moreover, we postulate conservativity at the junction:

$$\sum_{i=1}^{m} f_i(\rho_i(t, 0^-)) = \sum_{j=m+1}^{m+n} f_j(\rho_j(t, 0^+)).$$

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"Solutions"

Fix $\vec{\rho}_0 = (\rho_0^1, \ldots, \rho_0^{m+n})$ s.t. $\rho_0^h \in L^\infty(\Omega_h, [0,R])$, $\forall h \in \{1, \ldots, m+n\}$.
We call "solution" a $(m+n)$-uple $\vec{\rho} = (\rho_1, \ldots, \rho_{m+n})$ s.t.

- $\forall h$ ρ_h is a Kruzhkov entropy solution in $\mathbb{R}_+ \times \{\Omega_h \setminus \partial\Omega_h\}$.
  Namely $\forall k \in [0,R]$ and $\forall \xi \in \mathcal{C}_c^1(\mathbb{R}_+ \times \Omega_h)$, $\xi \geq 0$

$$\int_{\mathbb{R}_+} \int_{\Omega_h} |\rho_h - k|\xi_t + q_h(\rho_h, k)\xi_x \, dx \, dt \geq 0$$

  (with $q_h(\rho_h, k) = \text{sign}(\rho_h - k)(f_h(\rho_h) - f_h(k))$ the Kruzhkov entropy flux)

  Idea: k is an obvious solution...
  the above inequalities are "Kato inequalities" between ρ_h and k

- conservation at the junction holds.

There is no hope to prove well-posedness for "solutions".
Analogy:
junction coupling conditions (JCC) play the role of boundary conditions (BC). Imposing mere conservativity condition as JCC leaves the Cauchy problem underdetermined ! [A. ESAIM Proc.'15].

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Many different approaches to single out "suitable" solutions

For the Riemann problem at the junction
(road-wise constant initial conditions):

- [Holden, Risebro SIMA'95]
  maximize a concave "entropy" function at the junction ;
- [Coclite, Piccoli SIMA'02], [Coclite, Garavello, Piccoli SIMA'05]
  traffic distribution matrix + optimization ;
- [Lebacque '96], [Lebacque, Khoshyaran '02]
  Supply-Demand model ;
- ...

We prove well-posedness for solutions to the Cauchy problem
which are limits of vanishing viscosity (VV) approximations.

Essential ingredient: intrinsic characterization of VV limits
(a notion of solution, expressed e.g. via some "entropy inequalities")

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Vanishing viscosity approximations, Coclite-Garavello 2010

Fix $\varepsilon > 0$. Consider convection-ε-diffusion + JunctionCouplingCondition:

$$\begin{cases} \rho^\varepsilon_{h,t} + f_h(\rho^\varepsilon_h)_x = \varepsilon \rho^\varepsilon_{h,xx}, \\ \sum_{i=1}^m \Big(f_i(\rho^\varepsilon_i(t,0)) - \varepsilon \rho^\varepsilon_{i,x}(t,0) \Big) = \sum_{j=m+1}^{m+n} \Big(f_j(\rho^\varepsilon_j(t,0)) - \varepsilon \rho^\varepsilon_{j,x}(t,0) \Big), \\ \rho^\varepsilon_h(t,0) = \rho^\varepsilon_{h'}(t,0), \\ \rho^\varepsilon_h(0,x) = \rho^0_{h,\varepsilon}(x), \end{cases}$$

where the approximated initial conditions $\vec{\rho}_{0,\varepsilon}$ satisfy

$$\rho^0_{h,\varepsilon} \in W^{2,1}(\Omega_h) \cap C^\infty(\Omega_h), \quad [0,R]\text{-valued},$$
$$\rho^0_{h,\varepsilon} \longrightarrow \rho^0_h, \qquad \text{a.e. and in } L^p(\Omega_h),\ 1 \le p < \infty, \quad \text{as } \varepsilon \to 0,$$
$$\big\|\rho^0_{h,\varepsilon}\big\|_{L^1(\Omega_h)} \le \big\|\rho^0_h\big\|_{L^1(\Omega_h)}, \quad \big\|(\rho^0_{h,\varepsilon})_x\big\|_{L^1(\Omega_h)} \le TV(\rho^0_h),$$
$$\varepsilon \big\|(\rho^0_{h,\varepsilon})_{xx}\big\|_{L^1(\Omega_h)} \le C_0,$$

with $C_0 > 0$ independent from ε, h.

Coclite and Garavello, 2010

Theory of semigroups $\Rightarrow$ $\forall \varepsilon > 0$ there exists a unique $\vec{\rho^\varepsilon}$ s.t.

$$\rho_h^\varepsilon \in C([0,\infty); L^2(\Omega_h)) \cap L^1_{loc}((0,\infty); W^{2,1}(\Omega_h)), \quad h \in \{1,\ldots,m+n\},$$

$$0 \le \rho_h^\varepsilon \le R, \qquad \sum_{h=1}^{m+n} \|\rho_h^\varepsilon(t,\cdot)\|_{L^1(\Omega_h)} \le \sum_{h=1}^{m+n} \|\rho_h^0\|_{L^1(\Omega_h)}, \quad \forall t \ge 0,$$

+ additional a priori estimates like $\|\sqrt{\varepsilon}\rho_x^\varepsilon\|_{L^2} \le C$.

Compensated compactness $\Rightarrow$ existence of a sequence $\{\varepsilon_\ell\}_{\ell\in\mathbb{N}}$, $\varepsilon_\ell \to 0$ and a solution $\vec{\rho}$ of the inviscid Cauchy problem such that

$$\forall h \;\; \rho_h^{\varepsilon_\ell} \longrightarrow \rho_h, \qquad \text{a.e. and in } L^p_{loc}(\mathbb{R}_+ \times \Omega_h),\ 1 \le p < \infty. \qquad (1)$$

Uniqueness of VV solutions for the inviscid problem ?
It is proved [Coclite, Garavello SIMA'10] in the case $m = n$, $f_h \equiv f$ based on comparison with the obvious solution $\vec{k} = (k,\ldots,k)$.

"Local objective": extend these results to general junctions
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Monotonicity of the *VV* solver

Not only the viscous problem of Coclite-Garavello is well posed.
The key property, independent of ε, can be expressed as :

(i) monotonicity (order-preservation) of the solver:
$\vec{\rho}_0 \geq \hat{\vec{\rho}}_0$ (componentwise) $\Rightarrow$ $\forall t > 0\ \vec{\rho}^{\,\varepsilon}(t,\cdot) \geq \hat{\vec{\rho}}^{\,\varepsilon}(t,\cdot)$;

(ii) L^1-contractivity of the solver:

$$\sum_{h=1}^{m+n} \|\rho_h^\varepsilon(t,\cdot) - \hat{\rho}_h^\varepsilon(t,\cdot)\|_{L^1(\Omega_h)} \leq \sum_{h=1}^{m+n} \|\rho_{h,\varepsilon}^0 - \hat{\rho}_{h,\varepsilon}^0\|_{L^1(\Omega_h)};$$

(iii) Kato inequality: for all test function $\xi \in \mathcal{D}((0,+\infty)\times\mathbb{R})$, $\xi \geq 0$

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Links: (iii) $\Rightarrow$ (ii) (with $\xi \sim \mathbf{1}_{[0,t]}$); (ii) $\Leftrightarrow$ (i) (Crandall-Tartar lemma)

These properties are inherited by VV admissible solutions.

Kato inequality $\begin{cases} \text{guarantees uniqueness of VV solutions} \\ \text{provides their intrinsic characterization} \end{cases}$

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At least heuristically, JCC $\Leftrightarrow$ Riemann solver at junction.
Different solution semigroups for LWR on networks originate from different Riemann solvers at junction (Garavello, Piccoli,...)

Byproduct of our analysis:

a subclass of these semigroups shares key features of VV solutions

Required property: monotonicity of the junction Riemann solver (larger data on a road lead to larger solutions on the whole network)

General principle:

monotone, Lipschitz Riemann solver at junction
$\Longrightarrow$ an intrinsic notion of solution + well-posedness.

Notion of solution and uniqueness:
mimic tools developed for discontinuous-flux scalar conservation laws [A., Karlsen, Risebro ARMA'11]: admissibility germ, adapted entropies

Construction of solutions:
approximations, e.g. by the Godunov finite volume scheme

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Godunov's numerical flux

Consider the Riemann problem for pure SCL:

$$\begin{cases} u_t + f(u)_x = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ u_0(x) = \begin{cases} a, & \text{if } x < 0, \\ b, & \text{if } x > 0. \end{cases} \end{cases}$$

Denote by $\mathcal{R}[a,b]$ its Kruzhkov entropy solution.
The Godunov flux is the function $(a,b) \mapsto f(\mathcal{R}[a,b])|_{x=0^\pm}$.
Analytically

$$G(a,b) = \begin{cases} \min_{s\in[a,b]} f(s) & \text{if } a \le b, \\ \max_{s\in[b,a]} f(s) & \text{if } a \ge b. \end{cases}$$

Key properties of the Godunov flux:

- Consistency: for all $a \in [0,R]$, $G(a,a) = f(a)$;
- Monotonicity and Lipschitz continuity:
  $\exists L > 0 : \forall (a,b) \in [0,R]^2$ there holds

$$0 \le \partial_a G(a,b) \le L, \qquad -L \le \partial_b G(a,b) \le 0.$$

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Bardos-LeRoux-Nédélec boundary conditions

Consider the initial and boundary value problem

$$\begin{cases} u_t + f(u)_x = 0, & \text{for } t > 0,\ x < 0 \\ u(t, 0^-) \approx u_b(t), \\ u(0, x) = u_0(x), \end{cases}$$

u is an entropy solution for the IBVP if

- u is a Kruzhkov entropy solution in the interior of $\mathbb{R}_+ \times \mathbb{R}_-$,
- u satisfies the boundary condition in the (BLN) sense

$$\text{for a.e. } t > 0 \ \ \forall k \in \text{conv}\{u(t, 0^-), u_b(t)\}$$
$$q(u(t, 0^-), k) \equiv \text{sign}(u(t, 0^-) - k)\left(f(u(t, 0^-)) - f(k)\right) \geq 0 \qquad (*)$$

A reformulation of BLN:

u satisfies BLN $(*)$ $\iff f(u(t, 0^-)) = G(u(t, 0^-), u_b(t))$.

(known since [Dubois, LeFloch JDE'88])

The junction as a family of IBVPs

Fix $\vec{\rho}_0 = (\rho_0^1, \ldots, \rho_0^{m+n})$ s.t. $\rho_0^h \in L^\infty(\Omega_h, [0, R])$, $\forall h \in \{1, \ldots, m+n\}$.
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where $\vec{v} = (v_1, \ldots, v_{m+n}) : \mathbb{R}_+ \to [0, R]^{m+n}$ is to be chosen

- depending on the JCC one wants to express,
- in particular, ensuring conservation at the junction.

The case of VV limits: require that v_h be the same on all roads

[A., Cancès JHDE'15],[A., Mitrović AnnIHP'15]:
motivations in the "discontinuous-flux" case $m = n = 1$.
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Admissibility at the junction

Continuity of ρ at the junction - up to boundary layers! - is desired.

Definition I (Formalized from heuristics of the model)

Given $\vec{\rho}_0$ i.c., we say that $\vec{\rho} = (\rho_1, \ldots, \rho_{m+n})$ is an admissible solution for the Cauchy problem on the network, if $\exists p$ in $L^\infty(\mathbb{R}_+, [0, R])$ s.t.

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  (this includes the "density-at-junction" condition $\forall h \; v_h(t) = p(t)$)
- and "flux-at-junction" condition (conservativity) holds, i.e.

$$\sum_{i=1}^{m} G_i(\rho_i(t,0^-), p(t)) = \sum_{j=m+1}^{m+n} G_j(p(t), \rho_j(t,0^+)), \quad \text{for a.e. } t.$$

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Given $\vec{\rho}_0$ i.c., we say that $\vec{\rho} = (\rho_1, \ldots, \rho_{m+n})$ is an admissible solution for the Cauchy problem on the network, if $\exists p$ in $L^\infty(\mathbb{R}_+, [0, R])$ s.t.

- each component ρ_h is entropy solution for the IBVP

$$\begin{cases} \rho_{h,t} + f_h(\rho_h)_x = 0, & \text{on } \mathbb{R}_+ \times \Omega_h, \\ \rho_h(t,0) \approx p(t), & \text{on } \mathbb{R}_+, \\ \rho_h(0,x) = \rho_0^h(x), & \text{on } \Omega_h, \end{cases}$$

  (this includes the "density-at-junction" condition $\forall h\ v_h(t) = p(t)$)
- and "flux-at-junction" condition (conservativity) holds, i.e.

$$\sum_{i=1}^{m} G_i(\rho_i(t,0^-), p(t)) = \sum_{j=m+1}^{m+n} G_j(p(t), \rho_j(t,0^+)), \quad \text{for a.e. } t.$$

Drawback: with this definition { uniqueness is not obvious...
existence is not obvious.

The vanishing viscosity germ

JCC ⇔ "Germ" ∼ { stationary road-wise constant admissible sol. }

The germ underlying the above description of admissibility is

$$\mathcal{G}_{VV} = \left\{ \begin{array}{l} \vec{u} = (u_1, \ldots, u_{m+n}) : \exists p \in [0, R] \text{ s.t. :} \\ \qquad \sum_{i=1}^{m} G_i(u_i, p) = \sum_{j=m+1}^{m+n} G_j(p, u_j) \\ \qquad G_i(u_i, p) = f_i(u_i), \quad G_j(p, u_j) = f_j(u_j) \ \forall i, j \end{array} \right\}.$$

Proposition (the Riemann solver $\mathcal{R}_{VV}$ at junction)

Given any $\vec{u} = (u_1, \ldots, u_{m+n}) \in [0, R]^{m+n}$ the corresponding Riemann problem at the junction has an admissible solution $\vec{\rho} = \mathcal{R}_{VV}[\vec{u}]$. The vector of traces $\vec{\gamma\rho} = (\rho_1(0^-), \ldots, \rho_{m+n}(0^+))$ belongs to $\mathcal{G}_{VV}$.

Idea of proof: given Riemann data $\vec{u}$, construct p (the common boundary-value in the def. of admissibility) by solving

$$\text{find } p_{\vec{u}} \text{ s.t. } \sum_{i=1}^{m} G_i(u_i, p_{\vec{u}}) = \sum_{j=m+1}^{m+n} G_j(p_{\vec{u}}, u_j)$$

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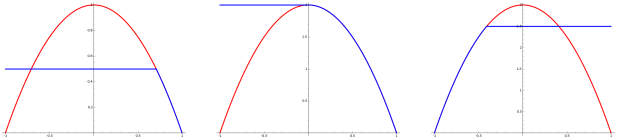
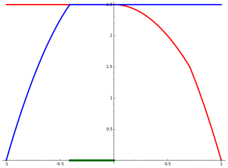
Example

Consider a junction consisting of two incoming and one outgoing roads.

$$f_1(x) = -x^2 + 1, \qquad f_2(x) = -2x^2 + 2, \qquad f_3(x) = -3x^2 + 3.$$

Given the initial condition $(\rho_0^1 = -\sqrt{1/2}, \rho_0^2 = 1/4, \rho_0^3 = \sqrt{1/6})$ one can trace $p \mapsto G_i(\rho_0^i, p)$, $i = 1, 2$, $p \mapsto G_3(p, \rho_0^3)$





For all $p \in [-\sqrt{1/6}, 0]$,

$$\sum_{i=1}^{2} G_i(\rho_0^i, p) = G_3(p, \rho_0^3) = 2.5.$$

The fluxes $G_{1,2}(\rho_0^i, p)$, $G_3(p, \rho_0^3)$ are independent of $p \in [-\sqrt{1/6}, 0]$.

NB: In practice, $p_{\vec{u}}$ can be found by *regula falsi* method.

Germ-based equivalent definitions of admissibility

A function $\vec{\rho} = (\rho_1, \ldots, \rho_{m+n})$ is an admissible solution if and only if

Definition II (trace-based: used to prove uniqueness)

- $\forall h \in \{1, \ldots, m+n\}$, ρ_h is a Kruzhkov solution on the road Ω_h;
- traces-in-germ condition holds:

$$\text{for a.e. } t \in \mathbb{R}_+,\ \vec{\gamma\rho}(t) := (\rho_1(t, 0^-), \ldots, \rho_{m+n}(t, 0^+)) \in \mathcal{G}_{VV}.$$

cf. [Garavello, Natalini, Piccoli, Terracina '07]
(admissibility in terms of Riemann solver at junction)

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- $\forall h \in \{1, \ldots, m+n\}$, ρ_h is a Kruzhkov solution on the road Ω_h;
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$$\forall \vec{k} \in \mathcal{G}_{VV} \quad \sum_{h=1}^{m+n} \left(\int_{\mathbb{R}_+} \int_{\Omega_h} \{ |\rho_h - k_h| \xi_t + q_h(\rho_h, k_h) \xi_x \} \, dx \, dt \right) \geq 0.$$

cf. [Baiti, Jenssen'97],[Audusse, Perthame'05]. These are Kato ineq.!

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Crucial properties of $\mathcal{G}_{VV}$

- The germ $\mathcal{G}_{VV}$ is "complete": namely, the Riemann solver $\mathcal{R}_{VV}$ is defined for all data.
- The germ $\mathcal{G}_{VV}$ is "dissipative": namely, for all $\vec{k}, \vec{c} \in \mathcal{G}_{VV}$

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Lemma (Oleinik-like condition)

$\mathcal{G}_{VV}$ is characterized by a "graph above-graph below" condition (cf. [Diehl JHDE'09] in the $n = m = 1$ case) + conservativity condition.

The maximality can be refined: consider

$$\mathcal{G}^o_{VV} = \{\text{strict "graph above-graph below" condition}\}.$$

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Well-posedness in the frame of admissible solutions

Theorem (Main result)

- *There exists an admissible solution for all L^∞ datum. Moreover, such solutions are VV limits.*
- *If $\vec{\rho}$ and $\hat{\vec{\rho}}$ are admissible solutions corresponding to $\vec{\rho}_0$ and $\hat{\vec{\rho}}_0$,*

$$\sum_{h=1}^{m+n} \|\rho_h(t) - \hat{\rho}_h(t)\|_{L^1(\Omega_h)} \leq \sum_{h=1}^{m+n} \|\rho_h^0 - \hat{\rho}_h^0\|_{L^1(\Omega_h)}.$$

(L^1-contractivity, monotonicity). Also Kato inequality holds.
$\Longrightarrow$ uniqueness of an admissible solution to Cauchy problem.

Proof of uniqueness: Kruzhkov-per-road (doubling of variables) + existence of junction traces $\gamma\vec{\rho}, \gamma\hat{\vec{\rho}} \Longrightarrow$ up-to-junction Kato inequality:

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Existence of admissible solutions

Proof of existence:

- Recall some of [Coclite, Garavello SIAM'10] results:
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- All $\vec{k} \in \mathcal{G}_{VV}^o$ can be obtained as VV limits. Tool: explicit construction of viscosity profiles $\vec{k}^{\,\varepsilon}$ (based upon the Oleinik "graph above-graph below" condition).
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Alternative proof: (for monotone junction Riemann solvers)

- use Godunov numerical scheme to construct solutions
  - Godunov scheme is well-balanced: $\vec{k}$ in the germ (stationary solutions) are exact discrete solutions
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Grazie!

Thank you for your attention!