Well-posedness
for some LWR models on a junction

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ANalysis and COntrol on NETworks: trends and perspectives
Padua, March 9-11, 2016
Statement of the problem

We consider a junction where $m$ incoming and $n$ outgoing roads meet.

- Incoming roads: $x \in \Omega_i = \mathbb{R}_-, i = 1, \ldots, m$;
- Outgoing roads: $x \in \Omega_j = \mathbb{R}_+, j = m + 1, \ldots, m + n$;
- The junction is located at $x = 0$. 
Statement of the problem

On each road the evolution of traffic is described by

$$\partial_t \rho_h + \partial_x f_h(\rho_h) = 0, \quad h = 1, \ldots, m + n,$$

- $\rho_h$ density of vehicles, $[0, R]$-valued for all $h$
- $f_h$ bell-shaped, non linearly non degenerate, Lipschitz fluxes

$$\forall h \quad f_h(0) = 0 = f_h(R)$$

Moreover, we postulate conservativity at the junction:

$$\sum_{i=1}^{m} f_i(\rho_i(t, 0^-)) = \sum_{j=m+1}^{m+n} f_j(\rho_j(t, 0^+)).$$
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\[ \sum_{i=1}^{m} f_i(\rho_i(t, 0^-)) = \sum_{j=m+1}^{m+n} f_j(\rho_j(t, 0^+)). \]
Fix $\bar{\rho}_0 = (\rho_0^1, \ldots, \rho_0^{m+n})$ s.t. $\rho_0^h \in L^\infty(\Omega_h, [0, R])$, $\forall h \in \{1, \ldots, m+n\}$. We call “solution” a $(m+n)$-uple $\bar{\rho} = (\rho_1, \ldots, \rho_{m+n})$ s.t.

- $\forall h \rho_h$ is a Kruzhkov entropy solution in $\mathbb{R}_+ \times \{\Omega_h \setminus \partial\Omega_h\}$. Namely $\forall k \in [0, R]$ and $\forall \xi \in C^1_c(\mathbb{R}_+ \times \Omega_h)$, $\xi \geq 0$

$$\int_{\mathbb{R}_+} \int_{\Omega_h} |\rho_h - k|\xi_t + q_h(\rho_h, k)\xi_x \, dx \, dt \geq 0$$

(with $q_h(\rho_h, k) = \text{sign}(\rho_h - k)(f_h(\rho_h) - f_h(k))$ the Kruzhkov entropy flux)

**Idea:** $k$ is an obvious solution...
the above inequalities are “Kato inequalities” between $\rho_h$ and $k$

- conservation at the junction holds.

There is no hope to prove well-posedness for “solutions”.

**Analogy:**
junction coupling conditions (JCC) play the role of boundary conditions (BC). Imposing mere conservativity condition as JCC leaves the Cauchy problem underdetermined! [A. ESAIM Proc.’15].
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We call “solution” a $(m + n)$-uple $\vec{\rho} = (\rho_1, \ldots, \rho_{m+n})$ s.t.

- $\forall h$ $\rho_h$ is a Kruzhkov entropy solution in $\mathbb{R}^+ \times \{\Omega_h \setminus \partial\Omega_h\}$.
  
  Namely $\forall k \in [0, R]$ and $\forall \xi \in C^1_c(\mathbb{R}^+ \times \Omega_h)$, $\xi \geq 0$

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Many different approaches to single out “suitable” solutions

For the Riemann problem at the junction (road-wise constant initial conditions):

- [Holden, Risebro SIMA’95] maximize a concave “entropy” function at the junction;
- [Coclite, Piccoli SIMA’02], [Coclite, Garavello, Piccoli SIMA’05] traffic distribution matrix + optimization;
- [Lebacque ’96], [Lebacque, Khoshyaran ’02] Supply-Demand model;
- ...

We prove well-posedness for solutions to the Cauchy problem which are limits of vanishing viscosity (VV) approximations.

Essential ingredient: intrinsic characterization of VV limits (a notion of solution, expressed e.g. via some “entropy inequalities”)

VV limits obey rather artificial JCC... but the study is instructive!
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Fix $\varepsilon > 0$. Consider convection-$\varepsilon$-diffusion + JunctionCouplingCondition:

$$\begin{cases}
\rho_{h,t}^\varepsilon + f_h(\rho_h^\varepsilon)x = \varepsilon \rho_{h,xx}^\varepsilon, \\
\sum_{i=1}^m \left( f_i(\rho_i^\varepsilon(t,0)) - \varepsilon \rho_{i,x}^\varepsilon(t,0) \right) = \sum_{j=m+1}^{m+n} \left( f_j(\rho_j^\varepsilon(t,0)) - \varepsilon \rho_{j,x}^\varepsilon(t,0) \right), \\
\rho_h^\varepsilon(t,0) = \rho_h'(t,0), \\
\rho_h^\varepsilon(0,x) = \rho_h^0(x),
\end{cases}$$

where the approximated initial conditions $\tilde{\rho}_0,\varepsilon$ satisfy

$$\rho_h^0,\varepsilon \in W^{2,1}(\Omega_h) \cap C^\infty(\Omega_h), \quad [0, R] \text{-valued},$$

$$\rho_h^0,\varepsilon \longrightarrow \rho_h^0, \quad \text{a.e. and in } L^p(\Omega_h), \quad 1 \leq p < \infty, \quad \text{as } \varepsilon \rightarrow 0,$$

$$\|\rho_h^0,\varepsilon\|_{L^1(\Omega_h)} \leq \|\rho_h^0\|_{L^1(\Omega_h)}, \quad \|(\rho_h^0,\varepsilon)_x\|_{L^1(\Omega_h)} \leq TV(\rho_h^0),$$

$$\varepsilon\|(\rho_h^0,\varepsilon)_{xx}\|_{L^1(\Omega_h)} \leq C_0,$$

with $C_0 > 0$ independent from $\varepsilon, \ h$. 
Coclite and Garavello, 2010

Theory of semigroups ⇒ ∀ε > 0 there exists a unique ρ^ε s.t.

ρ^ε_h ∈ C([0, ∞); L^2(Ω_h)) ∩ L^1_{loc}((0, ∞); W^{2,1}(Ω_h)), h ∈ {1, . . . , m + n},

0 ≤ ρ^ε_h ≤ R, \sum_{h=1}^{m+n} ||ρ^ε_h(t, ·)||_{L^1(Ω_h)} ≤ \sum_{h=1}^{m+n} ||ρ^0_h||_{L^1(Ω_h)}, ∀t ≥ 0,

+ additional a priori estimates like ||√ερ^ε_x||_{L^2} ≤ C.

Compensated compactness ⇒ existence of a sequence {ε_ℓ}_ℓ∈N,

ε_ℓ → 0 and a solution ̄ρ of the inviscid Cauchy problem such that

∀h ρ^ε_ℓ_h → ρ_h, a.e. and in L^p_{loc}(R_+ × Ω_h), 1 ≤ p < ∞.  (1)

Uniqueness of VV solutions for the inviscid problem?
It is proved [Coclite, Garavello SIMA’10] in the case m = n, f_h ≡ f
based on comparison with the obvious solution ̄k = (k, . . . , k).

“Local objective”: extend these results to general junctions
“Global objective”: better understand solvers for different JCC
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\]

\[
0 \leq \rho^\varepsilon_h \leq R, \quad \sum_{h=1}^{m+n} \|\rho^\varepsilon_h(t, \cdot)\|_{L^1(\Omega_h)} \leq \sum_{h=1}^{m+n} \|\rho^0_h\|_{L^1(\Omega_h)}, \quad \forall t \geq 0,
\]

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Compensated compactness ⇒ existence of a sequence \( \{\varepsilon_\ell\}_{\ell \in \mathbb{N}} \), \( \varepsilon_\ell \to 0 \) and a solution \( \bar{\rho} \) of the inviscid Cauchy problem such that

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“Global objective”: better understand solvers for different JCC
Monotonicity of the $VV$ solver

Not only the viscous problem of Coclite-Garavello is well posed. The key property, independent of $\varepsilon$, can be expressed as:

(i) monotonicity (order-preservation) of the solver:
\[ \bar{\rho}_0 \geq \hat{\rho}_0 \text{ (componentwise)} \Rightarrow \forall t > 0 \quad \bar{\rho}^\varepsilon(t, \cdot) \geq \hat{\rho}^\varepsilon(t, \cdot); \]

(ii) $L^1$-contractivity of the solver:
\[ \sum_{h=1}^{m+n} \| \rho_h^\varepsilon(t, \cdot) - \hat{\rho}_h^\varepsilon(t, \cdot) \|_{L^1(\Omega_h)} \leq \sum_{h=1}^{m+n} \| \rho_{h,\varepsilon} - \hat{\rho}_{h,\varepsilon} \|_{L^1(\Omega_h)}; \]

(iii) Kato inequality: for all test function $\xi \in D((0, +\infty) \times \mathbb{R})$, $\xi \geq 0$
\[ - \int_{\mathbb{R}^+} \int_{\Omega_h} \left( |\rho_h^\varepsilon - \hat{\rho}_h^\varepsilon| \xi_t + q_h(\rho_h^\varepsilon, \hat{\rho}_h^\varepsilon) \xi_x + \varepsilon |\rho_h^\varepsilon - \hat{\rho}_h^\varepsilon| x \xi_x \right) \leq 0, \]

Links: (iii) $\Rightarrow$ (ii) (with $\xi \sim 1_{[0,t]}$); (ii) $\Leftrightarrow$ (i) (Crandall-Tartar lemma)

These properties are inherited by $VV$ admissible solutions.

Kato inequality guarantees uniqueness of $VV$ solutions, provides their intrinsic characterization.
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Monotonicity of the Riemann solver at junction

At least heuristically, \( JCC \iff \text{Riemann solver at junction} \).
Different solution semigroups for LWR on networks originate from different Riemann solvers at junction (Garavello, Piccoli, . . .)

Byproduct of our analysis:

a subclass of these semigroups shares key features of \( VV \) solutions

Required property: monotonicity of the junction Riemann solver
(larger data on a road lead to larger solutions on the whole network)

General principle:

monotone, Lipschitz Riemann solver at junction
\( \implies \) an intrinsic notion of solution + well-posedness.

Notion of solution and uniqueness:
mimic tools developed for discontinuous-flux scalar conservation laws
[A., Karlsen, Risebro ARMA’11]: admissibility germ, adapted entropies

Construction of solutions:
approximations, e.g. by the Godunov finite volume scheme
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approximations, e.g. by the Godunov finite volume scheme
Consider the Riemann problem for pure SCL:

\[
\begin{aligned}
    u_t + f(u)_x &= 0, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\
    u_0(x) &= \begin{cases} 
        a, & \text{if } x < 0, \\
        b, & \text{if } x > 0.
    \end{cases}
\end{aligned}
\]

Denote by \( \mathcal{R}[a, b] \) its Kruzhkov entropy solution.
The **Godunov flux is the function** \((a, b) \mapsto f(\mathcal{R}[a, b])|_{x=0\pm} \).

Analytically

\[
    G(a, b) = \begin{cases} 
        \min_{s \in [a, b]} f(s) & \text{if } a \leq b, \\
        \max_{s \in [b, a]} f(s) & \text{if } a \geq b.
    \end{cases}
\]

**Key properties of the Godunov flux:**
- **Consistency:** for all \( a \in [0, R] \), \( G(a, a) = f(a) \);
- **Monotonicity and Lipschitz continuity:**
  \( \exists L > 0 : \forall (a, b) \in [0, R]^2 \) there holds
  \[
      0 \leq \partial_a G(a, b) \leq L, \quad -L \leq \partial_b G(a, b) \leq 0.
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  0 \leq \partial_a G(a, b) \leq L, \quad -L \leq \partial_b G(a, b) \leq 0.
  \]
Consider the initial and boundary value problem

\[
\begin{cases}
    u_t + f(u)_x = 0, & \text{for } t > 0, \ x < 0 \\
    u(t, 0^-) \approx u_b(t), \\
    u(0, x) = u_0(x),
\end{cases}
\]

\(u\) is an entropy solution for the IBVP if

1. \(u\) is a Kruzhkov entropy solution in the interior of \(\mathbb{R}_+ \times \mathbb{R}_-\),
2. \(u\) satisfies the boundary condition in the (BLN) sense

for a.e. \( t > 0 \) \( \forall k \in \text{conv}\{u(t, 0^-), u_b(t)\} \)

\[q(u(t, 0^-), k) \equiv \text{sign}(u(t, 0^-) - k) \ (f(u(t, 0^-)) - f(k)) \geq 0 \quad (\ast)\]

**A reformulation of BLN:**

\(u\) satisfies BLN (\(\ast\)) \(\iff\) \(f(u(t, 0^-)) = G(u(t, 0^-), u_b(t)).\)

(known since [Dubois, LeFloch JDE’88])
The junction as a family of IBVPs

Fix $\tilde{\rho}_0 = (\rho_0^1, \ldots, \rho_0^{m+n})$ s.t. $\rho_0^h \in L^\infty(\Omega_h, [0, R]), \forall h \in \{1, \ldots, m + n\}$. We look for $\tilde{\rho} = (\rho_1, \ldots, \rho_{m+n})$ s.t. $\forall h, \rho_h \in L^\infty(\mathbb{R}_+ \times \Omega_h, [0, R])$ is an entropy solution of

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where $\tilde{v} = (v_1, \ldots, v_{m+n}) : \mathbb{R}_+ \to [0, R]^{m+n}$ is to be chosen

- depending on the JCC one wants to express,
- in particular, ensuring conservation at the junction.

The case of VV limits: require that $\nu_h$ be the same on all roads

[A., Cancès JHDE’15], [A., Mitrović AnnIHP’15]:
motivations in the “discontinuous-flux” case $m = n = 1$.

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[A., Cancès CompGS’13, JHDE’15]: examples of different admissibility criteria (e.g. vanishing capillarity).
Admissibility at the junction

Continuity of $\rho$ at the junction - up to boundary layers! - is desired.

Definition I (Formalized from heuristics of the model)

Given $\vec{\rho}_0$ i.c., we say that $\vec{\rho} = (\rho_1, \ldots, \rho_{m+n})$ is an admissible solution for the Cauchy problem on the network, if $\exists p$ in $L^\infty(\mathbb{R}_+, [0, R])$ s.t.

- each component $\rho_h$ is entropy solution for the IBVP
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  \]
  (this includes the "density-at-junction" condition $\forall h \ \nu_h(t) = p(t)$)

- and "flux-at-junction" condition (conservativity) holds, i.e.
  \[
  \sum_{i=1}^{m} G_i(\rho_i(t, 0^-), p(t)) = \sum_{j=m+1}^{m+n} G_j(p(t), \rho_j(t, 0^+)), \quad \text{for a.e. } t.
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Drawback: with this definition

- uniqueness is not obvious...
- existence is not obvious.
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Continuity of $\rho$ at the junction - up to boundary layers! - is desired.

**Definition I (Formalized from heuristics of the model)**

Given $\vec{\rho}_0$ i.c., we say that $\vec{\rho} = (\rho_1, \ldots, \rho_{m+n})$ is an admissible solution for the Cauchy problem on the network, if $\exists p$ in $L^\infty(\mathbb{R}_+,[0,R])$ s.t.

- each component $\rho_h$ is entropy solution for the IBVP
  
  \[
  \begin{cases}
  \rho_{h,t} + f_h(\rho_h)x = 0, & \text{on } \mathbb{R}_+ \times \Omega_h, \\
  \rho_h(t,0) \approx p(t), & \text{on } \mathbb{R}_+, \\
  \rho_h(0,x) = \rho^h_0(x), & \text{on } \Omega_h,
  \end{cases}
  \]

  (this includes the “density-at-junction” condition $\forall h \nu_h(t) = p(t)$)

- and “flux-at-junction” condition (conservativity) holds, i.e.
  
  \[
  \sum_{i=1}^{m} G_i(\rho_i(t,0^-),p(t)) = \sum_{j=m+1}^{m+n} G_j(p(t),\rho_j(t,0^+)), \quad \text{for a.e. } t.
  \]

**Drawback:** with this definition

- uniqueness is not obvious...
- existence is not obvious.
**The vanishing viscosity germ**

\[ \text{JCC} \Leftrightarrow \text{“Germ”} \sim \{ \text{stationary road-wise constant admissible sol.} \} \]

The germ underlying the above description of admissibility is

\[ G_{VV} = \left\{ \tilde{u} = (u_1, \ldots, u_{m+n}) : \exists p \in [0, R] \text{ s.t. :} \right\} \]

\[ \sum_{i=1}^{m} G_i(u_i, p) = \sum_{j=m+1}^{m+n} G_j(p, u_j) \]

\[ G_i(u_i, p) = f_i(u_i), \quad G_j(p, u_j) = f_j(u_j) \quad \forall i, j \]

**Proposition (the Riemann solver \( R_{VV} \) at junction)**

*Given any \( \tilde{u} = (u_1, \ldots, u_{m+n}) \in [0, R]^{m+n} \) the corresponding Riemann problem at the junction has an admissible solution \( \tilde{\rho} = R_{VV}[\tilde{u}] \).*

*The vector of traces \( \tilde{\gamma} \tilde{\rho} = (\rho_1(0^-), \ldots, \rho_{m+n}(0^+)) \) belongs to \( G_{VV} \).*

**Idea of proof:** given Riemann data \( \tilde{u} \), construct \( p \)

(the common boundary-value in the def. of admissibility) by solving

\[ \text{find } p_{\tilde{u}} \text{ s.t. } \sum_{i=1}^{m} G_i(u_i, p_{\tilde{u}}) = \sum_{j=m+1}^{m+n} G_j(p_{\tilde{u}}, u_j) \]
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\[ \text{JCC} \iff \text{“Germ” } \sim \{ \text{stationary road-wise constant admissible sol.} \} \]

The germ underlying the above description of admissibility is

\[ \mathcal{G}_{VV} = \left\{ \tilde{u} = (u_1, \ldots, u_{m+n}) : \exists p \in [0, R] \text{ s.t.} \right. \]
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Proposition (the Riemann solver \( \mathcal{R}_{VV} \) at junction)

Given any \( \tilde{u} = (u_1, \ldots, u_{m+n}) \in [0, R]^{m+n} \) the corresponding Riemann problem at the junction has an admissible solution \( \bar{\rho} = \mathcal{R}_{VV}[\tilde{u}] \).

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Example

Consider a junction consisting of two incoming and one outgoing roads.

\[ f_1(x) = -x^2 + 1, \quad f_2(x) = -2x^2 + 2, \quad f_3(x) = -3x^2 + 3. \]

Given the initial condition \((\rho_0^1 = -\sqrt{1/2}, \rho_0^2 = 1/4, \rho_0^3 = \sqrt{1/6})\) one can trace \(p \mapsto G_i(\rho_0^i, p), \quad i = 1, 2, \quad p \mapsto G_3(p, \rho_0^3)\)

For all \(p \in [-\sqrt{1/6}, 0]\),

\[ \sum_{i=1}^{2} G_i(\rho_0^i, p) = G_3(p, \rho_0^3) = 2.5. \]

The fluxes \(G_{1,2}(\rho_0^i, p), \quad G_3(p, \rho_0^3)\) are independent of \(p \in [-\sqrt{1/6}, 0]\).

**NB:** In practice, \(\rho_{\bar{u}}\) can be found by *regula falsi* method.
Germ-based equivalent definitions of admissibility

A function $\mathbf{\bar{\rho}} = (\rho_1, \ldots, \rho_{m+n})$ is an admissible solution if and only if

**Definition II (trace-based: used to prove uniqueness)**

- $\forall h \in \{1, \ldots, m + n\}$, $\rho_h$ is a Kruzkhov solution on the road $\Omega_h$;
- traces-in-germ condition holds:
  
  for a.e. $t \in \mathbb{R}_+$, $\mathbf{\gamma}(t) := (\rho_1(t, 0^-), \ldots, \rho_{m+n}(t, 0^+)) \in G_{VV}$.

cf. [Garavello, Natalini, Piccoli, Terracina ’07]
(admissibility in terms of Riemann solver at junction)

**Definition III (integral formulation: used to prove existence)**

- $\forall h \in \{1, \ldots, m + n\}$, $\rho_h$ is a Kruzkhov solution on the road $\Omega_h$;
- adapted entropy inequalities hold: $\forall \xi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$, $\xi \geq 0$

$$\forall \mathbf{k} \in G_{VV}, \sum_{h=1}^{m+n} \left( \int_{\mathbb{R}_+} \int_{\Omega_h} \{|\rho_h - k_h| \xi_t + q_h(\rho_h, k_h) \xi_x\} \, dx \, dt \right) \geq 0.$$ 

cf. [Baiti, Jenssen’97], [Audusse, Perthame’05]. These are Kato ineq.!
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\forall \vec{k} \in \mathcal{G}_{VV} \quad \sum_{h=1}^{m+n} \left( \int_{\mathbb{R}_+} \int_{\Omega_h} \left| \rho_h - k_h \right| \xi_t + q_h(\rho_h, k_h) \xi_x \, dx \, dt \right) \geq 0.
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Crucial properties of $G_{VV}$

- The germ $G_{VV}$ is “complete”: namely, the Riemann solver $R_{VV}$ is defined for all data.
- The germ $G_{VV}$ is “dissipative”: namely, for all $\vec{k}, \vec{c} \in G_{VV}$
  \[ \sum_{i=1}^{m} q_i(k_i, c_i) - \sum_{j=m+1}^{m+n} q_j(k_j, c_j) \geq 0. \] (#)
- The germ $G_{VV}$ is “maximal”: namely, if $\vec{k}$ fulfills (#) for all $\vec{c} \in G_{VV}$, then $\vec{k} \in G_{VV}$.

Lemma (Oleinik-like condition)

$G_{VV}$ is characterized by a “graph above-graph below” condition (cf. [Diehl JHDE’09] in the $n = m = 1$ case) + conservativity condition.

The maximality can be refined: consider

$G^o_{VV} = \{\text{strict “graph above-graph below” condition}\}$.

- The subset $G^o_{VV}$ of $G_{VV}$ is “definite”: namely,
  
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Theorem (Main result)

There exists an admissible solution for all $L^\infty$ datum. Moreover, such solutions are VV limits.

If $\bar{\rho}$ and $\hat{\rho}$ are admissible solutions corresponding to $\bar{\rho}_0$ and $\hat{\rho}_0$,

$$\sum_{h=1}^{m+n} \|\rho_h(t) - \hat{\rho}_h(t)\|_{L^1(\Omega_h)} \leq \sum_{h=1}^{m+n} \|\rho^0_h - \hat{\rho}^0_h\|_{L^1(\Omega_h)}.$$ 

($L^1$-contractivity, monotonicity). Also Kato inequality holds. $\implies$ uniqueness of an admissible solution to Cauchy problem.

Proof of uniqueness: Kruzhkov-per-road (doubling of variables) + existence of junction traces $\gamma\bar{\rho}, \gamma\hat{\rho} \implies$ up-to-junction Kato inequality:

$$- \int_{\mathbb{R}^+} \int_{\Omega_h} \left( |\rho_h - \hat{\rho}_h| \xi_t + q_h(\rho_h, \hat{\rho}_h) \xi_x \right) \leq \text{RHS}[\gamma\bar{\rho}, \gamma\hat{\rho}] \quad \forall \xi \geq 0$$

By Def.II, $\gamma\bar{\rho}, \gamma\hat{\rho} \in G_{VV}$. Then dissipativity (#) $\implies$ RHS[$\gamma\bar{\rho}, \gamma\hat{\rho}] \leq 0.$
## Well-posedness in the frame of admissible solutions

### Theorem (Main result)

- **There exists an admissible solution** for all $L^\infty$ datum. Moreover, **such solutions are VV limits**.
- If $\rho$ and $\hat{\rho}$ are admissible solutions corresponding to $\rho_0$ and $\hat{\rho}_0$,

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$\implies$ **uniqueness of an admissible solution to Cauchy problem**.

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By Def.II, $\gamma\rho, \gamma\hat{\rho} \in G_{VV}$. Then dissipativity ($\#$) \[\Rightarrow\] $RHS[\gamma\rho, \gamma\hat{\rho}] \leq 0$. 
Existence of admissible solutions

Proof of existence:

- Recall some of [Coclite, Garavello SIAM’10] results:
  - (subseq. of) VV approximations $\vec{\rho}_\varepsilon$ converges a.e. to a limit $\vec{\rho}$;  
  - for all $\varepsilon > 0$, the semigroup of VV approximations is order-preserving / $L^1$-contractive / fulfills Kato inequality.

- Standard theory: each component $\rho_h$ of $\vec{\rho}$, s.t. $\rho_h^\varepsilon \to \rho_h$, is a Kruzhkov entropy solution in $\Omega_h$.

- All $\vec{k} \in \mathcal{G}_{VV}$ can be obtained as VV limits.
  
  Tool: explicit construction of viscosity profiles $\vec{k}_\varepsilon$ (based upon the Oleinik “graph above-graph below” condition).

- Pass to the limit $\varepsilon \to 0$ in Kato ineq. written for $\vec{\rho}_\varepsilon$ and $\vec{k}_\varepsilon$.
  We get adapted entropy inequalities $\Rightarrow \rho$ fulfills Def. III.

Alternative proof: (for monotone junction Riemann solvers)

- use Godunov numerical scheme to construct solutions
  - Godunov scheme is well-balanced:
    $\vec{k}$ in the germ (stationary solutions) are exact discrete solutions
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Grazie!
Thank you for your attention!