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Some Recent Results on Nonlocal Geometric Equations and Applications

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Journées Numériques de Besançon, May 2010

Three Model Examples of Front Propagation Problems with Nonlocal Normal Velocities

In these examples, the front, denoted by Γ_t , is the boundary of an open subset Ω_t. Typically, in phase transitions problems, Ω_t is a phase and Γ_t the interface between two phases.

Model problem 1 : dislocation type equations

$$V_n = c_0(\cdot, t) \star 1\!\!1_{\Omega_t} + c_1(x, t) + \varepsilon\kappa(x)$$

where c_0, c_1 are given function and $\kappa(x)$ is the mean curvature of Γ_t at x. The parameter ε will be 0 or 1

NB : dislocation lines are defects in crystals.

Model problem 2 : a Fitzhugh-Nagumo type system

$$V_n = \alpha(v) + \varepsilon\kappa(x)$$

where v solves an equation like

$$v_t - \Delta v = g^+(v)\mathbb{1}_{\Omega_t} + g^-(v)(1 - \mathbb{1}_{\Omega_t}) \text{ in } \mathbb{R}^N \times (0, T)$$

NB : This system is obtained as the asymptotics of a Fitzhugh-Nagumo system arising in neural wave propagation or chemical kinetics (cf. Soravia-Souganidis).

Model problem 3 : Volume dependent velocities

$$V_n = \beta(\mathcal{L}^N(\Omega_t)) + \varepsilon\kappa(x)$$

where the function $\beta : I\!R \to I\!R$ is Lipschitz continuous.

NB : Such fronts arise as the asymptotic limits of Allen-Cahn type equations with integral terms (cf. Chen-Hilhorst-Logak, Da Lio-Kim-Slepcev).

The General Framework : Level Set Formulation

The idea is to represent Ω_t by setting $\Omega_t = \{u(\cdot,t) > 0\}$ and $\Gamma_t = \{u(\cdot,t) = 0\}$: then u solves the "level-sets equation"

$$\frac{\partial u}{\partial t} + H[\mathbb{1}_{\Omega_t}](x,t,Du,D^2u) = 0 \quad \text{in } \mathbb{R}^N \times (0,T)$$

$$u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^N$$

The nonlinearity $H[\chi]$ depends in a nonlocal way on the function $\chi \in L^\infty(\mathbb{R}^N \times (0,T), [0,1])$ and is a "good" level set equation for any fixed χ, i.e. it is degenerate parabolic and geometric.

Typically, for the dislocations case

$$\frac{\partial u}{\partial t} - c[\chi](x,t)|Du| - \varepsilon\left(\Delta u - \frac{D^2uDu \cdot Du}{|Du|^2}\right) = 0$$

where

$$c[\chi](x,t) = (c_0(\cdot,t) \star \chi)(x) + c_1(x,t)$$

Important point : to have good informations on the "standard" level set equation

$$\frac{\partial u}{\partial t} - c(x,t)|Du| - \varepsilon\left(\Delta u - \frac{D^2uDu \cdot Du}{|Du|^2}\right) = 0$$

Some basic results for the "standard" LSA

– For any continuous initial data u_0, there exists a unique continuous solution of the level set equation

– If $\{u_0 > 0\} = \{v_0 > 0\}$ and $\{u_0 < 0\} = \{v_0 < 0\}$, then $\{u(\cdot,t) > 0\} = \{v(\cdot,t) > 0\}$ and $\{u(\cdot,t) < 0\} = \{v(\cdot,t) < 0\}$ for all t.

– Therefore $\{u(\cdot,t) = 0\} = \{v(\cdot,t) = 0\}$ and the "moving front" $\Gamma_t := \{u(\cdot,t) = 0\}$ does not depend on the "representation" we have chosen for Ω_0.

– Γ_t is well-defined and inherit the "good" stability properties of viscosity solutions.

A first key remark : Monotonicity

The level-set approach satisfies the property

$$\Omega_t^1 \subset \Omega_t^2 \;\Rightarrow\; \Omega_{t+h}^1 \subset \Omega_{t+h}^2 \text{ for all } h \geq 0$$

a geometric version of the Maximum Principle.

Nonlocal normal velocities can be handled as well through the Slepcev's approach BUT only if this monotonicity property holds... and this is not always the case !

Consequence : when the motion is not monotone, we have to combine level-set and viscosity solutions method with non-monotone arguments (contraction properties, for example).

Second remark : Γ_t is well-defined BUT it may have a "non-empty interior"

Main consequence : if $u_k \to u$ locally uniformly then we do not have in general

$$1\!\!1_{\{u_k(\cdot,t)\geq 0\}} \to 1\!\!1_{\{u(\cdot,t)\geq 0\}} \quad \text{in } L^1(I\!\!R^N)$$

Main difficulty : the nonlocal equation does not have in general a good dependence w.r.t. u through the nonlocal term...

Remark : If $\epsilon = 0$ (no curvature dependence) AND if c does not change sign, Γ_t has an empty-interior for all t (Soner-Souganidis-GB) and even a 0-Lebesgue measure (Ley). Less problem in that case !
But this also shows that either c changing sign and/or curvature dependence is a problem...

CONCLUSIONS : **Difficulties with**

(i) Suitable definition of "weak" solution

(ii) Existence

(iii) Uniqueness

Definition of "Weak Solutions"

A function u is said to be a weak solution of the nonlocal geometric equation if u is a viscosity solution in the L^1– sense of

$$\frac{\partial u}{\partial t} + H[\chi](x, t, Du, D^2u) = 0 \quad \text{in } \mathbb{R}^N \times (0, T)$$

for some function χ satisfying

$$\mathbb{1}_{\{u(\cdot,t)>0\}}(x) \ \leq \ \chi(x,t) \ \leq \mathbb{1}_{\{u(\cdot,t)\geq 0\}}(x) \text{ in } \mathbb{R}^N \times (0, T)$$

Existence of "Weak Solutions"

Key Additional Assumption : if $\chi_n \rightharpoonup \chi$ weakly-$*$ in $X := L^\infty(\mathbb{R}^N \times [0,T]; [0,1])$ with $\chi_n, \chi \in X$ for all n, then

$$\int_0^t H[\chi_n](x,s,p,M)ds \underset{n\to+\infty}{\longrightarrow} \int_0^t H[\chi](x,s,p,M)ds$$

locally uniformly for $t \in [0,T]$.

Theorem : Under general assumptions, there exists a weak solution of the nonlocal HJ Equation.

Main steps of the proof :
(i) Use Kakutani's fixed point theorem for the set-valued map $\xi : X \rightrightarrows X$

$$\xi(\chi) = \left\{\chi' \ : \ \mathbb{1}_{\{u(\cdot,t)>0\}} \le \chi'(\cdot,t) \le \mathbb{1}_{\{u(\cdot,t)\ge 0\}}\right\}$$

where u is the L^1-viscosity solution of the nonlocal HJ Equation associated to $H[\chi]$.

(ii) In the Hausdorff convex space $L^\infty(\mathbb{R}^N \times [0,T]; \mathbb{R})$, the subset X is convex and compact for the L^∞-weak-$*$ topology (since it is closed and bounded) and, for any $\chi \in X$, $\xi(\chi)$ is a non-empty convex compact subset of X for the L^∞-weak-$*$ topology.

(iii) ξ is upper semicontinuous for this topology, i.e. if

$$\chi_n \in X \underset{L^\infty\text{-weak-}*}{\rightharpoonup} \chi \quad \text{and} \quad \chi_n' \in \xi(\chi_n) \underset{L^\infty\text{-weak-}*}{\rightharpoonup} \chi',$$

then $\chi' \in \xi(\chi)$.

If u_n is the unique L^1-viscosity solution of the nonlocal Equation associated to χ_n, one has to show that u_n converges to the unique solution u of the nonlocal Equation associated to χ.

Consequence of :

1. the half-relaxed limit method
2. a new stability result for L^1-viscosity solutions
3. strong comparison results for the limiting equation

Uniqueness ? (and other approaches)

Case 1 : the "monotone" case and Slepčev's formulation

Assumption : $H[\chi] \le H[\chi']$ if $\chi \ge \chi'$ a.e.

The "natural" formulation (in terms of the "level-sets approach" and viscosity solutions) should be

$$\frac{\partial v}{\partial t} + H[\mathbb{1}_{\{v(\cdot,t)\ge v(x,t)\}}](x,t,Dv,D^2v) = 0 \quad \text{in } \mathbb{R}^N \times (0,T)$$

Theorem : Under general assumptions, there exists a **unique** solution $v \in C(\mathbb{R}^N \times [0,T])$ of this equation such that $v(x,0) = u_0(x)$ in $\mathbb{R}^N$.

Remark : connections with “weak solutions ?

The maximal and minimal weak solutions are the solutions associated respectively to

$$\chi^+ = 1\!\!1_{\{v(\cdot,t)\geq 0\}} \quad \text{and} \quad \chi^- = 1\!\!1_{\{v(\cdot,t)> 0\}}$$

The associated solutions $u^\pm$ satisfy

$$\{u^\pm(\cdot,t)\geq 0\} = \{v(\cdot,t)\geq 0\} \to \chi^+ = 1\!\!1_{\{u^+(\cdot,t)\geq 0\}}$$

$$\{u^\pm(\cdot,t)\leq 0\} = \{v(\cdot,t)\leq 0\} \to \chi^- = 1\!\!1_{\{u^-(\cdot,t)> 0\}}$$

and the nonlocal equation has a unique weak solution if and only if the set $\{v(\cdot,t) = 0\}$ has a zero-Lebesgue measure for almost all $t\in(0,T)$

A counter-example is available for the dislocations' equation if this condition is not satisfied !
(It is based on a counter-example of Soner-Souganidis-GB showing that $\{v(\cdot,t) = 0\}$ may have a non-empty interior.)

Case 2 : the "non-monotone" first-order case (without curvature term)

Here the equation is $\frac{\partial u}{\partial t} - c[\chi](x,t)|Du| = 0$ and a key assumption is

$$c[\chi](x,t) \geq 0 \text{ in } \mathbb{R}^N \times (0,T)$$

for any characteristic function χ.

WHY ?

Because if $c[\chi]$ does not change sign, Γ_t has an empty-interior for all t (Soner-Souganidis-GB) and even a 0-Lebesgue measure (Ley)

Main consequence : if $u_k \to u$ locally uniformly then

$$\mathbb{1}_{\{u_k(\cdot,t)\geq 0\}} \to \mathbb{1}_{\{u(\cdot,t)\geq 0\}} \quad \text{in } L^1(\mathbb{R}^N)$$

What is the difficulty to prove uniqueness ?

To connect sup-norms of u (or other solutions) and L^1-norms of $1\!\!1_{\{u(\cdot,t)\geq 0\}}$ (or characteristic functions of other solutions).

Key computation to prove uniqueness : (in the case of the dislocation type equation)

If u_1, u_2 are two solutions, by a classical "continuous dependence" result, we have

$$\sup_{t\in[0,T]} |(u_1 - u_2)(\cdot,t)|_\infty \leq KT \sup_{t\in[0,T]} |(c[1\!\!1_{\{u_1(\cdot,t)>0\}}] - c[1\!\!1_{\{u_2(\cdot,t)>0\}}])|_\infty$$

and, since $c[\chi] = c_0 \star \chi + c_1$

$$|(c[\mathbb{1}_{\{u_1(\cdot,t)>0\}}] - c[\mathbb{1}_{\{u_2(\cdot,t)>0\}}])(\cdot,t)|_\infty \leq$$
$$|c_0|_{L^1}|\mathbb{1}_{\{u_1(\cdot,t)>0\}} - \mathbb{1}_{\{u_2(\cdot,t)>0\}})(\cdot,t)|_{L^1}$$

On the other hand, if $\delta_T = \sup_{t\in[0,T]} |(u_1 - u_2)(\cdot,t)|_\infty$,

$$|\mathbb{1}_{\{u_1(\cdot,t)>0\}} - \mathbb{1}_{\{u_2(\cdot,t)>0\}})(\cdot,t)|_{L^1} = \mathcal{L}^N(\{-\delta_T \leq u_1(\cdot,t) < 0\}) +$$
$$\mathcal{L}^N(\{-\delta_T \leq u_2(\cdot,t) < 0\})$$

Need to estimate the measure of sets like

$$\{a \leq u(\cdot, t) \leq b\}$$

where $-\overline{\delta} \leq a < b \leq \overline{\delta}$ for some small enough $\overline{\delta}$.

NB : one can do it only for the "simple" Eikonal Equation

$$\frac{\partial u}{\partial t} = c(x,t)|Du| \quad \text{in} \quad \mathbb{R}^N \times (0,T)$$

Such estimates are related with perimeter estimates.

Formal computation : by the co-area formula

$$\int_{\mathbb{R}^N} \mathbb{1}_{\{a\leq u(\cdot,t)\leq b\}} dx = \int_a^b \int_{\{u(\cdot,t)=s\}} |Du|^{-1} d\mathcal{H}^{n-1} ds$$
$$\leq \frac{b-a}{\bar{\eta}} \sup_{a\leq s\leq b} \mathrm{Per}(\{u(\cdot,t)=s\}) ,$$

where $\bar{\eta}$ is the lower bound on $|Du|$ on the set $\{x : |u(x,t)| \leq \bar{\delta}\}$ (which is also needed).

This computation shows the two key points

– a lower bound on $|Du|$

– perimeter estimates on the fronts.

Olivier Ley's result :

If $c(x,t) \geq 0$ and u_0 satisfies

$$-|u_0(x)| - |Du_0(x)| \leq -\eta_0 < 0 \quad \text{in } \mathbb{R}^N$$

$$\Rightarrow -|u(x,t)| - \frac{e^{\gamma t}}{4}|Du(x,t)|^2 \leq \eta < 0 \ \text{ in } \mathbb{R}^N \times [0,T]$$

The gradient of u does not vanish on the front !

= first key ingredient

Then two ways to conclude

If u_0 and c are more regular, a curvature estimate is available which implies the perimeter estimates : Alvarez, Cardaliaguet and Monneau (geometrical arguments) or Ley and GB (pde arguments).

Without further regularity, an interior cone condition is preserved : Cardaliaguet, Ley, Monteillet and GB (control type arguments, rather technical...)

Remark : This provides uniqueness results for all times.

Case 3 : the "non-monotone" second-order case (with a curvature term)

Theorem : Under general assumptions on $H[\chi]$, one has a short time uniqueness result of weak solutions provided that the initial data u_0 satisfies

there exist constants $\lambda_0 \in (0,1)$, $\eta_0 > 0$ *and* $\nu \in C(\mathbb{R}^N, \mathbb{R}^N)$ *such that*

$u_0(x+\lambda\nu(x)) \geq u_0(x)+\lambda\eta_0$ *in a neighb. of* $\{u_0(\cdot)=0\}$

for all $\lambda \in [0, \lambda_0]$.

Other results in this direction :

– Forcadel for the case of graphs (dislocations type equation)

– Forcadel - Monteillet (minimizing movement for dislocations type equation)

Proof : it consists in showing that $u(\cdot, t)$ satisfies the same property as u_0 for small enough t.

$\Longrightarrow$ lower gradient bound + interior cone condition.

Key Ingredients of the proof : a general continuous dependence result for the "standard" level equation equation + a suitable change of variable

Remark : The perimeter estimate does not play a so important role in this case : in "simple" situations, we conclude almost directly with the lower gradient bound and, in more difficult cases, the interior cone condition is the main ingredient.