Nonlocal models : from dune morphodynamics to signal processing.

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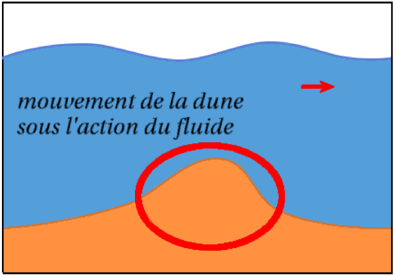
Outline

1. A nice and strange PDE for morphodynamics.
2. Where does the strange nonlocal term come from ?

A nice and strange PDE for morphodynamics.

Dune morphodynamics



Dune profile $u(t,x)$ satisfies Fowler's equation

$$\begin{cases} u_t(t,x) + u\,u_x(t,x) + \int_0^{+\infty} s^{-\frac{1}{3}} u_{xx}(t,x-s)\,ds - u_{xx}(t,x) &= 0, \\ u(0,x) &= u_0(x) \end{cases}$$

A conservative nonlinear nonlocal and non-monotone model

$$\begin{cases} \partial_t u + \partial_x \left(\left(\frac{u^2}{2}\right) + \int_0^{+\infty} s^{-\frac{1}{3}} u_x(t, x-s)\,ds - u_x \right) & = & 0, \\ u(0,x) & = & u_0(x) \end{cases}$$

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Main results

- Existence, uniqueness and continuous dependence of the solution u w.r.t. initial datum in $L^2(\mathbb{R})$
- Maximum principle violation $\Rightarrow$ can describe both erosion and accretion phenomena, contrary to hyperbolic models.
- Existence of travelling waves $u(x,t) = \varphi_c(x - c\,t)$, (may not be of solitary type). Local existence and uniqueness for initial datum in $\mathcal{C}_b^1$

N. Alibaud, P. Azerad, D. Isèbe, *A non-monotone conservation law for dune morphodynamics*, Differential Integral Equations, 2010.
B. Alvarez-Samaniego, P. Azerad, *Travelling wave solutions of the Fowler equation*, Discrete and Continuous Dynamical Systems, B, 2009.

Some expressions for the non local term $\mathcal{I}$

$$\mathcal{I}[\varphi](x) = \int_0^{+\infty} s^{-\frac{1}{3}} \varphi_{xx}(x-s)ds$$

Lévy-Khinchine

For all $\varphi \in \mathcal{S}(\mathbb{R})$ and $x \in \mathbb{R}$,

$$\mathcal{I}[\varphi](x) = \frac{4}{9} \int_0^{\infty} \frac{\varphi(x-s) - \varphi(x) - \varphi'(x)\, s}{s^{7/3}}\, ds.$$

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in Fourier variable

For all $\varphi \in \mathcal{S}(\mathbb{R})$ and $\xi \in \mathbb{R}$,

$$\mathcal{F}\left(\mathcal{I}[\varphi] - \varphi''\right)(\xi) = \left(\xi^2 - (a \pm i\, b)|\xi|^{\frac{4}{3}}\right) \mathcal{F}(\varphi)(\xi)$$

$$(a \pm ib) = \Gamma(2/3)\left(1/2 + i\,\mathrm{sgn}(\xi)\,\sqrt{3}/2\right)$$

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- $\mathcal{I}$ anti-diffusive differential operator of order $\frac{4}{3}$
- $\sim \xi^2$ when $|\xi| \to +\infty$ regularizing effect on initial datum

Existence, uniqueness and regularity.

Theorem

Let $T > 0$ and $u_0 \in L^2(\mathbb{R})$. There exists a unique mild solution $u \in L^\infty((0,T); L^2(\mathbb{R}))$. Moreover,

i) $u \in \mathcal{C}^\infty((0,T] \times \mathbb{R})$ *and for all* $t_0 \in (0,T]$, *u as well as its derivatives of any order belong to* $\mathcal{C}([t_0,T]; L^2(\mathbb{R}))$.

ii) *u satisfies* $u_t + (\frac{u^2}{2})_x + \mathcal{I}[u] - u_{xx} = 0$, *on* $(0,T] \times \mathbb{R}$, *in the classical sense.*

iii) $u \in \mathcal{C}([0,T]; L^2(\mathbb{R}))$ *et* $u(0,.) = u_0$ *a.e.*

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iii) *$u \in \mathcal{C}([0,T]; L^2(\mathbb{R}))$ et $u(0,.) = u_0$ a.e.*

Continuous dependence

Let two solutions (u,v) with initial data (u_0,v_0) in $L^2(\mathbb{R})$. Then, u and v fulfill

$$||u - v||_{C([0,T];L^2(\mathbb{R}))} \leq C\left(T, ||u_0||_{L^2(\mathbb{R})}, ||v_0||_{L^2(\mathbb{R})}\right) ||u_0 - v_0||_{L^2(\mathbb{R})}$$

The linearized problem

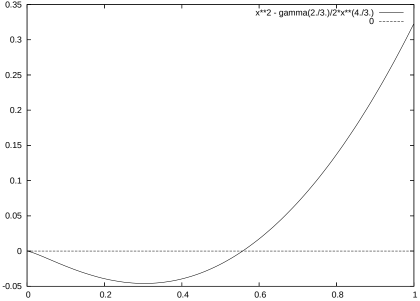
- Solution given by

$$u(t,x) = K(t,\cdot) * u_0(x).$$

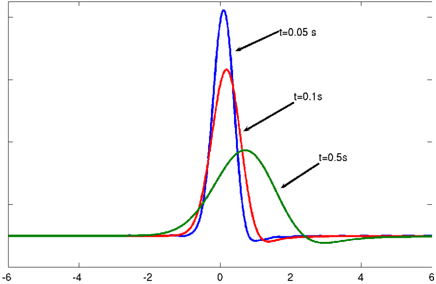
where the kernel

$$\mathcal{F}(K(t,\cdot))(\xi) := \exp\left(-t\left(\xi^2 - a|\xi|^{\frac{4}{3}} + i\, b\xi|\xi|^{\frac{1}{3}}\right)\right), \quad t > 0$$

$$\psi(\xi) = \xi^2 - a|\xi|^{\frac{4}{3}}.$$



$K(\cdot, t)$ is not positive



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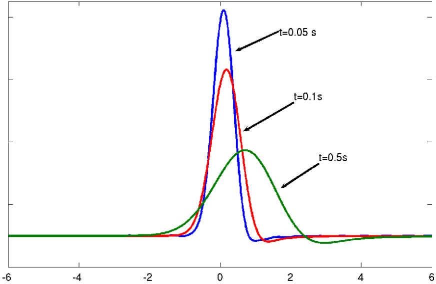

No maximum principle for $u_t + \mathcal{I}[u] - u_{xx} = 0$ but

$$\|K(t, \cdot) * u_0\|_{L^2(\mathbb{R})} \leq e^{\omega_0 t} \|u_0\|_{L^2(\mathbb{R})}$$

where $\omega_0 = -\min \psi$.

Properties of $K(t,\cdot), \quad t>0$

- $\mathcal{C}^0-$ **Semi-group** :

$$\forall u_0 \in L^2(\mathbb{R}), \qquad \begin{array}{l} K(t) * K(s) = K(t+s) \\ \lim_{t\to 0} K(t) * u_0 = u_0 \text{ in } L^2(\mathbb{R}). \end{array}$$

- **Regularity** :

$$K(t,x) \in C^\infty((0,+\infty)\times\mathbb{R})$$

- **Estimates for the gradient** :

$$\begin{aligned} \|\partial_x K(t)\|_{L^2(\mathbb{R})} &\leq C\, t^{-\frac{3}{4}} \\ \|\partial_x K(t)\|_{L^1(\mathbb{R})} &\leq C\, t^{-\frac{1}{2}} \end{aligned}$$

Mild solution

Definition

Let $T > 0$ and $u_0 \in L^2(\mathbb{R})$. We say that $u \in L^\infty((0,T);L^2(\mathbb{R}))$ is a *mild* solution if for a.e. $t \in (0,T)$ and $x \in \mathbb{R}$,

$$u(t,x) = K(t,\cdot) * u_0(x) - \frac{1}{2}\int_0^t \partial_x K(t-s,\cdot) * u^2(s,\cdot)(x)ds.$$

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$\Rightarrow$ local existence by contracting fixed point .

L^2 *a priori* estimate

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} u^2 dx + \int_{\mathbb{R}} \left(\mathcal{I}[u] - \partial_{xx}^2 u\right) u dx = 0.$$

$$\int_{\mathbb{R}} \left(\mathcal{I}[u] - \partial_{xx}^2 u\right) u dx = \int_{\mathbb{R}} \psi(\xi) \left|\mathcal{F} u(\xi)\right|^2 d\xi$$

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}} u^2 dx \leq \omega_0 \int_{\mathbb{R}} u^2 dx$$

$$\boxed{\|u(t,\cdot)\|_{L^2(\mathbb{R})} \leq e^{\omega_0 t}\|u_0\|_{L^2(\mathbb{R})}}$$

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$\Rightarrow$global existence .

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$\Rightarrow$global existence .

Violation of Maximum principle

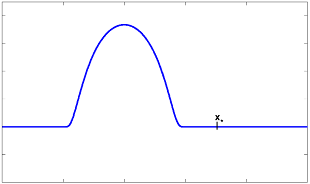
Theorem

Let $u_0 \in C^2(\mathbb{R}) \cap H^2(\mathbb{R})$ *positive such that there exists* $x_* \in \mathbb{R}$ *with* $u_0(x_*) = u_0'(x_*) = u_0''(x_*) = 0$ *and*

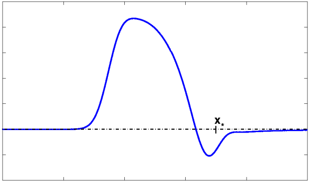
$$\int_0^\infty \frac{u_0(x_* - s)}{s^{7/3}}\, ds > 0.$$

then there exists $t_* > 0$ *such that* $u(t_*, x_*) < 0$.

time t=0



time t=t*



Travelling waves

- Existence but may not be solitary and $\notin L^2$
- Local existence theory in C_b^1
- global existence theory in C_b^1 ????OPEN

Let $\lambda > 0$ and $\eta > 0$. Define

$$u_\lambda(x,t) := \frac{1}{\lambda} u\Big(\frac{x}{\lambda}, \frac{t}{\lambda^2}\Big), \quad \text{for } x \in \mathbb{R} \text{ and } t \geq 0. \tag{1}$$

It is straightforward to check that if u is a solution to the general equation

$$\partial_t u(x,t) + \partial_x\Big(\frac{u^2}{2} - \partial_x u + \eta\, g[u]\Big)(x,t) = 0, \tag{2}$$

then u_λ satisfies equation (3).

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$\eta = 1$: Fowler's equation, $\eta = 0$: viscous Burgers equation

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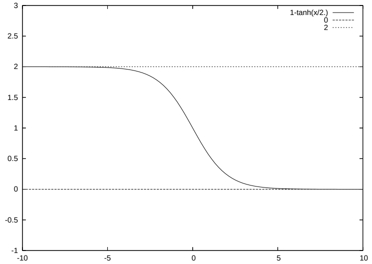
Case $\eta = 0$: For any $c \in \mathbb{R}$ the Taylor shock wave $u_c(x,t) = c\left[1 - \tanh\left(\frac{c}{2}(x - c\,t)\right)\right]$ is a travelling wave solution to viscous Burgers equation.



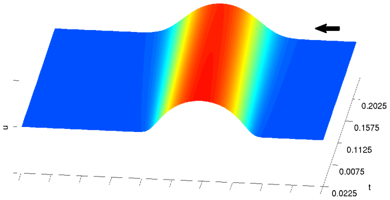
By implicit function theorem, it is possible to build travelling wave solution ϕ of equation (2) with speed c for very small η. Then by suitable scaling, then $\phi_\lambda(\cdot) = \frac{1}{\lambda}\phi(\frac{1}{\lambda}\cdot)$ is a travelling-wave solution of Fowler's equation (case $\eta = 1$) with speed c/λ.

Stability of the travelling waves

Ph. D. thesis A. Bouharguane,

- Global existence , uniqueness, regularity, continuous dependence for L^2- initial perturbation of a C_b^1 solution
- Instability of constant solutions , no flat bathymetry in nature !
- according to frequency of perturbation : quick dampening of high frequencies, slow amplification of low frequencies !

some numerical simulations



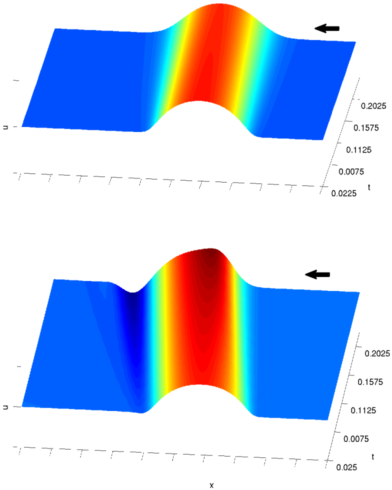
viscous Burgers equations

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) - \partial^2_{xx} u = 0$$

centered FD scheme

stability : CFL-Péclet

some numerical simulations



non-local term creates erosion and accretion

$$\partial_t u + \partial_x \left(\frac{u^2}{2} + g[u] \right) - \partial^2_{xx} u = 0$$

with

$$g[u](x) := \int_0^{+\infty} s^{-\frac{1}{3}} u_x(x-s)ds.$$

$$g[u_i^n] \approx \sum_{j=0}^{i} |j\Delta x|^{-\frac{1}{3}} \frac{(u^n_{i-j+1} - u^n_{i-j-1})}{2}$$

Application to Signal Processing.

$$\begin{cases} \partial_t u(t,x) - a\,\partial^2_{xx} u(t,x) + b\,\mathcal{I}_\lambda[u(t,.)](x) = 0 & t \in (0,T), x \in \mathbb{R}, \\ u(0,x) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{4}$$

where $0 < \lambda < 2$

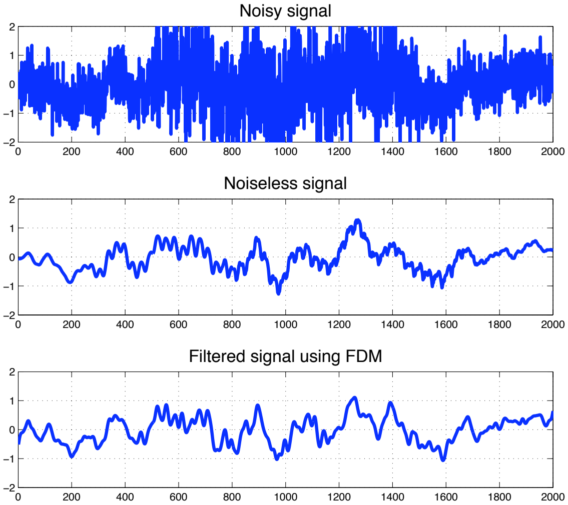
$$\mathcal{F}(\mathcal{I}_\lambda[\varphi])(\xi) := -|\xi|^\lambda \mathcal{F}(\varphi)(\xi) \tag{5}$$

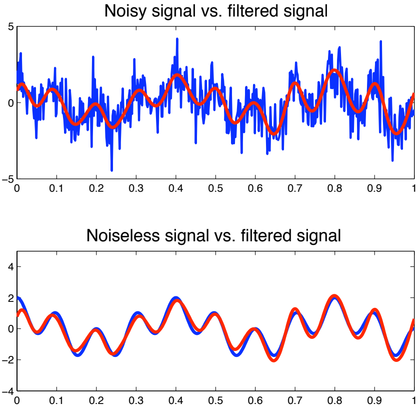
For $1 < \lambda < 2$, explicit formula (Imbert, JDE, 2005)

$$\mathcal{I}_\lambda[\varphi](x) = \int_{\mathbb{R}} \frac{\varphi''(x-s)}{|s|^{\lambda-1}} d\xi \tag{6}$$

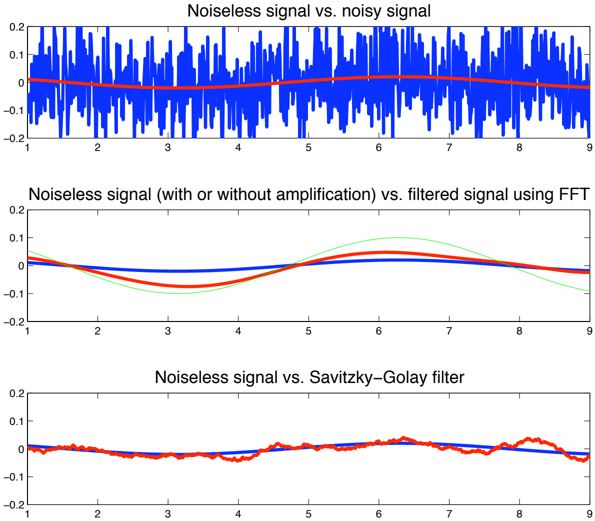
Alternatively, (slightly different definition inspired by fractional calculus) :

$$\mathcal{I}_\lambda[\varphi](x) = \int_0^{+\infty} \frac{\varphi''(x-s)}{|s|^{\lambda-1}} d\xi. \tag{7}$$


Noisy signal
Noiseless signal
Filtered signal using FDM
2
1
0
−1
−2
0
200
400
600
800
1000
1200
1400
1600
1800
2000



P. Azerad, A. Bouharguane and J.-F. Crouzet *Simultaneous denoising and enhancement of signals by a fractal conservation law* , submitted 2010.


Noiseless signal vs. noisy signal
Noiseless signal (with or without amplification) vs. filtered signal using FFT
Noiseless signal vs. Savitzky−Golay filter
0.2
0.1
0
−0.1
−0.2
1
2
3
4
5
6
7
8
9

Where does the strange nonlocal term come from ?

Define the antiderivative

$$\frac{d^{-1}f}{dx^{-1}} = \int_0^x f(t)\,dt$$

Again

$$\frac{d^{-2}f}{dx^{-2}} = \int_0^x \int_0^t f(s)\,dsdt = \int\int_{0<s<t<x} f(s)\,dsdt = \int_0^x f(s)\left(\int_s^x dt\right) ds$$

$$\frac{d^{-2}f}{dx^{-2}} = \int_0^x f(s)(x-s)\,ds$$

And again

$$\frac{d^{-3}f}{dx^{-3}} = \int_0^x \int_0^u \int_0^t f(s)\,dsdtdu = \int\int\int_{0<s<t<u<x} f(s)\,dsdtdu$$

$$= \int_0^x f(s)\left(\int\int_{s<t<u<x} dudt\right) ds = \int_0^x f(s)\left(\int_s^x (x-t)\,dt\right) ds$$

$$\frac{d^{-3}f}{dx^{-3}} = \int_0^x f(s)\frac{(x-s)^2}{2!}\,ds$$

What is the link with our nonlocal term ?

This can be extended for real negative exponents : Riemann Liouville formula (circa 1847, “Versuch einer Auffassung der Integration und Differentiation”).

$$\frac{d^{-q} f}{dx^{-q}}(x) = \int_0^x f(s) \frac{(x-s)^{q-1}}{\Gamma(q)}\, ds$$

Rediscovered many times,see e.g. Caputo’s differintegral
Other generalization possible, see e.g. Oldham and Spanier, “’the fractional calculus”, Academic Press, 1974
Now take f causal, i.e. $f(t) = 0, \quad \forall t < 0$. Compute the $4/3 = 2 - 2/3$ derivative (right way) :

$$\frac{d^{4/3}}{dx^{4/3}} f(x) = \frac{d^{-2/3}}{dx^{-2/3}} f'' = \int_0^x f''(s) \frac{(x-s)^{-1/3}}{\Gamma(2/3)}\, ds = \int_{-\infty}^x f''(s) \frac{(x-s)^{-1/3}}{\Gamma(2/3)}\, ds$$

$$\boxed{\frac{d^{4/3}}{dx^{4/3}} f(x) = \frac{1}{\Gamma(2/3)} \int_0^{+\infty} f''(x-s)\, s^{-1/3}\, ds}$$

Modelling

English school asymptotics, Fowler, Oxford [1]

Couette flow over a bump. $U' = U/h$, $Re = \frac{UL}{h}\frac{L}{\nu} = \frac{U'L^2}{\nu}$

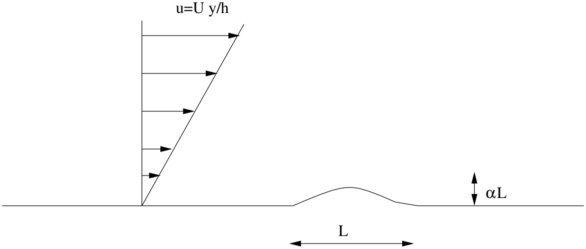


FIGURE: Shear flow perturbed by a small bump $\alpha << 1$.

suitable scaling : Double Deck theory Van Dyke, thin layer of size ϵ

1. thanks to P.-Y. Lagrée for french rigorous explanations.

Principle of least degeneracy :

Stretch the vertical scale

$$x = \bar{x}, \quad y = \epsilon \bar{y}, \quad u = \epsilon \bar{u}$$

Balance of convective term and diffusive terms reads

$$u \frac{\partial u}{\partial x} \sim \frac{1}{Re} \frac{\partial^2 u}{\partial y^2}$$

$$\epsilon^2 \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} \sim \frac{\epsilon}{\epsilon^2} \frac{1}{Re} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\epsilon \sim Re^{-1/3}$$

Prandtl equations

$$v = \epsilon^2 \bar{v}, \quad p = \epsilon^2 \bar{p}$$

Dropping bars

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} & = & 0 \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} & = & -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \\ 0 & = & -\frac{\partial p}{\partial y} \end{cases}$$

Boundary conditions :

- bottom no slip $u = v = 0$ on $y = f(x)$
- far upstream no perturbation $u \to y$ when $x \to -\infty$,
- matching $u \to y$ when $y \to \infty$

Linearization

Bump is small $\Rightarrow f(x) = \alpha f_1$ with $\alpha \leq \epsilon << 1$.

$$u = y + \alpha\, u_1, \quad v = \alpha\, v_1 \quad p = \alpha\, p_1$$

$$\left\{ \begin{array}{ccc} \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} & = & 0 \\ y\frac{\partial u_1}{\partial x} + v_1 & = & -\frac{\partial p_1}{\partial x} + \frac{\partial^2 u_1}{\partial y^2} \\ 0 & = & -\frac{\partial p_1}{\partial y} \end{array} \right.$$

Boundary conditions : Bottom no slip

$$0 = u(x, \alpha\, f_1) = u(x, 0) + \alpha\, f_1 \frac{\partial u}{\partial y}(x, 0) + O(\alpha^2)$$

Since $u = y + \alpha\, u_1$, $u(x, 0) = 0 + \alpha u_1(x, 0)$ and $\frac{\partial u}{\partial y}(x, 0) = 1 + \alpha \frac{\partial u_1}{\partial y}$

$$\Rightarrow u_1(x, 0) = -f_1(x, 0)$$

Fourier transform w.r.t. x : $\widehat{\phi}(k,y) = \frac{1}{\sqrt{2\pi}} \int \phi(x,y) e^{-ikx}\, dx, \;\Rightarrow \partial_x \to \times ik$

$$ik\widehat{u_1} + \frac{\partial \widehat{v_1}}{\partial y} = 0 \qquad (8)$$

$$ik\, y\, \widehat{u_1} + \widehat{v_1} = -ik\widehat{p_1} + \frac{\partial^2 \widehat{u_1}}{\partial y^2} \qquad (9)$$

$$0 = -\frac{\partial \widehat{p_1}}{\partial y} \qquad (10)$$

Let the shear stress

$$\widehat{\tau_1}(k) = \frac{\partial \widehat{u_1}}{\partial y}(k,y)$$

Differentiate w.r.t. y horizontal momentum equation (9)

$$ik\, \widehat{u_1} + ik\, y\, \widehat{\tau_1} + \frac{\partial \widehat{v_1}}{\partial y} = -ik\, \frac{\partial \widehat{p_1}}{\partial y} + \frac{\partial^2 \widehat{\tau_1}}{\partial y^2}$$

With incompressibility (8) and hydrostatic pressure (10)

$$ik\, y\, \widehat{\tau_1} = \frac{\partial^2 \widehat{\tau_1}}{\partial y^2} \qquad (11)$$

Airy equation

$$\phi''(z) = z\,\phi(z)$$

Solution vanishing at $+\infty$ $\quad \propto Ai(z)$
Scaling property :

$$\frac{d^2\phi(\lambda\, y)}{dy^2} = \lambda^3\, y\,\phi(\lambda\, y)$$

$$\widehat{\tau_1}(k,y) = C_k Ai((ik)^{1/3}\, y)$$

Integrate $\int_0^\infty \widehat{\tau_1}(k,y)\,dy = \widehat{u_1}(k,\infty) - \widehat{u_1}(k,0)$ with matching condition $\widehat{u_1}(k,\infty) = 0$ and bottom b.c. $u_1(x,0) = -f_1(x,0)$ we get

$$\int_0^\infty \widehat{\tau_1}(k,y)\,dy = \widehat{f_1}(k)$$

Ask `wolframalpha.com` for
$\int_0^\infty Ai(z)\,dz = 1/3 \Rightarrow \int_0^\infty C_k\, Ai((ik)^{1/3}\, y)\,dy = C_k\, \frac{ik^{-1/3}}{3}$
Finally we get $C_k = 3(ik)^{1/3}\widehat{f_1}(k)$ and

$$\widehat{\tau_1}(k,y) = 3(ik)^{1/3} Ai((ik)^{1/3}\, y)\,\widehat{f_1}(k)$$

Back to Physical variables

Bottom shear stress

$$\widehat{\tau_1}(k,0) = 3(ik)^{1/3} Ai(0)\,\widehat{f_1}(k)$$

Ask again `wolframalpha.com` $Ai(0) = \frac{1}{3^{2/3}\,\Gamma(2/3)}$

$$\widehat{\tau_1}(k,0) = \frac{3^{1/3}}{\Gamma(2/3)}(ik)^{1/3}\,\widehat{f_1}(k)$$

Use the slope of the bump f_1' instead of f_1

$$\widehat{\tau_1}(k) = \frac{3^{1/3}}{\Gamma(2/3)}(ik)^{-2/3}\,\widehat{f_1'}(k)$$

Since $\int_0^\infty x^{-1/3}\,e^{-ikx} = (ik)^{-2/3}\Gamma(2/3)$ we obtain

$$\tau_1(x,0) = \frac{3^{1/3}}{\Gamma(2/3)^2}\left(H(x)\,x^{-1/3} * f_1'\right) = \frac{3^{1/3}}{\Gamma(2/3)^2}\int_0^\infty \frac{f_1'(x-\xi)}{\xi^{1/3}}\,d\xi$$

since $\bar{\tau} = 1 + \alpha\tau_1$, $\boxed{\tau = U'\left(1 + \frac{3^{1/3}}{\Gamma(2/3)^2}\int_0^\infty \frac{f'(x-s)}{s^{1/3}}\,ds\right)}$

Outline

1. A nice and strange PDE for morphodynamics.
   - Mathematical analysis of the model.
   - Numerical simulations.
2. Where does the strange nonlocal term come from ?
   - Did you say differintegral ?
   - The basal shear stress