The long trapezoid property

joint work with Abraham Rueda Zoca (Universidad de Granada)

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Svratka, January 16, 2017

Definition (Godefroy)

The norm $\|\cdot\|$ on a Banach space X is *octahedral* if,
$\forall\, \varepsilon > 0\, \forall\, Y \subseteq X,\ \dim Y < \infty \Longrightarrow$

$$\exists x \in S_X \text{ such that } \|y + \lambda x\| \geq (1-\varepsilon)(\|y\| + |\lambda|)$$

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- Easy argument: if $\|\cdot\|$ is octahedral $\implies \ell_1 \underset{\sim}{\subset} X$
- Godefroy '89: converse: if $\ell_1 \underset{\sim}{\subset} X \implies X \in \langle$octahedral$\rangle$

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$\|\cdot\|$ on X is octahedral iff $\forall\ \varepsilon > 0\ \forall\ y_1, \ldots, y_n \in S_X\ \exists y \in S_X$ *such that* $\|y_i + x\| \geq 2 - \varepsilon$ *for all* $i = 1, \ldots, n$.

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$\Longrightarrow \forall\ f \in \mathrm{Lip}_0(M_0) = \mathcal{F}(M_0)^*$, $\|f\| = 1$ s.t. $\langle f, \mu_i \rangle = 1$ admits $\tilde{f} \in \mathrm{Lip}_0(M)$, $\tilde{f} \restriction_{M_0} = f$, $\left\|\tilde{f}\right\| \leq 1 + \varepsilon$ and

$$\|\mu_i + m_{uv}\| \geq \left\langle \tilde{f}, \mu_i + m_{uv} \right\rangle \geq 2 - \varepsilon$$

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3. $\forall\, \varepsilon > 0\ \forall\, N \subset M$ *finite,* $\exists\, u, v \in M, u \neq v$*, such that*

$$(1-\varepsilon)(d(x,y) + d(u,v)) \leq d(x,u) + d(y,v)$$

for all $x, y \in N$*.*

Proof of (1)$\Longrightarrow$(3).

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Let $\|\cdot\|$ *on* X *be octahedral, let* $V \subseteq S_X$ *norming for* X^**. Then,* $\forall\, \varepsilon > 0$ *and* $\forall\, y_1, \ldots, y_n \in S_X \Longrightarrow \exists\, x \in V$ *such that* $\|y_i + x\| > 2 - \varepsilon$ *for every* $i \in \{1, \ldots, n\}$*.*

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$$\begin{aligned}\tilde{f}(u) - \tilde{f}(v) &= f(x) - f(y) + (1+\varepsilon)(d(x,u) + d(y,v)) \\ &\geq f(x) - f(y) + \frac{1+\varepsilon}{1+\varepsilon}(d(x,y) + d(u,v)) \geq d(u,v)\end{aligned}$$

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Let $N \subset M$ finite and $\varepsilon > 0$. $\implies u, v \in M$, $u \neq v$, such that

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for all $x, y \in N$. For $f \in B_{\mathrm{Lip}_0(N)}$ we define

$\tilde{f}(u) = \inf_{x \in N} f(x) + (1+\varepsilon) d(x,u)$,

$\tilde{f}(v) = \sup_{x \in N \cup \{u\}} \tilde{f}(x) - (1+\varepsilon) d(x,v)$.

$\tilde{f}$ is $(1+\varepsilon)$-Lipschitz on $N \cup \{u,v\} \implies$ it admits an $(1+\varepsilon)$-Lipschitz extension to the whole of M.

N finite $\implies \exists\, x \in N$ and $y \in N \cup \{u\}$ such that $\tilde{f}(u) = f(x) + (1+\varepsilon)d(x,u)$ and $\tilde{f}(v) = \tilde{f}(y) - (1+\varepsilon)d(y,v)$.

If $y = u \implies \tilde{f}(u) - \tilde{f}(v) = (1+\varepsilon)d(u,v)$

If $y \neq u \implies$

$$\begin{aligned}\tilde{f}(u) - \tilde{f}(v) &= f(x) - f(y) + (1+\varepsilon)(d(x,u) + d(y,v)) \\ &\geq f(x) - f(y) + \frac{1+\varepsilon}{1+\varepsilon}(d(x,y) + d(u,v)) \geq d(u,v)\end{aligned}$$

Proof of (2) $\Rightarrow$ (1) is like the example, except simpler. □

Sets without the LTP

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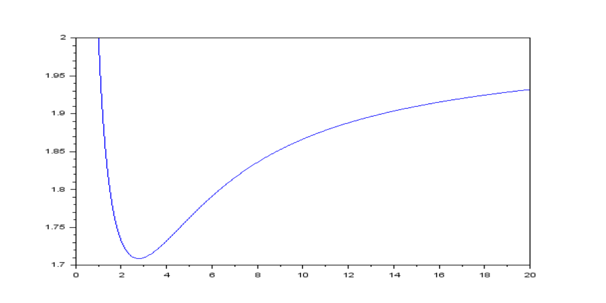

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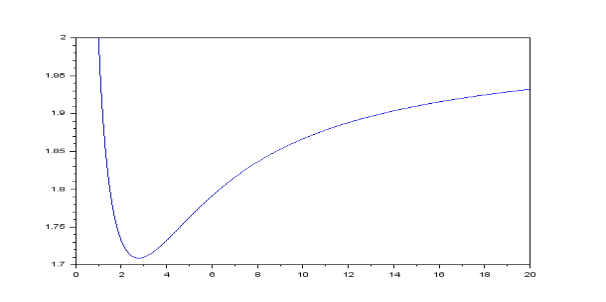

- What about ℓ_1?

Negation of the LTP

Let (M, d) fail the LTP, i.e. $\exists \varepsilon > 0\ \exists N \subset M$ finite such that $\forall\ u \neq v \in M\ \exists\ x \neq y \in N$

$$(1-\varepsilon)(d(x,y)+d(u,v)) > \min\left\{d(x,u) + d(y,v), d(x,v) + d(y,u)\right\}.$$

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Let (M, d) fail the LTP. Then $\exists A \subset M$ infinite $\exists \varepsilon > 0\ \exists x \neq y \in A$ such that $\forall\ u \neq v \in M$

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Theorem

Every infinite subset of ℓ_1 has the long trapezoid property.

Proof of the LTP for infinite subsets of ℓ_1

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(1-\varepsilon)(\|x-y\|+\|u-v\|) &> \min\{\|x-u\|+\|y-v\|\,,\|x-v\|+\|y-u\|\} \\
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$\square$

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- Question: if every infinite $A \subset X$ has the LTP $\Longrightarrow$ $\overline{\delta}_X(t) = t$?

Tnahk you for yuor antteoitn!