Some geometric properties of Lipschitz-free spaces over ultrametric spaces

joint work with Aude Dalet and Pedro Kaufmann

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Proof.

$\forall\, f \in \mathrm{Lip}_0(M)\ \exists!\ \widehat{f} \in \mathcal{F}(M)^*$ such that $\widehat{f}\restriction_M = f$. Moreover $\left\|\widehat{f}\right\| = \|f\|_L$. □

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- Every ultrametric space is isometric to a subset of an $\mathbb{R}$-tree.

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Proof.

$$\mathcal{F}(M) \equiv \ell_1(\Gamma) \Longrightarrow \mathrm{Lip}_0(M) \equiv \ell_\infty(\Gamma) \Longrightarrow$$
$$\Longrightarrow \forall\, f, g \in \mathrm{Ext}\left(B_{\mathrm{Lip}_0(M)}\right) : \|f - g\|_L = 2.$$

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$\Longrightarrow \forall\, f, g \in \mathrm{Ext}\left(B_{\mathrm{Lip}_0(M)}\right) : \|f - g\|_L = 2.$
We will find $f, g \in \mathrm{Ext}\left(B_{\mathrm{Lip}_0(M)}\right) : \|f - g\|_L < 2.$

Extensions of Lipschitz functions

Let (M,d) be a metric space, $A \subset M$, $0 \in A$ and $f \in \mathrm{Lip}_0(A)$.
Then $\exists\, \widetilde{f} \in \mathcal{F}(M)$ such that $\widetilde{f} \restriction_A = f$ and $\|f\|_L = \left\|\widetilde{f}\right\|_L$.

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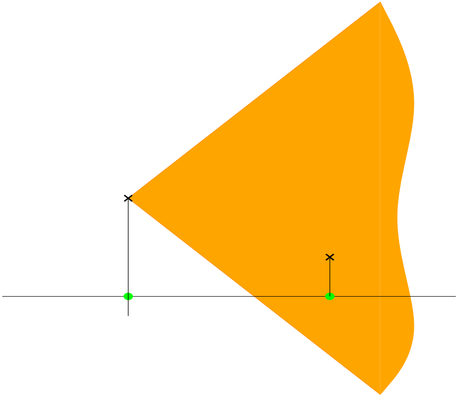

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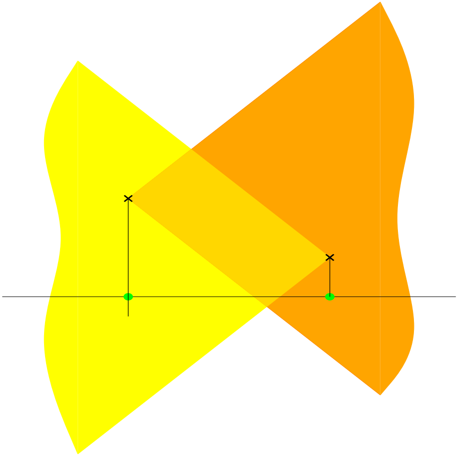

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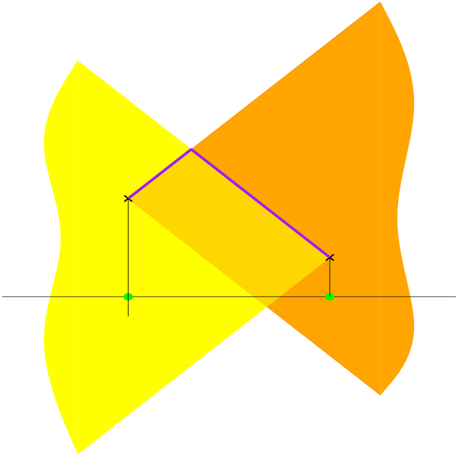
Then $\exists\, \widetilde{f} \in \mathcal{F}(M)$ such that $\widetilde{f} \restriction_A = f$ and $\|f\|_L = \left\|\widetilde{f}\right\|_L$.

$$\widetilde{f}_I(y) = \inf_{z \in A} f(z) + d(y, z)$$



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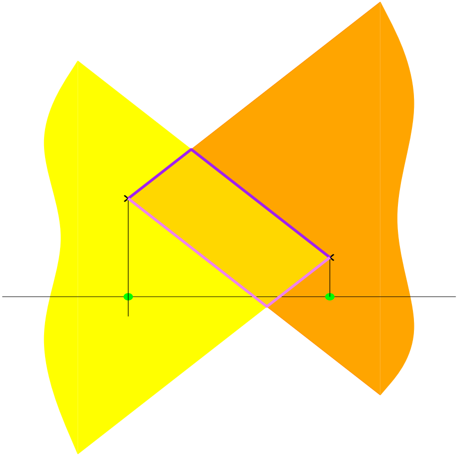

$$\widetilde{f_I}(y) = \inf_{z \in A} f(z) + d(y,z)$$

$$\widetilde{f_S}(y) = \sup_{z \in A} f(z) - d(y,z)$$

Lemma

Let (M, d) *be a metric space,* $A \subset M$, $0 \in A$ *and* $f \in Ext(B_{\mathrm{Lip}_0(A)})$. *Then* $\widetilde{f_I}, \widetilde{f_S} \in Ext(B_{\mathrm{Lip}_0(M)})$.

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Proof.

Let $\widetilde{f_S} = \frac{p+q}{2}$, $p, q \in B_{\mathrm{Lip}_0(M)}$. If $x \in A$, then $p(x) = q(x) = f(x)$ as $f \in \mathrm{Ext}(B_{\mathrm{Lip}_0(A)})$. If $x \in M \setminus A$, then $\forall\, z \in A$:

$$f(z) - p(x) = p(z) - p(x) \leq d(z, x).$$

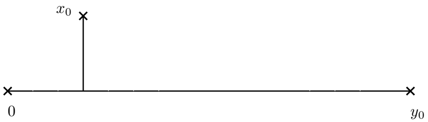
Thus

$$\widetilde{f_S}(x) = \sup_{z \in A} f(z) - d(z, x) \leq p(x)$$

Also $\widetilde{f_S}(x) \leq q(x)$. So $f_S(x) = p(x) = q(x)$. □

Two extreme points at distance < 2

Let $0, x_0, y_0 \in M$ be three different points.
WLOG $d(0, y_0) = d(x_0, y_0) \geq d(0, x_0)$.



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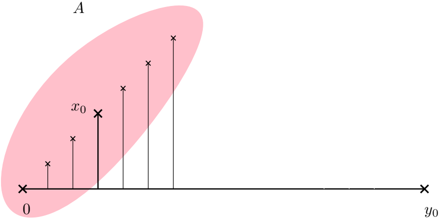

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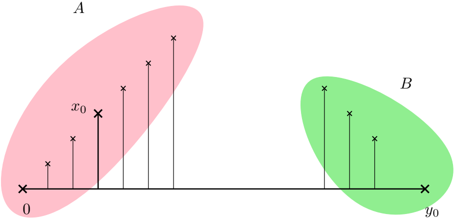

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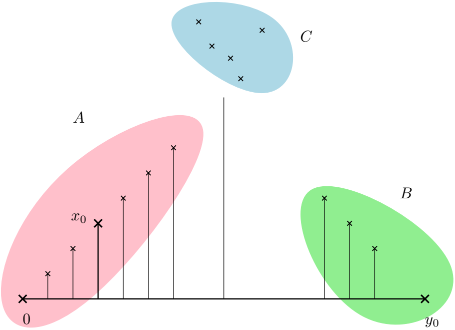

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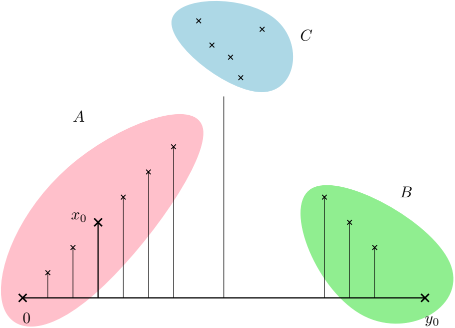

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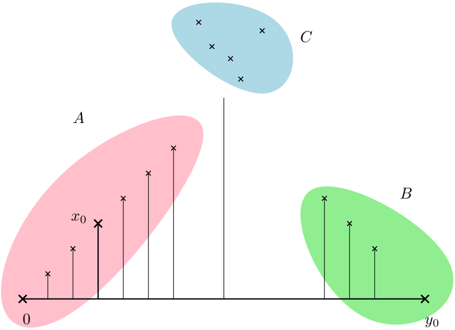

$$\begin{aligned} g(x) &:= \widetilde{(f\restriction_{A\cup C})}_S \\ &= d(0,x) \text{ when } x \in A \cup C \\ &= \sup_{z \in A\cup C} d(0,z) - d(z,x),\ x \in B \\ &= \sup_{z \in A\cup C} d(0,z) - d(z,y_0) \end{aligned}$$

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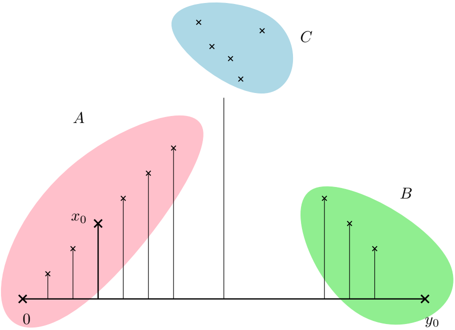

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because $0 \geq g(y) \geq d(0,x_0) - d(x_0,y_0) = d(0,x_0) - d(0,y_0)$ with $x \in A \cup C$, $y \in B$.

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Let X be a Banach space and $C = \bigcap_{i=1}^{n} x_i^{*-1}(-\infty, 1)$. Let $A \subset X \setminus C$ such that $\forall\, x \neq y \in A$ we have $\frac{x+y}{2} \in C$. Then $|A| \leq n$.

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Proof.

Put $\varphi(x) := i$ such that $x_i^*(x) \geq 1$. We have $1 > x_{\varphi(x)}^*\left(\frac{x+y}{2}\right)$. Thus $x_{\varphi(x)}^*(y) < 1$ and φ is injective. □

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Let $f_1, \ldots, f_{2n+1} \in S_Y$ such that $\left\|\frac{f_i+f_j}{2}\right\| \leq \frac{1}{1+\varepsilon}$. Then $d_{BM}(Y, \ell_\infty^n) > 1 + \varepsilon$.

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Proof.

Let $T : Y \to \ell_\infty^n$ such that $\|f\| \leq \|Tf\|_\infty \leq (1+\varepsilon)\|f\|$. Then $\|Tf_i\| \geq 1$, $\left\|\frac{Tf_i+Tf_j}{2}\right\| < 1$ and $B^O_{\ell_\infty^n}$ is the intersection of $2n$ halfspaces. □

Proposition

Let $M = \{0, x_1, \ldots, x_n\}$, $n \geq 2$, be an ultrametric space. Then

$$d_{BM}(\mathcal{F}(M), \ell_1^n) > (1 - \frac{m}{4\,\mathrm{diam}(M)})^{-1}$$

where $m = \min_{x \neq y} d(x, y)$.

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Proof.

For every $i \neq j \leq n$ $\exists\, f_{ij} \in B_{\mathrm{Lip}_0(M)}$ such that $\frac{f_{ij}(x_k) - f_{ij}(x_l)}{d(x_k, x_l)} = 1$ iff $k = i$ and $l = j$. □

Thank you for your attention!