Renorming non-separable Banach spaces by smooth LUR norms

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1. General problem

1) Given a Banach space X on which there exist two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$, the former being C^1 and the latter locally uniformly rotund (LUR), is it possible to find a third equivalent norm $\|\cdot\|_3$ on X which is both C^1 and LUR?

YES if moreover $\|\cdot\|_1^*$ on X^* is LUR, we say $\|\cdot\|_1$ is LUR* (Asplund, 1967). This is the case for example for X separable or $X = c_0(\Gamma)$. A fundamental reason behind this result is the residuality of both LUR and LUR* norms.

2) If $\|\cdot\|_1$ is C^k-smooth, $k \geq 2$, can $\|\cdot\|_3$ be made C^k-smooth?

NO unless the space is super-reflexive (Fabian, Whitfield, Zizler, 1983)

3) In the non-super-reflexive case, can $\|\cdot\|_3$ be at least approximated by C^k-smooth norms?

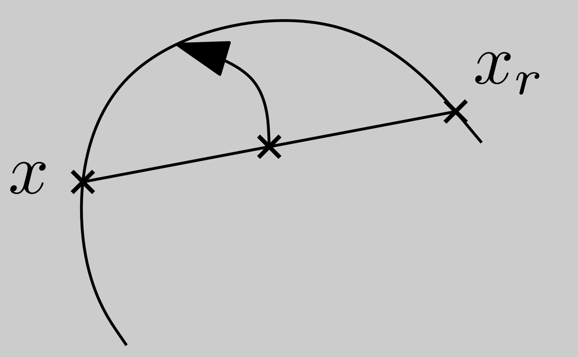
YES if X is separable (McLaughlin, Poliquin, Vanderwerff, Zizler, 1993)
YES if $X = c_0(\Gamma)$ (Pechanec, Whitfield, Zizler, 1981)

A norm $\|\cdot\|$ is LUR if for any $x \in X$, $(x_r)_{r=1}^{\infty} \subset X$ **the LUR hypothesis at** x

$$\frac{\|x\|^2 + 2\|x_r\|^2}{2} - \left\|\frac{x+x_r}{2}\right\|^2 \longrightarrow 0$$

implies $x_r \longrightarrow x.$ The LUR hypothesis is equivalent to a (better known) condition:

$$\lim_{r\to\infty}\left\|\frac{x+x_r}{2}\right\| = \lim_{r\to\infty}\|x_r\| = \|x\|.$$



For example, the canonical norm on ℓ_2 is LUR.

2. Some more positive answers (P. Hájek, A.P., 2009)

- $\|\cdot\|_3$ is LUR, C^1 and limit of C^∞-smooth norms when $X = C[0,\alpha]$, with α an ordinal. Note that if $\alpha \geq \omega_1$ then X does not admit any LUR* norm.
- $\|\cdot\|_3$ is LUR, C^1, and limit of C^k-smooth norms when X is WCG and admits a C^k smooth norm
- some other classes of Banach spaces (e.g. Haydon's $C_0(\Upsilon)$ where Υ is a tree s.t. $C_0(\Upsilon)$ admits a C^1 norm) ...

In the proof we do not manipulate the existing LUR norm $\|\cdot\|_2$ on X. Instead, we "smoothen up" the known constructions of LUR norms on these spaces. These constructions (although coming from different authors and are originally done by different methods) have been all unified (see Chapter VII in Deville-Godefroy-Zizler) so that they obey the same pattern – see the example of a LUR renorming of $c_0(\Gamma)$ below.

3. A LUR norm on $c_0(\Gamma)$ (Day, Rainwater)

For $x = (x(\gamma)) \in c_0(\Gamma)$ we define

$$|\!|\!|x|\!|\!|_n := \left(\|x\|_\infty^2 + \sum_{m=1}^{\infty} 2^{-m} \sup_{A\subset\Gamma, |A|=n}\left\{\overbrace{\sum_{\gamma\in A} x(\gamma)^2}^{\varphi_A(x)^2} + \frac{1}{m}\underbrace{\|x - x\restriction_A\|_\infty^2}_{\psi_A(x)^2}\right\}\right)^{1/2}$$

and $|\!|\!|x|\!|\!|^2 := \sum_{n=1}^{\infty} \frac{|\!|\!|x|\!|\!|_n^2}{2^n}$. Then $|\!|\!|\cdot|\!|\!|$ is LUR.

Sketch of the proof (DGZ style):

1) Consider the LUR hypothesis at $0 \neq x \in c_0(\Gamma)$ for $|\!|\!|\cdot|\!|\!|$ and $(x_r) \subset c_0(\Gamma)$. By a standard convexity/positivity argument for an ℓ_2-sum we get the LUR hypothesis at x for $|\!|\!|\cdot|\!|\!|_n$ for *every* $n \geq 1$. Fix $\varepsilon > 0$.

2) Fix n large (see step 3), set $I = \{A \subset \Gamma : |A| = n\}$ and apply the following lemma.

Key Lemma VII.1.1 in [DGZ] aka Deville's Master Lemma

Let $(\varphi_i)_{i\in I}$ and $(\psi_i)_{i\in I}$ be two families of real valued convex nonnegative symmetric functions defined on a Banach space $(X, \|\cdot\|)$, which are uniformly bounded on bounded sets. For $i \in I$ and $m \in \mathbb{N}$, let us denote

$$\theta_{i,m} = \varphi_i^2(x) + \frac{1}{m}\psi_i^2(x),$$
$$\theta_m(x) = \sup\{\theta_{i,m}(x) : i \in I\} \text{ and}$$
$$\theta(x) = \|x\|^2 + \sum_{m=1}^{\infty} 2^{-m}\theta_m(x).$$

If $|\!|\!|\cdot|\!|\!|$ denotes the Minkowski functional of $B = \{x \in X : \theta(x) \leq 1\}$, then $|\!|\!|\cdot|\!|\!|$ is an equivalent norm on X with the following property:
If $x_r, x \in X$ satisfy $\lim_r \left(\frac{|\!|\!|x|\!|\!|^2 + |\!|\!|x_r|\!|\!|^2}{2} - |\!|\!|\frac{x+x_r}{2}|\!|\!|^2\right) = 0$, then there is a sequence $(i_n) \subset I$ such that:

(i) $\lim_r \left(\frac{\psi_{i_r}(x)^2 + \psi_{i_r}(x_r)^2}{2} - \psi_{i_r}\left(\frac{x+x_r}{2}\right)^2\right) = 0$ and

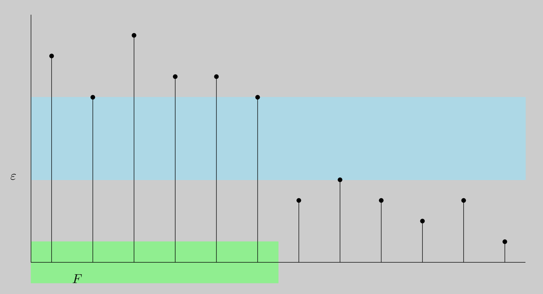
(ii) $\lim_r \varphi_{i_r}(x) = \lim_r \varphi_{i_r}(x_r) = \lim_r \varphi_{i_r}\left(\frac{x+x_r}{2}\right) = \sup\{\varphi_i(x) : i \in I\}$.

3) There exists a finite $F \subset \Gamma$ such that

a) $\|x - x\restriction_F\|_\infty < \varepsilon$

b) $\min |x|\restriction_F > \max |x|\restriction_{\Gamma\setminus A}$

Fix $n = |F|$.



3. continued

4) The condition (ii) together with properties of F show that necessarily $i_r = F$ eventually. We say that $A \in I \mapsto \varphi_A(x)$ attains a *rigid* maximum at F.

5) Now the LUR hypotesis at x *propagates* onto ψ_F and φ_F. But φ_F is the canonical norm on $\ell_2(F)$ which is LUR so $x_r\restriction_F \longrightarrow x\restriction_F$ and $\psi_F(x) < \varepsilon$ so $\limsup_r \psi_F(x_r) < \varepsilon$.

4. LUR, C^1, limit of C^k

It is clear that if we want the resulting norm $|\!|\!|\cdot|\!|\!|$ to have good smoothness properties we need smooth input φ_i and ψ_i, $i \in I$ and a smooth version of the key lemma.

Smooth Deville's Master Lemma (P.Hájek, A.P.)

Let $(X, \|\cdot\|)$ be a Banach space. Let I be an index set. Let $\{\varphi_i\}_{i\in I}$ be a family of non-negative, convex, symmetric mappings from X to $\mathbb{R}$ such that

- for each $i \in I$, φ_i is C^1-smooth in $\{x \in X : \varphi_i > 0\}$,
- $(\varphi_i(x))_{i\in I} \in c_0(I)$ for every $x \in X$,
- $\{\varphi_i\}_{i\in I}$ is uniformly bounded on bounded sets of X.

Let $\{\psi_A : A \subset I, 1 \leq |A| < \infty\}$ be a family of non-negative convex symmetric C^1-smooth functions from X to $\mathbb{R}$ such that for each $n \in \mathbb{N}$ the family $\{\psi_A : A \subset I, 1 \leq |A| \leq n\}$ is uniformly bounded on bounded sets of X.
Assume that the norm $\|\cdot\|$ is C^1-smooth. Then there exists en equivalent C^1-smooth norm $|\!|\!|\cdot|\!|\!|$ on X which for every $x, x_r \in X$, $r \in \mathbb{N}$, satisfies the following property: If $2|\!|\!|x|\!|\!|^2 + 2|\!|\!|x_r|\!|\!|^2 - |\!|\!|x + x_r|\!|\!|^2 \to 0$, then

$$\frac{\psi_A(x)^2 + \psi_A(x_r)^2}{2} - \psi_A\left(\frac{x+x_r}{2}\right)^2 \to 0$$

for each finite $A \subset I$ for which $0 \notin \{\varphi_i(x) : i \in A\}$.
Moreover, if, for $k \in \mathbb{N} \cup \{\infty\}$, the norm $\|\cdot\|$ is C^k-smooth and all the functions φ_i, ψ_A are approximated uniformly on bounded sets by C^k-smooth convex non-negative functions, then $|\!|\!|\cdot|\!|\!|$ is approximated uniformly on bounded sets of X by C^k-smooth norms.

Main differneces:

- The role of I has changed - now I corresponds rather to the "coordinates" than to their finite subsets. Thus the step 1) above has been included inside the smooth DML. The reason is the following: $\left(\sum_{\gamma\in A} x(\gamma)^2\right)_{|A|=n} \in \ell_\infty(\{|A| = n\})$ can have uncountably many maximizing coordinates (A). This is a problem for the differentiability. Working rather on $\ell_\infty(\{|A| \leq n\})$ and by tweaking the ℓ_2-sum we can penalize those maximizing coordinates A which have larger cardinality. Thus we'll end up again with only finitely many maximizing coordinates of *small* cardinality.
- The functions φ_i play only the role of the rigidity condition trigger, the LUR hypothesis is not passed onto them.
- The strong assumption $(\varphi_i(x))_{i\in I} \in c_0(I)$ is our main technical tool allowing us to achieve the differentiability of the resulting norm. As a byproduct, we are able to build in the steps 3) and 4) above (the rigidity condition) inside the smooth DML, thus arriving at the stronger conclusion
$$\frac{\psi_A(x)^2 + \psi_A(x_r)^2}{2} - \psi_A\left(\frac{x+x_r}{2}\right)^2 \to 0$$
(as compared to the expected $\frac{\psi_{A_r}(x)^2 + \psi_{A_r}(x_r)^2}{2} - \psi_{A_r}\left(\frac{x+x_r}{2}\right)^2 \to 0$).

Examples

In the case of $X = c_0(\Gamma)$ we set $I = \Gamma$,

$$\varphi_\gamma(x) = |x(\gamma)| \quad \text{and} \quad \psi_A^l(x) = \sum_{\gamma\in A} \xi_l(x(\gamma))^2 + \frac{1}{m}\xi_l(\|x - x\restriction_A\|_\infty)^2$$

where $\xi_l : \mathbb{R} \to \mathbb{R}$ is convex, even, C^∞ such that $\xi_l([0, \frac{1}{l}]) = \{0\}$ and $\xi_l(t) = t - \frac{2}{l}$ for $t > 3l$. Thus we obtain $|\!|\!|\cdot|\!|\!|_l$, and we put $|\!|\!|\cdot|\!|\!| := \sum 2^{-l}|\!|\!|\cdot|\!|\!|_l$.

Let $k \in \mathbb{N} \cup \{\infty\}$. Let $(X, |\cdot|)$ be a Banach space with a projectional resolution of the identity $\{P_\alpha\}_{\omega\leq\alpha\leq\mu}$ such that for each $\omega \leq \alpha < \mu$, the subspace $(P_{\alpha+1} - P_\alpha)X$ admits an equivalent norm $\|\cdot\|_\alpha$ which is C^1-smooth, LUR, and limit of C^k-smooth norms. Assume that X admits an equivalent C^k-smooth norm $\|\cdot\|$.
Then X admits an equivalent norm $|\!|\!|\cdot|\!|\!|$ which is C^1-smooth, LUR, and limit of C^k-smooth norms.
Proof: We set $I = [\omega, \mu)$ and (with $Q_\gamma x = (P_{\gamma+1} - P_\gamma)x$)

$$\varphi_\gamma(x) = \left\|Q_\gamma x\right\|_\gamma \quad \text{and} \quad \psi_A^l(x) = \sum_{\gamma\in A} \xi_l(\left\|Q_\gamma x\right\|_\gamma)^2 + \frac{1}{m}\xi_l(\left\|x - \sum_{\gamma\in A} Q_\gamma x\right\|)^2$$

Thus we obtain $|\!|\!|\cdot|\!|\!|_l$, and we put $|\!|\!|\cdot|\!|\!| := \sum 2^{-l}|\!|\!|\cdot|\!|\!|_l$.