On low-distortion embeddings of metric spaces into reflexive spaces.

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Definition

Let (M, d) metric, X Banach, $D \geq 1$.

- $M \underset{D}{\hookrightarrow} X$ means $\exists\, f : M \to X$ such that

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- it is unknown if $\ell_1 \underset{D}{\hookrightarrow} X$, $D < 2 \Rightarrow \ell_1 \subseteq X$

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- Kalton: any network $N \subset c_0$: $N \hookrightarrow X \Rightarrow X$ non-reflexive

Proposition

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Stable metric spaces nearly isometrically embed into the class of reflexive spaces. I.e. for any continuous functions $\rho, \omega : [0, +\infty) \to [0, +\infty)$ s.t.

(i) $t \leq \omega(t)$ *for* $t \in [0, 1]$ *and* $\omega(t) = t$ *for* $t \in [1, \infty)$*,*

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there exist a reflexive space X and a map $f : M \to X$ such that for all $x, y \in M$

$$\rho(d(x, y)) \leq \|f(x) - f(y)\| \leq \omega(d(x, y)).$$

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So $\exists X$ reflexive such that $\Phi(M) \underset{1}{\hookrightarrow} X$, thus $M \underset{D}{\hookrightarrow} X$. □

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$$M = \{\mathbf{0}\} \cup \mathbb{N} \cup F$$

where

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$$\mathrm{d}(\overline{\mathrm{co}}\,\{f(i)\}_{i=1}^{n}\,, \overline{\mathrm{co}}\,\{f(i)\}_{i=n+1}^{\infty}) \geq 4 - 2D$$

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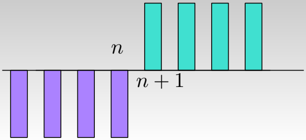

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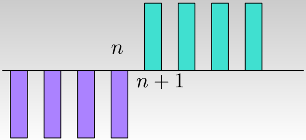

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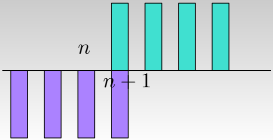

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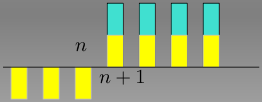

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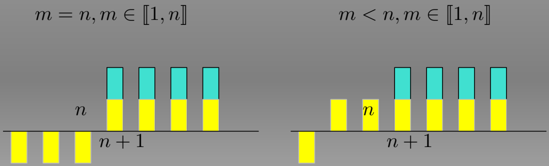

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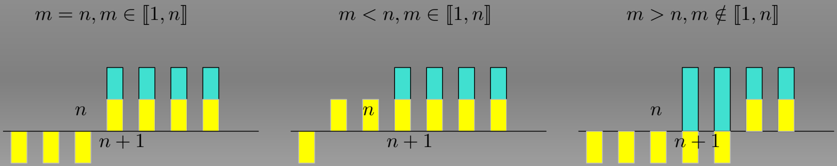

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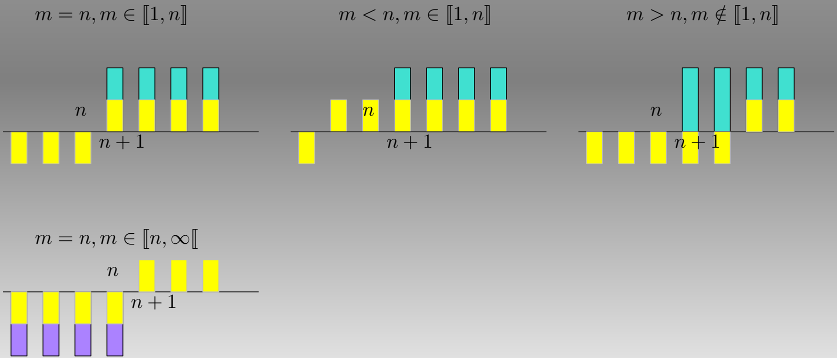

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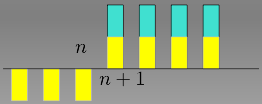

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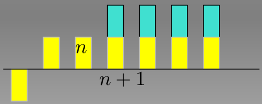
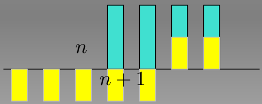
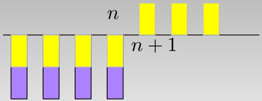


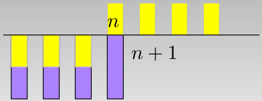
$m > n, m \notin \llbracket 1, n \rrbracket$



$m = n, m \in \llbracket n, \infty \llbracket$



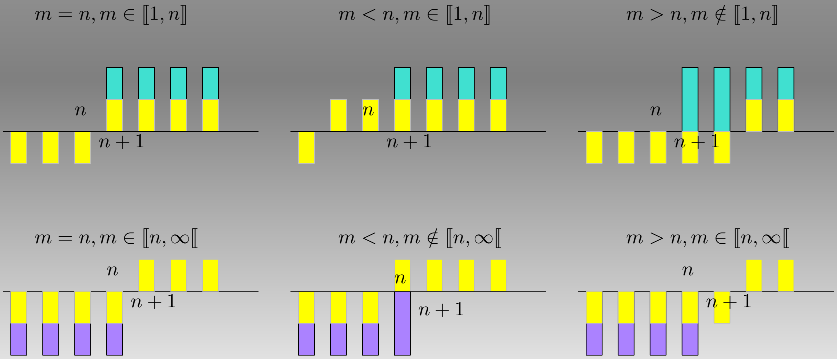
$m < n, m \notin \llbracket n, \infty \llbracket$



$$f(\llbracket 1, n \rrbracket) = 2\mathbf{1}_{\llbracket n+1, \infty \llbracket}$$

$$f(\llbracket n, \infty \llbracket) = -2\mathbf{1}_{\llbracket 1, n \rrbracket}$$

$$f(n) = -\mathbf{1}_{\llbracket 1, n \rrbracket} + \mathbf{1}_{\llbracket n+1, \infty \llbracket}$$



Theorem (yesterday)

It is equivalent for any Banach X

(i) *For any* $\varepsilon > 0$ *there is an equivalent norm* $|\cdot|$ *on* X *such that* $M \underset{1+\varepsilon}{\hookrightarrow} (X, |\cdot|)$

(ii) $M \underset{D}{\hookrightarrow} (X, \|\cdot\|)$ *for some* $1 < D < 2$ *and some equivalent norm* $\|\cdot\|$ *on* X

(iii) X *is not reflexive*

Dedicated to the memory
of
Luis Sánchez González

Thank you for your attention!