About Lipschitz free spaces of subsets of trees

joint work with Ramón Aliaga and Colin Petitjean

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For every Banach space X and every Lipschitz mapping $f : M \to X$ such that $f(0) = 0$ there is a unique linear $\hat{f} : \mathcal{F}(M) \to X$ such that $\hat{f} \circ \delta = f$. Moreover $\left\|\hat{f}\right\| = \|f\|_L$.

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$\Phi_* \delta(n) = e_n$. $\Phi f(0) = 0$ and $\Phi f(n) = f(n)$.

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- the Lebesgue measure on T is defined as

$$\lambda(E) = \sup \left\{ \sum_{k=1}^{n} \lambda(E \cap [x, y_k]) : [x_k, y_k] \subset T \text{ pairwise disjoint } \right\}$$

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Question

Assuming that $M \subset T$ is separable, complete, $\lambda(M) = 0$, does $\mathcal{F}(M) \subseteq \ell_1$?

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$\Longrightarrow$ M contains an infinite δ-separated family in M.

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- if $(\alpha_n, \beta_n) \cap \psi_{n-1} \circ \ldots \circ \psi_0(M) = \emptyset$.
- no condition on α_n, β_n.

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(iv) *No copy of* L_1 *in* $\mathcal{F}(M)$

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□

Enhorabuena Dr. Abraham!