Well-balanced Finite Volume schemes for scalar discontinuous-flux conservation laws

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based upon joint works with

Clément Cancès (Paris VI)
and many other co-authors

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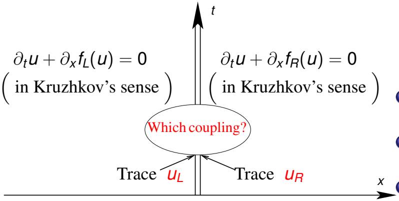
Discontinuous-flux setting and state-of-the-art

Prototype discontinuous flux problem

Prototype scalar conservation law with discontinuous flux (DFSCL):

$$\partial_t u + \partial_x \big(f_L(u)\,\mathbb{1}_{x<0} + f_R(u)\,\mathbb{1}_{x>0} \big) = 0$$



- Which notion(s) of solution ?
  Answer: depends on the model ! [Adimurthi,Mishra,V.Gowda'05]
- Uniqueness ?
- Existence (passage to the limit) ?
- Numerical approximation ?

In many examples, DFSCL can be seen as a *singular limit* problem.
What information is inherited at the limit ?
How can solutions of DFSCL be characterized *intrinsically* ?

Answer: the essential information is contained in stationary solutions
$\Rightarrow$ importance of well-balanced schemes for FV approximation of DFSCL

NB: away from the interface, we will always use
the Kruzhkov notion of entropy solution
+ Finite Volume approximations with two-point *monotone fluxes*.

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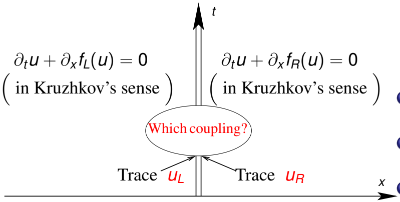

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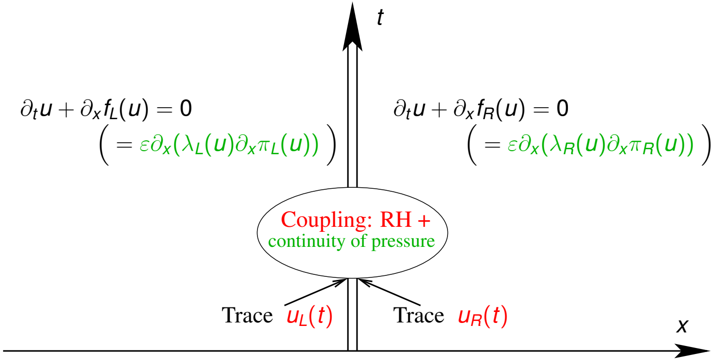
Example: Buckley-Leverett equation as vanishing capillarity limit

Consider Buckley-Leverett equation in 1D medium constituted of two rocks with distinct physical properties

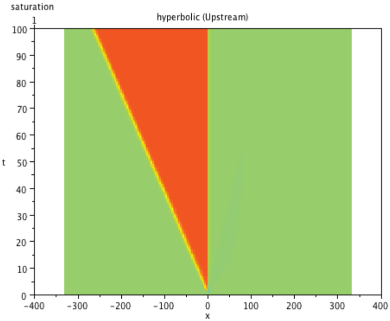
$$\partial_t u + \partial_x \left(f_L(u) \mathbb{1}_{x<0} + f_R(u) \mathbb{1}_{x>0} \right) = 0$$



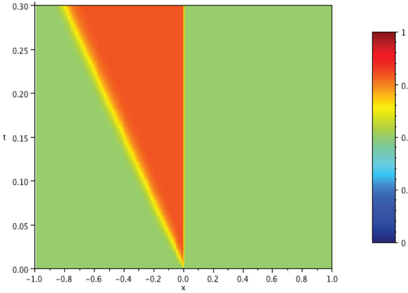
NB: the nonlinearities $\pi_{L,R}$ (capillary pressures) and $\lambda_{L,R}$ enter the model for $\varepsilon > 0$ but don't enter the limit model
$\Rightarrow$ should Interface Coupling keep memory of $\pi_{L,R}$ and $\lambda_{L,R}$?

Numerical examples: practical interest of the limit model

Constant initial condition, some choice of $f_{L,R}$ and $\pi_{L,R}$



(a) Numerical solution u_h
of the limit (hyperbolic) problem



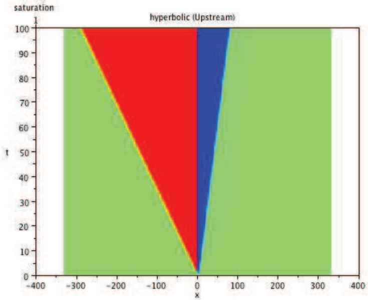
(b) Numerical solution u_h^ε
of the parabolic problem ($\varepsilon = 10^{-3}$)

Speed-up hyperbolic versus parabolic : factor 800.

The limit problem is approximated according to the recipes of
[A.,Cancès '12 and '14], [A.,Cancès'15]

Numerical examples: the limit model is under-determined

Same initial condition, same choice of $f_{L,R}$, but $\pi_{L,R}$ are changed



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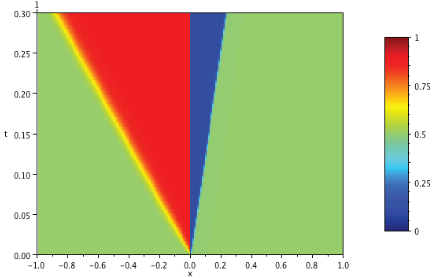

(d) Numerical solution u_h^ε of the parabolic problem ($\varepsilon = 10^{-3}$)

Conclusion: the limit DFSCL model should indeed depend on $\pi_{L,R}(\cdot)$

Goal : understand and formalize this dependence
in terms of ICC (Interface Coupling Conditions)
and find numerical strategies for approximating DFSCL + ICC

Focus on steady states for DFSCL.

[A.,Karlsen,Risebro'11] : understanding the model DFSCL equation

$$\partial_t u + \partial_x \big(f_L(u) \mathbb{1}_{x<0} + f_R(u) \mathbb{1}_{x>0} \big) = 0$$

One can characterize the Interface Coupling
by describing the set $\mathcal{G}$ of all couples $(u_L, u_R) \in \mathbb{R}^2$
that can appear as possible traces on the (left,right) at $x = 0$.

Scaling invariance $\Rightarrow$
$(u_L, u_R) \in \mathcal{G}$ iff $k(x) = u_L \mathbb{1}_{x<0} + u_R \mathbb{1}_{x>0}$ is an (admissible) solution
Thus, we are speaking about the piecewise constant steady states !

Algebraic property of $\mathcal{G}$ (called $L^1 D$ germ):

- (conservative coupling) $\forall (u_L, u_R) \in \mathcal{G} \quad f_L(u_L) = f_R(u_R)$
- (L^1-dissipative coupling) $\forall (u_L, u_R), (\hat{u}_L, \hat{u}_R) \in \mathcal{G}$

$$\mathrm{sign}(u_L - \hat{u}_L)(f_L(u_L) - f_L(\hat{u}_L)) - \mathrm{sign}(u_R - \hat{u}_R)(f_R(u_R) - f_R(\hat{u}_R)) \geq 0$$

- $\mathcal{G}$ is called maximal if it has no extension satisfying these constraints
- $\mathcal{G}$ is called definite if it has a unique maximal extension, called $\mathcal{G}^*$

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Notion of solution and well-posedness.

Definition

Assume $\mathcal{G}$ is a definite L^1D germ.

An L^∞ function u is a $\mathcal{G}$-entropy solution
if it is a local Kruzhkov solution away from $\{x=0\}$
and moreover, for a.e. $t>0$, the couple $(u(t,0^-),u(t,0^+))\in\mathcal{G}^*$.

Equivalently, the trace condition can be replaced by
adapted entropy inequalities:
$\forall (u_L,u_R)\in\mathcal{G}$, setting $k(x)=u_L 1\!\!1_{x<0}+u_R 1\!\!1_{x>0}$,

$$\partial_t|u-k(x)|+\partial_x\big(\mathrm{sign}(u-k(x))(f(x,u)-f(x,k(x)))\big)\le 0 \ \text{ in } \mathcal{D}'.$$

Theorem

For every definite L^1D germ,
Cauchy problem is well posed in the setting of $\mathcal{G}$-entropy solutions.

The Godunov scheme (including the $\mathcal{G}$-Godunov solver at $\{x=0\}$) converges to this solution.

Convergence of approximations.

Numerical (Godunov) or suitable viscosity approximations are proved to converge using the following arguments:

- The approximation method fulfills the (approximate) localized contraction inequality:

$$\partial_t|u^h - \hat{u}^h| + \partial_x\Big(\mathrm{sign}(u-\hat{u})(f(x,u^h) - f(x,\hat{u}^h))\Big) \leq \mathrm{Rem}^h \;\text{ in } \mathcal{D}'$$

- the steady states $k(x) = u_L 1\!\!1_{x<0} + u_R 1\!\!1_{x>0}$, $(u_L, u_R) \in \mathcal{G}$ are limits of the approximation method
- the inequality is used for $\hat{u}(t,x) = k(x)$; at the limit $h \to 0$, one gets adapted entropy inequalities for u.

Thus, crucial features for a numerical method are:

- Discrete contraction
- Preservation (exact, or at the limit $h \to 0$) of the steady states $k(x)$ defined from $\mathcal{G}$.

NB: **If** $\mathcal{G}$ is known... **If** the Godunov scheme is used... it converges.

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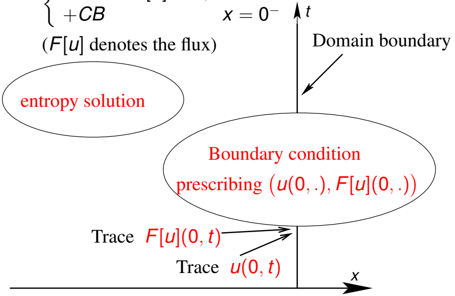
Boundary Conditions and Interface Coupling Conditions

General dissipative boundary conditions

$$\begin{cases} \partial_t u + \partial_x F[u] = 0, & x < 0 \\ +CB & x = 0^- \end{cases}$$

($F[u]$ denotes the flux)



Local "Kato inequality" obtained from the local entropy formulation:

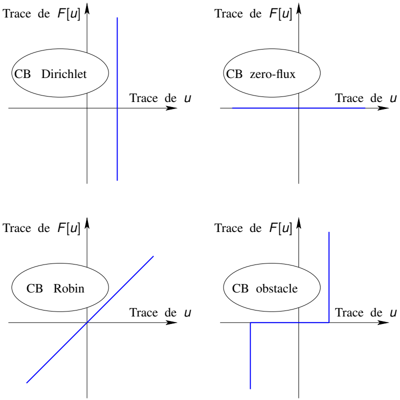
$$\forall \xi \in \mathcal{D}(\Omega)^+$$

$$\int_\Omega |u - \hat{u}|(T,x) - \int_\Omega |u_0 - \hat{u}_0| + \int_0^T \int_\Omega \text{sign}\,(u - \hat{u})(F[u] - F[\hat{u}]) \cdot \nabla \xi \leq 0$$

Exploit KI near the boundary: test fct. $\xi_n \to \mathbf{1}_\Omega$ with $\nabla \xi_n \rightharpoonup -\delta|_{\partial\Omega}\mathbf{n} \;\Rightarrow$

$$\int_\Omega |u - \hat{u}|(T,x) - \int_\Omega |u_0 - \hat{u}_0| \leq - \int_0^T \gamma_{ad\ hoc}\Big\{ \text{sign}\,(u - \hat{u})(F[u] - F[\hat{u}]) \cdot \mathbf{n} \Big\}(t)\, dt$$

Classical boundary conditions



In these cases, $(u, F[u]) \in \beta$ for some maximal monotone graph β .

General framework: BC set up in terms of a maximal monotone dependence between the solution u and flux $F[u]$ at the boundary

Boundary dissipation:

$\mathrm{sign}\,(u - \hat{u})\big(F[u] - F[\hat{u}]\big) = \mathrm{sign}\,(u - \hat{u})\big(\beta(u) - \beta(\hat{u})\big) \geq 0$!

Dissipative BC for hyperbolic conservation law. Projection.

Hyperbolic equation $u_t + f(u)_x = 0$ + formal BC $(u, F[u]) \in \beta$:

- Uniqueness is obvious for the formal problem
- Formal problem ill-posed (in general, existence fails)
- Problem with ... $= \varepsilon\partial^2_{xx} u$ is well posed.
  The limit is a local entropy solution verifying effective BC $(u, F[u]) \in \tilde\beta$ where $\tilde\beta$ is a projection of β.
  Problem with effective BC (i.e., $\tilde\beta$ in BC) is well posed
- One can easily grasp the projection procedure by picturing $\tilde\beta$.
  One observes : $\tilde\beta$ is the maximal monotone subgraph of f which is the closest to β !

Example: BLN condition [Bardos,LeRoux,Nédélec'79] can be reformulated this way [Dubois,LeFloch'88]

- One can describe $\tilde\beta$ in terms of the "Godunov numerical flux":

$$\tilde\beta = \left\{ (u, \mathcal{F}) \,\middle|\, \mathcal{F} = f(u) = \mathrm{God}[u, \tilde u] \in \beta(\tilde u) \right\}$$

Détails : [Thesis Sbihi'06],[A.,Sbihi'15]

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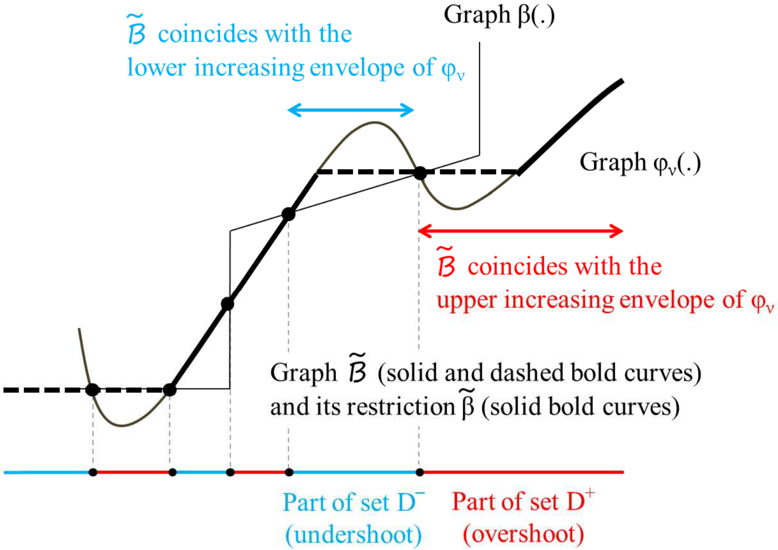
Example for a general BC: the projection procedure



Dissipative Interface Coupling Conditions (ICC)

Analogy : One assimilates inner interface to a "double boundary"

Interface Coupling Conditions (ICC) can be expressed, as in the BC case, by

$$\Big((u_L, u_R)\,,\,(F_L, F_R) \Big) \;\in\; \mathcal{H} \subset \mathbb{R}^2 \times \mathbb{R}^2$$

where $u_{L,R}$ are the traces (left and right) of the solution u
and $F_{L,R}$ are the normal traces (left and right) of the flux $F[u]$.

The ICC is conservative if $\forall\big((u_L, u_R), (F_L, F_R)\big) \in \mathcal{H},\ F_L + F_R = 0.$

The L^1-dissipativity of the CCI is equivalent to the monotonicity of $\mathcal{H}$ in the sense: $\mathcal{H}$ is called 1-monotone if

$$\forall \Big((u_L, u_R)\,,\,(F_L, F_R) \Big), \Big((\hat{u}_L, \hat{u}_R)\,,\,(\hat{F}_L, \hat{F}_R) \Big) \in \mathcal{H}$$

$$\operatorname{sign}_{max}(u_L - \hat{u}_L)(F_L - \hat{F}_L) + \operatorname{sign}_{max}(u_R - \hat{u}_R)(F_R - \hat{F}_R) \geq 0$$

Principle: The situation of ICC is fully analogous to that of BC!

NB : Idea comes from [Imbert,Monneau'14] (HJeqns on networks).
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The projection procedure for ICC. The return of the "germs".

In particular: a formally prescribed ICC is projected: $\mathcal{H} \longrightarrow \widetilde{\mathcal{H}}$,

$$\widetilde{\mathcal{H}} := \Big\{ (u_L, u_R\,;\, F_L, F_R) \in \mathbb{R}^2 \times \mathbb{R}^2 \,\Big|\, \exists (\bar u_L, \bar u_R\,;\, F_L, F_R) \in \mathcal{H} \\ F_L = f_L(u_L) = \mathrm{God}_L[u_L, \bar u_L],\ -F_R = f_R(u_R) = \mathrm{God}_R[\bar u_R, u_R] \Big\}$$

($\mathrm{God}_{L,R}[\cdot,\cdot]$ being the Godunov fluxes associated with $f_{L,R}$).

As for the BC case, $\widetilde{\mathcal{H}}$ should be seen as the effective ICC [A.,'15] .
One finds:

- $\mathcal{H}$ is conservative $\Rightarrow$ $\widetilde{\mathcal{H}}$ is also conservative
- $\mathcal{H}$ is L^1-dissipative $\Rightarrow$ $\widetilde{\mathcal{H}}$ is also L^1-dissipative ; moreover, the domain of $\widetilde{\mathcal{H}}$ is an L^1D germ

Example of ICC: "conservative inflow-outflow Robin conditions"
Given monotone continuous functions $A_{L,R} : \mathbb{R} \to \mathbb{R}$ (e.g., $A_{L,R}(u) = \frac{\lambda_{L,R}}{1-\lambda_{L,R}} u$ for some parameters $\lambda_{L,R} \in (0,1)$),

$$\mathcal{H} := \Big\{ (u_L, u_R\,;\, F, -F) \in \mathbb{R}^2 \times \mathbb{R}^2 \,\Big|\, u_{L,R} \in \mathbb{R},\ F = A_L(u_L) - A_R(u_R) \Big\}.$$

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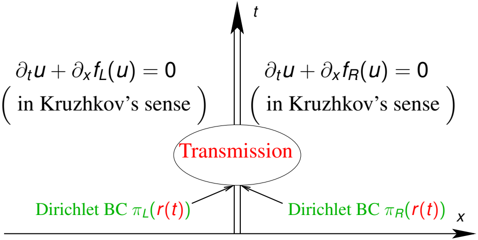
Examples of ICC, applications and well-balanced FV schemes

Conservative ICC defined by transmission maps

The example of vanishing capillarity suggests the following ICC:

$$(\, u_L, u_R\, ;\; F_L, F_R\,) \in \mathcal{H}(\pi_{L,R}) \;\Leftrightarrow\; F_L + F_R = 0, \;\; \pi_L(u_L) = \pi_R(u_R).$$

The interface coupling by transmission map $r \mapsto (\pi_L(r), \pi_R(r))$:



Transmission: two Dirichlet pbs (in the BLN sense) coupled by

- the Dirichet BC $\pi_{L,R}(r(t))$ ($r(t)$ being additional unknown)
- the conservativity relation
  $\mathrm{God}_L[u(t, 0^-), \pi(r(t))] = \mathrm{God}_R[\pi_R(r(t)), u(t, 0^+)].$

Well-balanced FV schemes for transmission-map ICC

[A.,Cancès'12,'14,'15] FV schemes for the transmission=map ICC:
the two-point interface flux $F_{int}(\cdot,\cdot)$ is defined by

$$F_{int}(u_-,u_+) = \mathrm{God}_L[u_-,\pi_L(r)] = \mathrm{God}_R[\pi_R(r),u_+]$$

where $r \in \mathbb{R}$ solves $\mathrm{God}_R[\pi_R(r),u_+] - \mathrm{God}_L[u_-,\pi_L(r)] = 0$.

Properties of the scheme:

- One implicit unknown per interface point;
  the equation to be solved is a scalar monotone equation
  (e.g., $\Rightarrow$ *regula falsi* method)
- The numerical flux F_{int} is monotone and Lipschitz
- The scheme is well balanced
  (it preserves the "germ" steady states) $\Rightarrow$ the scheme converges
  NB: we use Godunov fluxes of $f_{L,R}$...
  but not the Riemann solver at the interface !
- Moreover, $\mathrm{God}_{L,R}$ can be replaced by any classical num. flux !
  The scheme is "asymptotically well-balanced" and convergent.

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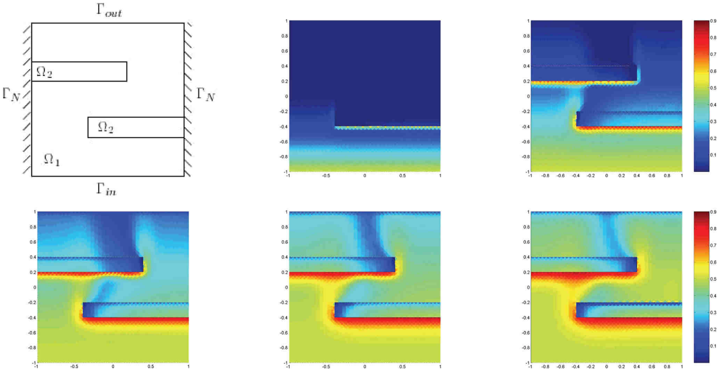
Numerical example in 2D (IMPES scheme)

Combination with 2D IMplicit Pressure - Explicit Saturation Scheme:



The two-rock domain is initially saturated in water. Two barriers (rock Ω_2) have a higher entry pressure. The vertical boundaries are impermeable. Bottom+top : a constant rate of a total flux is prescribed. Saturation $s = 0.5$ imposed on Γ_{in}. Details: [Andreianov,Brenner,Cancès'13] .

Traffic models with point constraint

[Colombo,Goatin'07] : LWR model $\partial_t u + \partial_x f(u) = 0$
with point constraint $f(u)|_{x=0} \leq q(t)$.
Models red lights, pay tolls, construction cites,...
The underlying ICC is:

$$\begin{aligned}\mathcal{H}(t) \; &= \{(k,k,F,-F) \mid k \text{ arbitrary}, \; F \leq q(t)\} \;\; \text{(the Kruzhkov part)} \\ &\bigcup \{(k_L,k_R,F,-F) \mid k_L > k_R, \; F = q(t)\} \;\; \text{(non-Kruzhkov jumps)}\end{aligned}$$

Given any monotone consistent Lipschitz numerical flux $F(\cdot,\cdot)$,
the interface numerical flux for the constrained model is defined by:

$$F_{int}(t; u_-, u_+) = \min\{F(u_-,u_+), q(t)\}.$$

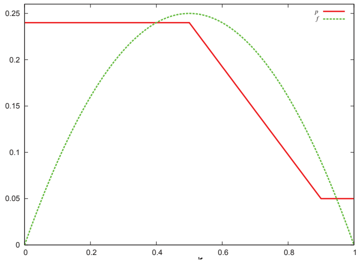
- the flux F_{int} is monotone and Lipshitz
- the scheme is asymptotically well balanced $\Rightarrow$ it converges [A.,Goatin,Seguin'10]
- if F is the Godunov flux of f, then the resulting scheme is the Godunov scheme also at the interface [Cancès,Seguin'12]

Application to pedestrian traffic modeling

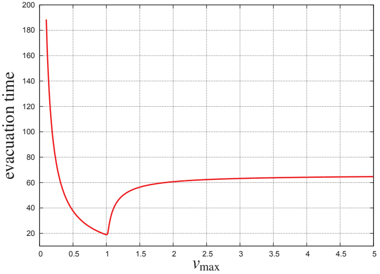
Let us make depend $q(t)$ on the solution $u(t,\cdot)$. We propose new pedestrian ("panic at the exit") models [A.,Donadello,Rosini'14] :

$$f(u)|_{t=0} \leq q(t) = P\Big(\int_{\mathbb{R}_-} w(x)u(t,x)\,dx\Big), \quad w \geq 0, \ \int_{\mathbb{R}_-} w(x)\,dx = 1.$$

$P(\cdot)$ non-increasing $\Rightarrow$ "Faster is Slower" and Braess paradoxes!
Simulations of [A.,Donadello,Razafison,Rosini prep.'15] .



(e) Flux $f(\cdot)$ of the LWR model and the "exit-clugging map" $P(\cdot)$



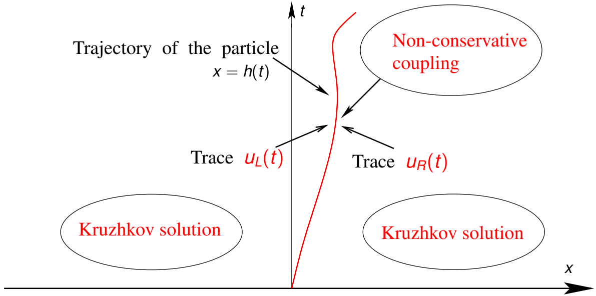
(f) Dependence of evacuation time at the exit on the speed v_{max} at the entrance

Conservation Laws with point source

Coupling of Burgers fluid and a point Particule via a drag force

Model proposed by [Lagoutière,Seguin,Takahashi'07] :

$$\begin{cases} \partial_t u + \partial_x \frac{u^2}{2} = -\big(u - h'(t)\big)\delta_0(x - h(t)), \\ h''(t) = u(t, h(t)) - h'(t) \end{cases}$$



– Splitting arguments or fixed point arguments $\Rightarrow$ decoupling

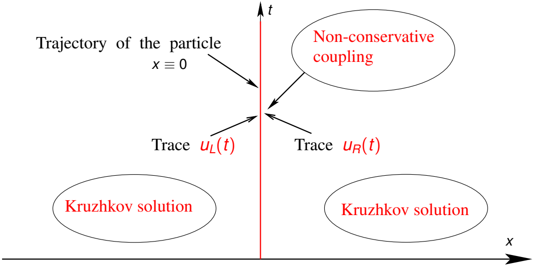
– Change of variable $\Rightarrow$ reduction to the case $h' \equiv 0$.

Theory: [A.,Lagoutière,Seguin,Takahashi'14] ;
numerics: [ALST'10],[Aguillon,Lagoutière,Seguin'14],[Towers'15]

Burgers equation perturbed by a singular source term

Simplified version of the previous problem:

$$\partial_t u + \partial_x \frac{u^2}{2} = -u\delta_0(x)$$



NB: Formal dissipativity $\Rightarrow$ the "germ"/ICC formalism can be used. [Lagoutière,Seguin,Takahashi'07] : the rigorous interpretation of the non-conservative product $u(t,x)\delta_0(x)$ reduces to finding steady states via $\delta_\varepsilon,\varepsilon \to 0$. Particular steady states:

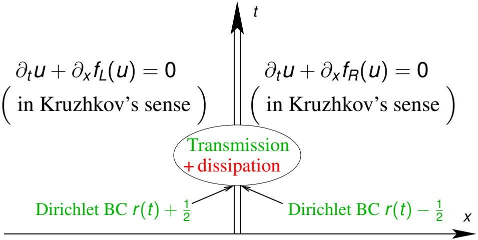
$$k(x) = u_L \mathbb{1}_{x<0} + u_R \mathbb{1}_{x>0}, \quad u_L = r + \tfrac{1}{2}, \; u_R = r - \tfrac{1}{2},$$

moreover, the corresponding defect of conservation equals r

Transmission maps for non-conservative coupling

One can attempt to encode the ICC using:

- the transmission map $r \mapsto (r+\frac{1}{2}, r-\frac{1}{2})$
- and the dissipation map $r \to \psi(r) = r$



Transmission: two Dirichlet pbs (in the BLN sense) coupled by

- the Dirichet BC $r(t) \pm \frac{1}{2}$ ($r(t)$: the additional unknown)
- the dissipativity relation
  $\mathrm{God}_R[r(t)+\frac{1}{2}, u(t,0^+)] - \mathrm{God}_L[u(t,0^-), r(t)-\frac{1}{2}] + \psi(r) = 0.$

Result: ψ monotone $\Rightarrow$ same recipes apply for the FV scheme

Numerics: drafting-kissing-tumbling. Extension to Euler system?

In fact, a simpler (fully explicit) but less robust scheme has already been proposed for Burgers-particle problem. A simulation:

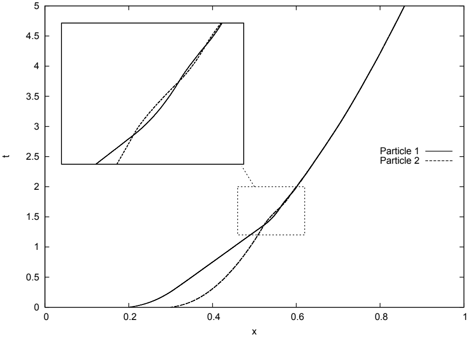


Figure: Trajectories of two particles

NB: For Euler-particle pb., extension of this scheme fails [Aguillon'14]
Preliminary results on the transmission-like scheme are encouraging.

Conclusions:

- in modeling with DFSCL,
  identification of Interface Coupling Conditions is essential
- well-balanced (asymptotically) monotone FV schemes converge
- no general strategy (except for transmission+dissipation ICC)
- successful examples

Perspectives:

- other examples of ICC that appear in practice ?
- (partial) extension of transmission strategies to some systems ??

GRAZIE !