A new fictitious domain method: optimal convergence without cut elements

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January 19, 2016

Abstract

We present a method of the fictitious domain type for the Poisson-Dirichlet problem. The computational mesh is obtained from a background (typically uniform Cartesian) mesh by retaining only the elements intersecting the domain where the problem is posed. The resulting mesh does not thus fit the boundary of the problem domain. Several finite element methods (XFEM, CutFEM) adapted to such meshes have been recently proposed. The originality of the present article consists in avoiding integration over the elements cut by the boundary of the problem domain, while preserving the optimal convergence rates, as confirmed by both the theoretical estimates and the numerical results.

1 Introduction and presentation of the method.

Consider the Poisson problem

$$-\Delta u = f \text{ in } \Omega, \qquad u = g \text{ on } \Gamma \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a domain with smooth boundary Γ, f and g are given functions on Ω and Γ respectively. The goal of the article is to construct a fictitious domain finite element (FE) discretization of problem (1) whose convergence rate is the same as that of a standard FE discretization on a mesh fitting the geometry of Ω. We start by embedding Ω into a simply shaped domain $\mathcal{O}$ and introduce a quasi-uniform mesh $\mathcal{T}_h^{\mathcal{O}}$ on $\mathcal{O}$ that can cut the boundary Γ in an arbitrary manner. Let

$$\mathcal{T}_h = \{T \in \mathcal{T}_h^{\mathcal{O}} : T \cap \Omega \neq \varnothing\}, \qquad \Omega_h = (\cup_{T\in\mathcal{T}_h} T)^\circ$$

$\Gamma_h = \partial\Omega_h$, as illustrated in Fig. 1. Several optimally convergent fictitious domain methods have been recently proposed following the XFEM or CutFEM paradigm. The FE approximation to u is sought there in a FE space defined over the mesh $\mathcal{T}_h$ and boundary conditions on Γ are imposed either through Lagrange multipliers [2, 5] or by Nitsche method [1, 3]. The common feature of all these methods is that the integrals over Ω are preserved in the FE formulation so that a non trivial numerical quadrature should be performed to compute the contributions to the stiffness matrix and to the right-hand side on the parts of mesh elements obtained by cutting $\mathcal{T}_h$ with Γ. We attempt, in the present paper, to circumvent this technical complication by introducing a reformulation of the problem that involves the integrals only over Ω_h, Γ_h and Γ.

Let us extend f from Ω to Ω_h and imagine (for the moment) that (1) can be solved on the extended domain Ω_h while still imposing the boundary conditions on Γ:

$$-\Delta u = f \text{ in } \Omega_h, \qquad u = g \text{ on } \Gamma. \tag{2}$$

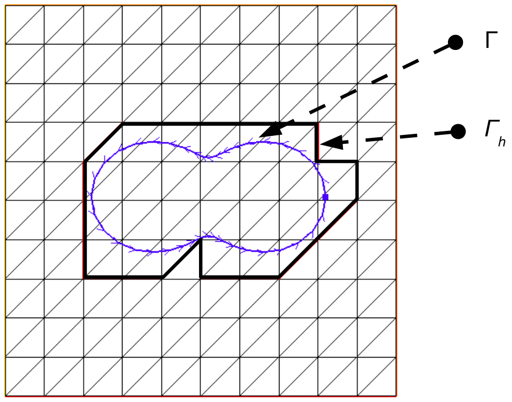

Figure 1 – The "background" mesh $\mathcal{T}_h^{\mathcal{O}}$, the "physical" domain Ω (inside Γ) and the computational domain Ω_h (inside Γ_h).

We keep here the same notations u and f for the functions on Ω_h as for the originals on Ω. Integration by parts over Ω_h yields

$$\int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v + \int_{\Gamma} u \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_{\Gamma} uv = \int_{\Omega_h} fv + \int_{\Gamma} g \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_{\Gamma} gv \tag{3}$$

for any $v \in H^1(\Omega_h)$ and $\gamma > 0$. Here, n on Γ or Γ_h denotes the unit normal looking outwards from Ω or Ω_h.

We inspire ourselves with the variational formulation (3) in writing the following FE discretization: introduce

$$V_h = \{v_h \in H^1(\Omega_h) : v_h|_T \in \mathbb{P}_1(T)\, \forall T \in \mathcal{T}_h\}$$

with $\mathbb{P}_1$ denoting the set of polynomials of degree ≤ 1 and search for $u_h \in V_h$ such that

$$a_h(u_h, v_h) = L_h(v_h) \quad \forall v_h \in V_h \tag{4}$$

where

$$a_h(u,v) = \int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v + \int_{\Gamma} u \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_{\Gamma} uv + \sigma h \sum_{E \in \mathcal{F}_\Gamma} \int_E \left[\frac{\partial u}{\partial n}\right] \left[\frac{\partial v}{\partial n}\right]$$

$$L_h(v) = \int_{\Omega_h} fv + \int_{\Gamma} g \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_{\Gamma} gv,$$

γ, σ are some positive numbers properly chosen in a manner independent of h,

$$\mathcal{F}_\Gamma = \{E \text{ (an internal edge of } \mathcal{T}_h) \text{ such that } \exists T \in \mathcal{T}_h : T \cap \Gamma \neq \varnothing \text{ and } E \in \partial T\}$$

and $[\cdot]$ stands for the jump over an edge. The last term in the definition of a_h represents the ghost penalty, as proposed in [1], and helps to assure the coerciveness of a_h. Our method (4) is in fact very close to the non-symmetric Nitsche fictitious domain method from [1], except for the idea to extend u_h from Ω to Ω_h.

The well-posedness and the optimal error estimates for (4) are proved in the next section. We restrict ourselves here to $P1$ continuous FE on a triangular mesh, but all the results would remain the same in the case of quadrilateral meshes and $Q1$ FE. An extension to higher-order FE seems less straightforward.

Note that the proofs below abandon eventually the assumption that (2) can be solved in Ω_h and rely rather on an arbitrary extension $\tilde{u}$ of u, i.e. the solution to (1), from Ω to Ω_h. This resembles the method of [4] where a smooth extension of u to the whole of $\mathcal{O}$ is constructed numerically by an iterative process. The basic difference between the method of [4] and that of the present paper (apart from the presence of stabilization terms) is that we need here the extension only in a narrow strip of width $\sim h$. This minimizes the effect of choosing a “wrong” extension and enables us to avoid its explicit construction.

2 Coerciveness of a_h and error bounds.

In what follows, C denotes a constant depending only on regularity of $\mathcal{T}_h$ and that of Γ.

Lemma 1. *Let* $\mathcal{T}_h^\Gamma = \{T \in \mathcal{T}_h : T \cap \Gamma = \varnothing\}$ and $\Omega_h^\Gamma = \left(\cup_{T\in\mathcal{T}_h^\Gamma} T\right)^\circ$. Then, for all $v_h \in V_h$

$$|v_h|^2_{1,\Omega_h^\Gamma} \leqslant \alpha |v_h|^2_{1,\Omega_h} + \beta h \sum_{E\in\mathcal{F}_\Gamma} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E} \tag{5}$$

with some $0 < \alpha < 1$ *and* $\beta > 0$ *that depend only on the mesh regularity. Moreover,*

$$\sum_{E\in\mathcal{F}_\Gamma} \|v_h\|^2_{0,E} \le C(\|v_h\|^2_{0,\Gamma} + h|v_h|^2_{1,\Omega_h^\Gamma}). \tag{6}$$

Proof. The boundary Γ can be covered by element patches $\{P_i\}_{i=1,\dots,N_P}$ having the following properties:

— Each patch is a connected set;

— $P_i = T_i \cup P_i^\Gamma$ where T_i is a triangle from $\mathcal{T}_h$ lying inside Ω and P_i^Γ contains at most M triangles from $\mathcal{T}_h^\Gamma$ (with M depending only on the mesh regularity);

— $\mathcal{T}_h^\Gamma = \cup_{i=1}^{N_P} P_i^\Gamma$;

— P_i and P_j are disjoint if $i \neq j$.

Choose any $\beta > 0$ and consider

$$\alpha := \max_{P_i, v_h \neq const} \frac{|v_h|^2_{1,P_i^\Gamma} - \beta h \sum_{E\in\mathcal{F}_i} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E}}{|v_h|^2_{1,P_i}} \tag{7}$$

where the maximum is taken over all the possible configurations of a patch P_i allowed by the mesh regularity and over all the piecewise linear functions on P_i, which are different from a constant on P_i. The subset $\mathcal{F}_i \subset \mathcal{F}_\Gamma$ gathers the edges internal to P_i. Note that the quantity under the max sign in (7) is invariant under the scaling transformation $x \mapsto hx$ and is homogeneous with respect to v_h. Thus, the maximum is indeed attained since it is taken over a bounded set in a finite dimensional space. Clearly, $\alpha \le 1$. Supposing $\alpha = 1$ would lead to a contradiction. Indeed, if $\alpha = 1$ then we can take P_i, v_h yielding this maximum and suppose without loss of generality $h = 1$ and $|v_h|_{1,P_i} = 1$. We observe then

$$1 = |v_h|^2_{1,P_i^\Gamma} - \beta \sum_{E\in\mathcal{F}_i} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E} = \frac{|v_h|^2_{1,P_i^\Gamma} - \beta \sum_{E\in\mathcal{F}_i} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E}}{|v_h|^2_{1,P_i^\Gamma} + |v_h|^2_{1,T_i}}$$

so that

$$|v_h|^2_{1,T_i} + \beta \sum_{E\in\mathcal{F}_i} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E} = 0$$

This implies $\nabla v_h = 0$ on T_i and $[\nabla v_h] = 0$ on all $E \in \mathcal{F}_i$, thus $\nabla v_h = 0$ on P_i, which contradicts $|v_h|_{1,P_i} = 1$. Thus $\alpha < 1$ so that

$$|v_h|^2_{1,P_i^\Gamma} \le \alpha |v_h|^2_{1,P_i} + \beta h \sum_{E\in\mathcal{F}_i} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E}$$

for all v_h and all admissible patches P_i. Summing this over P_i, $i = 1, \dots, N_P$ yields (5).

To prove (6), we observe first for any $v \in H^1(\Omega_h^\Gamma)$

$$\|v\|_{0,\Omega_h^\Gamma} \le C\left(\sqrt{h}\|v\|_{0,\Gamma} + h|v|_{1,\Omega_h^\Gamma}\right) \tag{8}$$

which is valid since Ω_h^Γ is a strip of width $\sim h$ around Γ (the proof goes essentially by a Taylor expansion of order 1 around Γ). We then recall the well-known trace inequality

$$\|v\|_{0,E}^2 \le C\left(\frac{1}{h}\|v\|_{0,T}^2 + h|v|_{1,T}^2\right) \tag{9}$$

valid on any edge E and the adjacent triangle T. Summing (9) over all $E \in \mathcal{F}_\Gamma$ and combining it with (8) gives (6). □

Lemma 2. *Provided σ is sufficiently big, there exists an h-independent constant $c > 0$ such that $\forall v_h \in V_h$*

$$a(v_h, v_h) \ge c\|\!|v_h|\!\|_h^2 \qquad \text{with} \qquad \|\!|v|\!\|_h^2 = |v|_{1,\Omega_h}^2 + \frac{1}{h}\|v\|_{0,\Gamma}^2$$

Proof. For any $v_h \in V_h$, We have by the definition of a_h

$$a_h(v_h, v_h) = \int_{\Omega_h} |\nabla v_h|^2 - \int_{B_h} |\nabla v_h|^2 - \sum_{F \in \mathcal{F}_\Gamma} \int_{F \cap B_h} v_h \left[\frac{\partial v_h}{\partial n}\right] + \frac{\gamma}{h}\int_\Gamma v_h^2 + \sigma h \sum_{E \in \mathcal{F}_\Gamma} \int_E \left[\frac{\partial v_h}{\partial n}\right]^2$$

where B_h denotes the strip between Γ and Γ_h. Noting that $B_h \subset \Omega_h^\Gamma$ we can use (5) combined with the Young inequality (for any $\varepsilon > 0$) and (6) to write

$$\begin{aligned} a(v_h, v_h) &\geqslant (1-\alpha)|v_h|_{1,\Omega_h}^2 + \left(\sigma - \beta - \frac{1}{2\varepsilon}\right) h \sum_{E \in \mathcal{F}_\Gamma} \left\|\left[\frac{\partial v_h}{\partial n}\right]\right\|_{0,E}^2 - \frac{\varepsilon}{2h} \sum_{E \in \mathcal{F}_\Gamma} \|v_h\|_{0,E}^2 + \frac{\gamma}{h}\|v_h\|_{0,\Gamma}^2 \\ &\geqslant \left(1 - \alpha - \frac{\varepsilon C}{2}\right)|v_h|_{1,\Omega_h}^2 + \left(\sigma - \beta - \frac{1}{2\varepsilon}\right) h \sum_{E \in \mathcal{F}_\Gamma} \left\|\left[\frac{\partial v_h}{\partial n}\right]\right\|_{0,E}^2 + \frac{\gamma - \varepsilon C/2}{h}\|v_h\|_{0,\Gamma}^2 \end{aligned}$$

Taking ε sufficiently small and σ sufficiently big this bounds $a(v_h, v_h)$ from below by $c\|\!|v_h|\!\|_h^2$ as claimed. □

It is easy to see that the coerciveness of a_h provided by the preceding lemma in combination with Galerkin orthogonality and interpolation estimates gives an *a priori* estimate

$$\|\!|u - u_h|\!\|_h \le Ch|u|_{2,\Omega_h}$$

for the solution u to (2). This is however not completely satisfactory since one cannot expect the usual elliptic regularity $|u|_{2,\Omega_h} \le C\|f\|_{0,\Omega_h}$. Fortunately, one can recover the optimal convergence at the expense of a stronger assumption on the right-hand side in (1) and its extension to Ω_h as shown in the following

Theorem 1. *Suppose $f \in H^1(\Omega_h)$, $g \in H^{5/2}(\Gamma)$ and let $u \in H^3(\Omega)$ be the solution to (1), $u_h \in V_h$ be the solution to (4). Provided σ is sufficiently big, there exists an h-independent constant $C > 0$ such that*

$$|u - u_h|_{1,\Omega} + \frac{1}{\sqrt{h}}\|u - u_h\|_{0,\Gamma} + \frac{1}{\sqrt{h}}\|u - u_h\|_{0,\Omega} \le Ch\left(\|f\|_{1,\Omega_h} + \|g\|_{5/2,\Gamma}\right). \tag{10}$$

Proof. Under the Theorem's assumptions, the solution to (1) is indeed in $H^3(\Omega)$ and it can be extended to a function $\tilde{u} \in H^3(\Omega_h)$ such that $\tilde{u} = u$ on Ω and $\|\tilde{u}\|_{3,\Omega_h} \le C\left(\|f\|_{1,\Omega} + \|g\|_{5/2,\Gamma}\right)$.

Clearly, $\tilde{u}$ satisfies (2) and (3) with u replaced by $\tilde{u}$ and f replaced by $\tilde{f} := -\Delta\tilde{u}$. We have then using the standard nodal interpolation $I_h : C(\bar{\Omega}_h) \to V_h$

$$\begin{aligned}\frac{1}{c}|\!|\!|u_h - I_h\tilde{u}|\!|\!|_h \le \sup_{v_h\in V_h}\frac{a_h(u_h - I_h\tilde{u}, v_h)}{|\!|\!|v_h|\!|\!|_h} &= \sup_{v_h\in V_h}\frac{a_h(\tilde{u} - I_h\tilde{u}, v_h) + (f-\tilde{f}, v_h)_{L^2(\Omega_h)}}{|\!|\!|v_h|\!|\!|_h}\\ &\le C\left(h|\tilde{u}|_{2,\Omega_h} + \|f-\tilde{f}\|_{0,\Omega_h}\right)\end{aligned}$$

thanks to the usual interpolation estimates and to the bound $\|v_h\|_{0,\Omega_h} \le C|\!|\!|v_h|\!|\!|_h$. We remind now that $f = \tilde{f}$ on Ω and conclude with the aid of a Poincaré-like inequality in the strip $B_h = \Omega_h \setminus \Omega$ of width $\sim h$

$$\|f-\tilde{f}\|_{0,\Omega_h} = \|f-\tilde{f}\|_{0,B_h} \le Ch|f-\tilde{f}|_{1,\Omega_h} \le Ch\left(|f|_{1,\Omega_h} + \|\tilde{u}\|_{3,\Omega_h}\right).$$

Combining the estimates above with the triangle inequality proves $|\!|\!|u_h - \tilde{u}|\!|\!|_h \le Ch\left(\|f\|_{1,\Omega_h} + \|g\|_{5/2,\Gamma}\right)$, i.e. the estimates in (10) in $H^1(\Omega)$ and $L^2(\Gamma)$ norms.

To prove the $L^2(\Omega)$ error estimate, let us introduce $z : \Omega \to \mathbb{R}$ such that

$$-\Delta z = u - u_h \text{ in } \Omega, \qquad z = 0 \text{ on } \Gamma.$$

By elliptic regularity, $\|z\|_{2,\Omega} \le C\|u-u_h\|_{0,\Omega}$. Let $\tilde{z}$ be an extension of z from Ω to Ω_h preserving the H^2 norm estimate and set $z_h = I_h\tilde{z}$. Applying inequality (8) to $\tilde{z}$ and to $\nabla\tilde{z}$ yields $\|\tilde{z}\|_{0,\Omega_h^\Gamma} \le Ch\|u-u_h\|_{0,\Omega}$ and $|\tilde{z}|_{1,\Omega_h^\Gamma} \le C\sqrt{h}\|u-u_h\|_{0,\Omega}$. Similarly, by a Taylor expansion of order 2 around Γ, one can prove $\|\tilde{z}\|_{0,\Gamma_h} \le Ch\|u-u_h\|_{0,\Omega}$. We combine now the bounds above with the interpolation estimates to obtain

$$\begin{aligned}|\tilde{z} - z_h|_{1,\Omega_h} + \sqrt{h}\left\|\frac{\partial(\tilde{z}-z_h)}{\partial n}\right\|_{0,\Gamma\cup\Gamma_h} + \frac{1}{\sqrt{h}}\|\tilde{z}-z_h\|_{0,\Gamma\cup\Gamma_h}&\\ + \sqrt{h}|\tilde{z}|_{1,\Omega_h^\Gamma} + \|\tilde{z}\|_{0,\Gamma_h} + \|z_h\|_{0,\Omega_h^\Gamma} \le Ch\|u-u_h\|_{0,\Omega}& \qquad (11)\end{aligned}$$

Using Galerkin orthogonality $a_h(\tilde{u}-u_h, z_h) = \int_{\Omega_h}(\tilde{f}-f)z_h$ and estimates (11) we arrive at

$$\begin{aligned}\|u-u_h\|_{0,\Omega}^2 &= \int_\Omega \nabla(u-u_h)\cdot\nabla z - \int_\Gamma (u-u_h)\frac{\partial z}{\partial n}\\ &= a_h(\tilde{u}-u_h, \tilde{z}-z_h) + \int_{\Gamma_h}\frac{\partial(\tilde{u}-u_h)}{\partial n}\tilde{z} - 2\int_\Gamma(u-u_h)\frac{\partial z}{\partial n} - \int_{B_h}\nabla(\tilde{u}-u_h)\cdot\nabla\tilde{z} + \int_{B_h}(\tilde{f}-f)z_h\\ &\le C\left(|\!|\!|\tilde{u}-u_h|\!|\!|_h + \left\|\frac{\partial(\tilde{u}-u_h)}{\partial n}\right\|_{0,\Gamma_h} + \frac{1}{h}\|u-u_h\|_{0,\Gamma} + \frac{1}{\sqrt{h}}|\tilde{u}-u_h|_{1,\Omega_h^\Gamma} + \|\tilde{f}-f\|_{0,B_h}\right)h\|u-u_h\|_{0,\Omega}\end{aligned}$$

which gives the announced error estimate in $L^2(\Omega)$ norm thanks to already proven estimates in $H^1(\Omega)$ and $L^2(\Gamma)$ norms. □

Note that the L^2 estimate in the preceding theorem is sub-optimal, although the numerical experiments reveal the optimal convergence rate $O(h^2)$, similar to the state of the art in the study of the non-symmetric Nitsche method.

3 Numerical experiments.

We have applied our method (4) to Problem (1) with $f = 1$, $g = 0$ and domain Ω with boundary Γ represented by the curve

$$\begin{pmatrix} x(t) \\ y(t)\end{pmatrix} = \begin{pmatrix} x_0 \\ y_0\end{pmatrix} + R(1+\delta\cos 2t)\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}, \qquad t\in[0,2\pi]$$

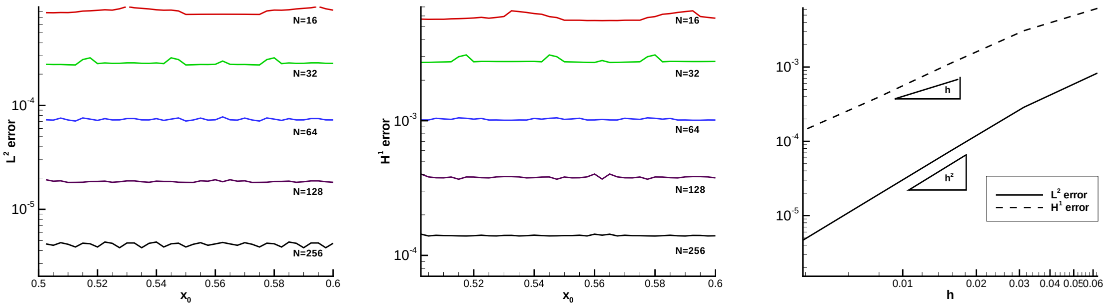

Figure 2 – The error $(u_h - u_{ref})$ in $L^2(\Omega)$ and $H^1(\Omega)$ norms as functions of the geometry parameter x_0 (left and middle) and as functions of h for x_0 fixed (right).

We take the parameters $R = 0.2$, $\delta = 0.5$ and vary (x_0, y_0) over the line $x_0 - 2y_0 + \frac{1}{2} = 0$. To set up the numerical method we embed Ω into the unit square $\mathcal{O} = (0,1)^2$ and introduce the uniform triangular mesh $\mathcal{T}_h^{\mathcal{O}}$ with $(N+1)\times(N+1)$ nodes. Both the domain and the background mesh (with $N = 10$) that we have used in our calculations are represented in Fig. 1. The natural extension $f = 1$ over Ω_h was chosen in (4) and the following stabilization parameters were used: $\gamma = 0.5$, $\sigma = 0.01$. To attest the accuracy of the numerical solution u_h, it was compared with a reference solution u_{ref} obtained by the standard $P1$ FEM on a mesh $\mathcal{T}_h^f$ fitting the geometry of Ω with the fine mesh size $h_f \approx h/5$, $h = \frac{1}{N}$ being the mesh size of $\mathcal{T}_h$. All the computations were done using FreeFem++ [6]. The results are reported in Fig. 2. We give there first the errors in $L^2(\Omega)$ and $H^1(\Omega)$ norms as functions of x_0 (the x-coordinate of the center of Ω), thus demonstrating the robustness of the method with respect to the placement of Ω across the background mesh. The optimal rates of convergence, i.e. $O(h^2)$ in the L^2 norm and $O(h)$ in the H^1 norm, are confirmed by the rightmost plot, where the errors are computed for the fixed placement of Ω: $x_0 = 0.58$, $y_0 = 0.54$. The H^1 error seems to behave even super-optimally. This is probably due to a Zienkiewicz-Zhu smoothing that we apply to ∇u_h on $\mathcal{T}_h$ before interpolating it to $\mathcal{T}_h^f$ on which the integral $\int_\Omega |\nabla u_h - \nabla u_{ref}|^2$ is evaluated.

Acknowledgements

I wish to thank Prof. Erik Burman for enlightening discussions and for the organization of a workshop on Unfitted FEM at University College London where a preliminary version of this work was presented.

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