

Weighted L^p -theory for Poisson, biharmonic and Stokes problems on periodic unbounded strips of \mathbb{R}^n

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Abstract

This paper establishes isomorphisms for Laplace, biharmonic and Stokes operators in weighted Sobolev spaces (WSS). The $W_\alpha^{m,p}(\mathbb{R}^n)$ -spaces are similar to standard Sobolev spaces $W^{m,p}(\mathbb{R}^n)$, but they are endowed with weights $(1 + |x|^2)^{\alpha/2}$ prescribing functions' growth or decay at infinity. Although well established in \mathbb{R}^n [3], these weighted results do not apply in the specific hypothesis of periodicity. This kind of problem appears when studying singularly perturbed domains (roughness, sieves, porous media, etc) : when zooming on a single perturbation pattern, one often ends with a periodic problem set on an infinite strip. We present a unified framework that enables a systematic treatment of such problems. We provide existence and uniqueness of solutions in our WSS. This gives a refined description of solution's behavior at infinity which is of importance in the multi-scale context. The isomorphisms are valid for any relative integer m , any p in $(1, \infty)$, and any real α out of a countable set of critical values.

Keywords: periodic infinite strip; weighted Sobolev spaces; Hardy inequality; isomorphisms; Laplace operator; Stokes equations; Green function; boundary layers;

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1 Introduction

In [9, 24, 25], the first author studied blood-flow in stented arteries using homogenization techniques. Blood flows were modeled through Laplace or Stokes equations (which is plausible since the Reynolds number in arteries is relatively small) and the stent device was introduced as a thin singular perturbation between separated domains. Asymptotic expansions separate scales and lead to microscopic boundary layer problems. These can be seen as a zoom on minimal periodic patches of the stent device, where the macroscopic effects are experienced as boundary condition at infinity. Thus the boundary layer problems are defined on infinite periodic strips with obstacles. In a great number of articles [21, 20, 1, 23, 27], the analysis of such problems is done in various ways, mostly in the Hilbertian context, out of a generic frame. The aim of this paper is to provide an adequate framework for a systematic analysis of these problems. To that end, we follow the ideas developed in [17, 3, 4] : since the operators we are investigating are linear, we first solve them in periodic strips without obstacles in order to only focus on the behavior at infinity of the solutions. Then using techniques from [4], we combine the latter results and solve the associated exterior problems. This will be done in a forthcoming paper.

Thus in this paper we consider the Stokes problem

$$\begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } \mathbb{R}^n, \\ \operatorname{div} \mathbf{u} = g & \text{in } \mathbb{R}^n, \end{cases} \quad (1)$$

where the velocity field $\mathbf{u} : \mathbb{R}^n \mapsto \mathbb{R}^n$, the pressure $\pi : \mathbb{R}^n \mapsto \mathbb{R}$ and the data $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $g : \mathbb{R}^n \mapsto \mathbb{R}$ are L_k periodic with respect to the first $n - 1$ directions, that is, for any $\mathbf{y} = (\mathbf{y}', y_n) \in \mathbb{R}^n$,

$$\mathbf{u}(\mathbf{y}' + \mathbf{L}, y_n) = \mathbf{u}(\mathbf{y}), \quad \pi(\mathbf{y}' + \mathbf{L}, y_n) = \pi(\mathbf{y}), \quad (2)$$

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and

$$\mathbf{f}(\mathbf{y}' + \mathbf{L}, y_n) = \mathbf{f}(\mathbf{y}), \quad g(\mathbf{y}' + \mathbf{L}, y_n) = g(\mathbf{y}), \quad (3)$$

where $\mathbf{L} = (L_j)_{j \in \{1, \dots, n-1\}}$ is a vector of positive real numbers.

We shall also consider the biharmonic problem:

$$-\Delta^2 u = f \quad \text{in } \mathbb{R}^n, \quad (4)$$

where, for any $\mathbf{y} \in \mathbb{R}^n$,

$$u(\mathbf{y}' + \mathbf{L}, y_n) = u(\mathbf{y}) \quad \text{and} \quad f(\mathbf{y}' + \mathbf{L}, y_n) = f(\mathbf{y}). \quad (5)$$

This problem is closely related to the Stokes problem.

In order to be able to define functional spaces on the strip $\prod_{k=1}^{k=n-1} [0, L_k] \times \mathbb{R}$, we follow the ideas developed in [22, 33]. We identify the strip $\prod_{k=1}^{k=n-1} [0, L_k] \times \mathbb{R}$ with the set $G := \prod_{k=0}^{k=n-1} (\mathbb{R}/L_k\mathbb{Z}) \times \mathbb{R}$, which together with addition as group operation and the canonical quotient topology inherited from \mathbb{R}^n yields a locally compact abelian group. The Haar measure associated to G is, modulo some normalization, the product of Lebesgue's measures associated to each direction. The Stokes problem (1)–(3) can be rephrased in G as:

$$\begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } G, \\ \operatorname{div} \mathbf{u} = g & \text{in } G. \end{cases} \quad (6)$$

Similarly the biharmonic problem (4)–(5) can be rewritten as:

$$-\Delta^2 u = f \quad \text{in } G. \quad (7)$$

Since the solutions of (6) and (7) correspond to the *microscopic* solutions of the boundary layer problems, their behavior at infinity in the y_n -direction are of importance: they provide averaged feed-backs on the macroscopic scale (see [12] and references therein). Therefore we choose to solve (6) and (7) in weighted Sobolev spaces in order to describe this behavior of the solutions and of the data. Moreover, this approach allows to consider a large type of behavior at infinity of the functions (polynomial growth or decay). The weights, when adapted to our problem, are polynomial functions at infinity and regular bounded functions in the neighborhood of the origin : they are powers of $\rho(\mathbf{y}) := (1 + y_n^2)^{1/2}$. The literature on the weighted Sobolev spaces is wide [18, 4, 16, 15, 11, 10, 7, 30, 5, 6] and deals with various types of domains. To our knowledge, this type of weights has not been applied to problems (6), (7) and (8) in the context of periodic strips.

This paper combines various tools from several fields mixing Fourier analysis on local abelian groups [22, 33], fonctionnal analysis with Muckenhoupt weights [34], weighted Sobolev spaces techniques for the whole space [3, 2, 29]. We prove these results in the framework of $W_\alpha^{m,p}(G)$ spaces as defined in [3], for $p \in (1, \infty)$ and α in \mathbb{R} out of a discrete set of critical values. The core of the paper are the isomorphisms of the Laplace operator for any relative integer m , any $p \in (1, \infty)$ any weight α : for a given data f we look for u solving

$$-\Delta u = f \quad \text{in } G. \quad (8)$$

Firstly, these results are obtained on a starting interval of weights $\alpha \in (-1/p, 1/p')$ for which we prove Theorem 4.1. Then using induction arguments higher regularity is obtained combined with a natural weight shift. Calderón-Zygmund-type results from [34] provide another set of regularity results, the weight index α being kept in the initial interval. The combination of these latter steps extends then the first isomorphisms to the general non critical case as stated in Theorem 4.3.

Composing isomorphisms of the Laplace operator, we provide similar results for the poly-harmonic operator Δ^m , m being any given positive integer. For sake of simplicity these results are shown in detail for the bi-harmonic operator.

At this stage, one solves the Stokes problem (6) : eliminating the divergence equation, the pressure is obtained using the isomorphisms of Laplace operator, then the velocity is obtained through another use of these isomorphism, the gradient of the pressure becoming part of the right hand side in the first equation of (6). The biharmonic operator is used in order to characterize the kernel of the Stokes operator.

The structure of the paper follows lines explained above. First in Section 2, we define the differentiable structure on our locally compact abelian group. Then in Section 3, we set up the weighted function spaces and establish their basic properties. In Section 4, we construct isomorphisms of the Laplace operator, in Sections 5 and 6 we then use the latter results in order to provide isomorphisms for the biharmonic and the Stokes operators. Finally, in the appendix, we provide some results on distributions with compact support defined on the Pontryagin's dual group \hat{G} .

2 Fourier analysis on the LCA-group G

As in [34], we briefly define the structure inherent to periodic strips. We denote Ξ_G the canonical quotient mapping

$$\Xi_G : \mathbb{R}^n \rightarrow G, \quad \Xi_G(\mathbf{y}) := ([y_1], \dots, [y_{n-1}], y_n), \quad (9)$$

where $[y_j] \in \mathbb{R}/L_j\mathbb{Z}$ is the equivalence class of $y_j \in \mathbb{R}$, for $j \in \{1, \dots, n-1\}$.

Let $\boldsymbol{\lambda} \in \mathbb{N}^n$, be a multi-index an n -uple of nonnegative integers λ_j , we denote $D_{y_j} = \partial_j$ the partial derivative wrt j -coordinate, then $D^{\boldsymbol{\lambda}} := \prod_{j=1}^n D_{y_j}^{\lambda_j}$ denotes the differential operator of order $|\boldsymbol{\lambda}| = \sum_{j=1}^n \lambda_j$. For $m \in \mathbb{N} \cup \{\infty\}$ let us define

$$C^m(G) := \{u : G \rightarrow \mathbb{C}, \text{ such that } \exists \tilde{u} \in C^m(\mathbb{R}^n) \text{ and } \tilde{u} = u \circ \Xi_G\}$$

the space of m -times differentiable functions on G . We define the derivatives via

$$D^{\boldsymbol{\lambda}} u = D^{\boldsymbol{\lambda}} \tilde{u}|_{\prod_{k=1}^{n-1} [0, L_k) \times \mathbb{R}},$$

where $\boldsymbol{\lambda} \in \mathbb{N}^n$ with $|\boldsymbol{\lambda}| \leq m$ and we identified G with $\prod_{k=1}^{n-1} [0, L_k) \times \mathbb{R}$. We define $C(G)$ the space of $C^0(G)$ functions that vanish when $|y_n| \rightarrow \infty$ and $\mathcal{D}(G) := C_0^\infty(G)$, the set of infinitely many differentiable functions with compact support. The Schwartz-Bruhat space associated to G reads

$$\mathcal{S}(G) := \{u \in C^\infty(G) \text{ such that } n_j(u) < \infty, \quad \forall j \in \mathbb{N}^*\}$$

where we set the semi-norm $n_j(u) := \sup_{\mathbf{y} \in G} (1 + |y_n|^2)^{j/2} |\mathbf{D}^j(u)|$ and $\mathbf{D}^j u := (D^{\boldsymbol{\lambda}} u)_{|\boldsymbol{\lambda}| \leq j}$ (see [36]). The corresponding dual is well defined with respect to the weak-* topology and denotes $\mathcal{S}'(G)$, the space of tempered distributions on G . This allows to define the derivatives of elements of $\mathcal{S}'(G)$ via the primal space :

$$\langle D^{\boldsymbol{\lambda}} T, u \rangle = (-1)^{|\boldsymbol{\lambda}|} \langle T, D^{\boldsymbol{\lambda}} u \rangle, \quad u \in \mathcal{S}(G), \boldsymbol{\lambda} \in \mathbb{N}^n$$

and proves that they are also tempered distributions. The Pontryagin dual of G is identified with $\hat{G} := \prod_{k=1}^{n-1} (2\pi/L_k\mathbb{Z}) \times \mathbb{R}$. It is also a locally compact abelian group see [32, Theorem 1.2.6]. By default \hat{G} is equipped with the compact-open topology which in this case coincides with the product of discrete topologies on each $2\pi/L_k\mathbb{Z}$ and the Euclidian topology on \mathbb{R} . The Haar measure on \hat{G} is simply the product of counting measures on $2\pi/L_k\mathbb{Z}$ for each $k \in \{1, \dots, n-1\}$ and Lebesgue's measure on \mathbb{R} . The associated differentiable structure similar to the group G is then

$$C^\infty(\hat{G}) := \left\{ u \in \hat{G} \rightarrow \mathbb{C} \text{ such that } u(\mathbf{k}, \cdot) \in C^\infty(\mathbb{R}), \forall \mathbf{k} \in \prod_{\ell=1}^{\ell=n-1} (2\pi\mathbb{Z}/L_\ell) \right\}$$

where we define the derivatives $D^\lambda u(\mathbf{k}, \zeta) = D_{\zeta^\lambda}^\lambda u(\mathbf{k}, \zeta)$ together with the norms $\hat{\mathbf{n}}_j(u) := \sup_{\boldsymbol{\eta} \in \hat{G}} (1 + |\boldsymbol{\eta}|)^j |\mathbf{D}^j u(\boldsymbol{\eta})|$ where $\mathbf{D}^j u := (D_{\xi^\lambda}^\lambda u)_{\lambda \leq j}$. The Schwartz space on \hat{G} then reads

$$\mathcal{S}(\hat{G}) := \{\hat{\mathbf{A}}u \in C^\infty(\hat{G}) \text{ s.t. } \hat{\mathbf{n}}_j(u) < \infty, \quad \forall j \in \mathbb{N}^*\}$$

and again this provides the correct setting in order to define the associated dual space. For more details see [22, 34] and references therein. By the same arguments as in [19, Theorem 1.5.2], one can prove that $\mathcal{E}'(\hat{G})$, the set of tempered distributions with compact support in \hat{G} is identical to the dual space of $C^\infty(\hat{G})$ with the topology defined by the semi norms $\hat{\mathbf{m}}_{j,K}(u) := \sup_{\boldsymbol{\eta} \in K} |\mathbf{D}^j u(\boldsymbol{\eta})|$ where K ranges over all compact sets of \hat{G} , and j over all non-negative integers.

We are in the position to define the Fourier transform $\mathcal{F}_G : L^1(G) \rightarrow C(\hat{G})$ as

$$\mathcal{F}_G(u)(\mathbf{k}, \xi) := \frac{1}{|\Sigma|} \int_{\Sigma} \int_{\mathbb{R}} u(\mathbf{y}', y_n) \exp(-i\mathbf{y}' \cdot \mathbf{k} - iy_n \xi) dy_n d\mathbf{y}'$$

where we denote $\Sigma := \prod_{\ell=1}^{n-1} (0, L_\ell)$ and $|\Sigma|$ its Lebesgue measure. Several remarks are to be made, \mathcal{F}_G maps $L^1(G)$ into $C(\hat{G})$ by the Riemann-Lebesgue lemma. One proves [13] that $\mathcal{F}_G : \mathcal{S}(G) \rightarrow \mathcal{S}(\hat{G})$ is a homeomorphism. The corresponding inverse Fourier transform reads $\mathcal{F}_G^{-1} : L^1(\hat{G}) \rightarrow C(G)$ where

$$\mathcal{F}_G^{-1}(u)(\mathbf{y}', y_n) := \sum_{\mathbf{k} \in \prod_{k=1}^{n-1} (2\pi/L_k \mathbb{Z})} \int_{\mathbb{R}} u(\mathbf{k}, \xi) \exp(i\mathbf{y}' \cdot \mathbf{k} + iy_n \xi) dy_n d\mathbf{y}'.$$

By the Pontryagin duality theorem, there exists also a Fourier transform $\mathcal{F}_{\hat{G}} : \mathcal{S}(\hat{G}) \rightarrow \mathcal{S}(G)$, which again has an inverse denoted $\mathcal{F}_{\hat{G}}^{-1} : \mathcal{S}(G) \rightarrow \mathcal{S}(\hat{G})$. We have the correspondence $\mathcal{F}_{\hat{G}}^{-1}(\mathcal{F}_G(f))(\mathbf{y}) = f(\mathbf{y}) = \mathcal{F}_{\hat{G}}(\mathcal{F}_G(f))(-\mathbf{y})$ for all $\mathbf{y} \in G$ and every $f \in \mathcal{S}(G)$. One is able to define the Fourier and the inverse Fourier transforms of a tempered distribution by duality $\mathcal{F}_G : \mathcal{S}'(G) \rightarrow \mathcal{S}'(\hat{G})$ and $\mathcal{F}_{\hat{G}}^{-1} : \mathcal{S}'(\hat{G}) \rightarrow \mathcal{S}'(G)$ as

$$\langle \mathcal{F}_G(T), \varphi \rangle = \langle T, \mathcal{F}_{\hat{G}}(\varphi) \rangle, \quad \forall \varphi \in \mathcal{S}(\hat{G})$$

and

$$\langle \mathcal{F}_{\hat{G}}^{-1}(T), \varphi \rangle = \langle T, \mathcal{F}_{\hat{G}}^{-1}(\varphi) \rangle, \quad \forall \varphi \in \mathcal{S}(G).$$

For all $T \in \mathcal{S}'(G)$ there is a correspondence between derivation and multiplication in the frequency domain : $\mathcal{F}_G(D^\lambda T) = i^{|\lambda|} \boldsymbol{\eta}^\lambda \mathcal{F}_G(T)$ for any $\lambda \in \mathbb{N}^n$ and $\boldsymbol{\eta} = (\mathbf{k}, \xi) \in \hat{G}$.

3 Properties of the weighted Sobolev spaces

We define the weight function with respect to the normal coordinate $\rho(\mathbf{y}) := \sqrt{1 + y_n^2}$.

The sequence $(U_k)_{k \in \mathbb{Z}}$ defined as

$$U_k := \prod_{j=1}^{j=n-1} ([0, r_{k,j}) \cup (L_j - r_{k,j}, L_j]) \times (-2^k, 2^k), \quad r_{k,j} := \min(2^k, L_j),$$

together with $A = 2$ and $\theta(k) = k + 1$, forms a local base of $0 \in G$ i.e.

- i) $\cup_{k \in \mathbb{Z}} U_k = G$
- ii) $U_k \subset U_{k'}$ if $k < k'$
- iii) there exist a positive constant A and a mapping $\theta : \mathbb{Z} \rightarrow \mathbb{Z}$ such that for all $k \in \mathbb{Z}$ and all $\mathbf{x} \in G$, $k < \theta(k)$, $U_k - U_k \in U_{\theta(k)}$, and $\mu(\mathbf{x} + U_{\theta(k)}) < A\mu(\mathbf{x} + U_k)$. The set difference stems for $U - V := \{z \in G; \exists u \in U \text{ and } v \in V, z = u - v\}$.

This proposition shows that the structure of G is compatible with Assumptions 1.1 in [33]. Let $p \in (1, \infty)$. A weight $w \in L^1_{\text{loc}}(G)$ belongs to the Muckenhoupt class $A_p(G)$ iff

$$\mathcal{A}_p(w) := \sup_{U \in G} \left(\frac{1}{\mu(U)} \int_U w d\mu \right) \left(\frac{1}{\mu(U)} \int_U w^{-\frac{p'}{p}} d\mu \right)^{\frac{p}{p'}} < \infty$$

where the supremum runs over all base sets $U \in G$.

For any $p \in (1, \infty)$ and any real number $\alpha \in (-1/p, 1/p')$, the weight $\rho^{\alpha p}(\mathbf{y})$ is of Muckenhoupt type $A_p(G)$.

Proof. As the weight ρ is constant wrt \mathbf{y}' , the Muckenhoupt criterion reduces to check whether $\rho^{\alpha p}(y_n)$ is in $A_p(\mathbb{R})$, i.e. :

$$\mathcal{A}_p(\rho^{\alpha p}) := \frac{1}{R} \int_{|y_n| \leq R} \rho(y_n)^{\alpha p} dy_n \left(\frac{1}{R} \int_{|y_n| \leq R} \rho(y_n)^{-\alpha p'} dy_n \right)^{p/p'} < \infty$$

for $R := 2^k > 0$ for any $k \in \mathbb{Z}$. If $\alpha > -1/p$ then the first term of the product behaves as $R^{\alpha p}$, whereas as $\alpha < 1/p'$ the second term is equivalent to $(R^{-\alpha p'})^{p/p'}$. Thus the product of both terms is bounded for $\alpha \in (-1/p, 1/p')$. \square

For $\alpha \in \mathbb{R}$, we introduce the weighted space :

$$L^p_\alpha(G) \equiv W_\alpha^{0,p}(G) := \{u \in L^p_{\text{loc}}(G) \text{ s.t. } \rho^\alpha u \in L^p(G)\},$$

which is a Banach space equipped with its natural norm

$$\|u\|_{L^p_\alpha(G)} = \|\rho^\alpha u\|_{L^p(G)}.$$

Proposition 3 allows to use [34, Lemma 3.1] and claim that $\mathcal{D}(G)$ is dense in $L^p_\alpha(G)$ for $\alpha \in (-1/p, 1/p')$. This yields the general density result stated below. For any real number α , $\mathcal{D}(G)$ is dense in $L^p_\alpha(G)$.

Proof. The proof is obvious considering that for any α , the mapping $u \in L^p_\alpha(G) \mapsto \rho^\gamma u \in L^p_{\alpha-\gamma}(G)$ is an isomorphism for any real γ . Then using the previous density result established in the range $\alpha \in (-1/p, 1/p')$, the claim is proved. \square

For any integer $m > 0$, any $p \in (1, \infty)$ and any real number α , we now define the weighted Sobolev space

$$W_\alpha^{m,p}(G) := \left\{ u \in \mathcal{S}'(G); \forall \boldsymbol{\lambda} \in \mathbb{N}^n : 0 \leq |\boldsymbol{\lambda}| \leq m, \rho^{\alpha-m+|\boldsymbol{\lambda}|} D^{\boldsymbol{\lambda}} u \in L^p(G) \right\},$$

which is a Banach space endowed with its natural norm

$$\|u\|_{W_\alpha^{m,p}(G)} = \left(\sum_{0 \leq |\boldsymbol{\lambda}| \leq m} \|\rho^{\alpha-m+|\boldsymbol{\lambda}|} D^{\boldsymbol{\lambda}} u\|_{L^p(G)}^p \right)^{1/p}.$$

We define the semi-norm

$$|u|_{W_\alpha^{m,p}(G)} = \left(\sum_{|\boldsymbol{\lambda}|=m} \|\rho^\alpha D^{\boldsymbol{\lambda}} u\|_{L^p(G)}^p \right)^{1/p}.$$

For detailed studies on this functional space defined in the whole \mathbb{R}^n , the reader can refer to [3, 17, 18]. Among several properties, we have the following algebraic and topological inclusions :

$$W_\alpha^{m,p}(G) \subset W_{\alpha-1}^{m-1,p}(G) \cdots \subset L^p_{\alpha-m}(G). \quad (10)$$

The mapping

$$u \in W_\alpha^{m,p}(G) \mapsto D^\lambda u \in W_\alpha^{m-|\lambda|,p}(G) \quad (11)$$

is continuous for $\lambda \in \mathbb{N}^n$. Using the fact that for any $\lambda, \gamma \in \mathbb{R}$,

$$|D_{y_n}^\lambda \rho^\gamma| \leq C \rho^{\gamma-\lambda}, \quad (12)$$

the mapping

$$u \in W_\alpha^{m,p}(G) \mapsto \rho^\gamma u \in W_{\alpha-\gamma}^{m,p}(G) \quad (13)$$

is an isomorphism for any $\alpha, \gamma \in \mathbb{R}$.

For any $(m, p, \alpha) \in \mathbb{N} \times (1, \infty) \times \mathbb{R}$, $\mathcal{D}(G)$ is dense in $W_\alpha^{m,p}(G)$.

Proof. The ideas of the proof come from [3] and [28]. Let u be in $W_\alpha^{m,p}(G)$.

- (i) We first approximate u by functions with compact support in the y_n -direction. As this is a standard procedure, we refer to [18] p. 230–231 for further details.
- (ii) We define ω , the y' -periodic transform of a test function $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$\omega\varphi(\mathbf{y}) := \sum_{\mathbf{k} \in \prod_{\ell=1}^{n-1} L_\ell \mathbb{Z}} \varphi(y_1 - k_1, \dots, y_{n-1} - k_{n-1}, y_n), \quad \forall \mathbf{y} \in G.$$

A simple computation shows that for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\|\omega\varphi\|_{L_\alpha^p(G)} \leq c(|\text{supp } \varphi|) \|\varphi\|_{L_\alpha^p(\mathbb{R}^n)}. \quad (14)$$

There exists a function $\theta \in \mathcal{D}(\mathbb{R}^{n-1})$ such that $\omega\theta = 1$. Indeed, for any non-negative function $\varphi \in \mathcal{D}(\mathbb{R}^{n-1})$ whose support contains at least one period Σ , one can choose $\theta := \varphi / (\omega\varphi)$. Such a function is called a y' -periodic partition of unity. Let $(\alpha_j)_{j \in \mathbb{N}}$ be a sequence such that $\alpha_j \in \mathcal{D}(\mathbb{R}^n)$, $\alpha_j \geq 0$, $\int_{\mathbb{R}^n} \alpha_j(\mathbf{x}) d\mathbf{x} = 1$ and the support of α_j is included in the closed ball of radius $r_j > 0$ and centered at 0 where $r_j \rightarrow 0$ as $j \rightarrow \infty$. It is well known that as $j \rightarrow \infty$, α_j converges in the distributional sense to the Dirac measure. We set $w(\mathbf{y}) = \theta(y') \tilde{u}(\mathbf{y})$, where \tilde{u} denotes the periodic extension of u in \mathbb{R}^n . Then w belongs to $W_\alpha^{m,p}(\mathbb{R}^n)$ and has a compact support. Moreover, since $\overline{\omega\theta} = 1$, we have $\omega w = \omega(\theta \tilde{u}) = (\omega\theta)u = u$. We define $\varphi_j = w * \alpha_j$. Then φ_j belongs to $\mathcal{D}(\mathbb{R}^n)$ and converges to w in $W_\alpha^{m,p}(\mathbb{R}^n)$ as j tends to ∞ . Let $\psi_j := \overline{\omega\varphi_j}$, then ψ_j belongs to $\mathcal{D}(G)$ and thanks to (14), ψ_j converges to $\overline{\omega} w = u$ in $W_\alpha^{m,p}(G)$ as j tends to ∞ . □

If $j \in \mathbb{Z}$ then we denote by \mathcal{P}_j the set of polynomials of degree less or equal to j that only depend on the y_n -direction, with the convention that if $j < 0$ then $\mathcal{P}_j = \{0\}$. For any integer $m \geq 1$, we denote by $\mathcal{P}_j^{\Delta^m}$ the set of polyharmonic polynomials of order m of \mathcal{P}_j . In particular, \mathcal{P}_j^Δ is the space of harmonic polynomials of \mathcal{P}_j .

Observe that since the polynomials of \mathcal{P}_j only depend on the y_n direction, then the degree of the polynomials of $\mathcal{P}_j^{\Delta^m}$ can not exceed $2m - 1$. In other words, for any integer $m \geq 1$, we have

$$\mathcal{P}_j^{\Delta^m} = \mathcal{P}_{\min(j, 2m-1)}.$$

Let $m \geq 0$ be an integer, $p \in (1, \infty)$ and α be real numbers. We introduce the integer

$$q(m, p, \alpha) = \begin{cases} \lfloor m - 1/p - \alpha \rfloor & \text{if } 1/p + \alpha \notin \mathbb{Z}, \\ m - 1/p - \alpha - 1 & \text{otherwise,} \end{cases}$$

where $\lfloor \cdot \rfloor$ denotes the floor part of its argument. Then $\mathcal{P}_{q(m,p,\alpha)}$ is the biggest set of polynomials included in $W_\alpha^{m,p}(G)$,

For any $\alpha \in \mathbb{R}$ and $p \in (1, \infty)$, one has the continuous embedding

$$\mathcal{S}(G) \hookrightarrow L_\alpha^p(G) \hookrightarrow \mathcal{S}'(G).$$

Moreover we identify the distribution with its pointwise $L_\alpha^p(G)$ associated function in order to define its action on Schwartz test functions.

Proof. If $\alpha < -1/p$ then a simple application of Hölder inequalities leads to

$$\|\varphi\|_{L_\alpha^p(G)}^p \leq c \sup_{\mathbf{y} \in G} |\varphi(\mathbf{y})| \leq c n_0(\varphi)$$

which proves continuous injection in these cases. Otherwise, if $\alpha \geq -1/p$, then for any $\varphi \in \mathcal{S}(G)$, we can write

$$\begin{aligned} \|\varphi\|_{L_\alpha^p(G)}^p &= \int_G |\varphi(\mathbf{y})|^p \rho^{\alpha p}(\mathbf{y}) d\mathbf{y} \leq \sup_{\mathbf{y} \in G} (1 + \rho(\mathbf{y}))^{jp} |\varphi(\mathbf{y})|^p \int_G \rho^{\alpha p - jp}(\mathbf{y}) d\mathbf{y} \\ &\leq c n_j(\varphi)^p \int_G \rho^{\alpha p - jp}(\mathbf{y}) d\mathbf{y} \end{aligned}$$

the latter integral is finite as soon as $j := \lfloor \alpha + 1/p \rfloor + 1 > \alpha + 1/p$, which ends the proof of the first continuous injections for all $\alpha \in \mathbb{R}$. Then using these injections one has directly that

$$\left| \int_G u(\mathbf{y}) \varphi(\mathbf{y}) d\mathbf{y} \right| \leq \|u\|_{L_\alpha^p(G)} \|\varphi\|_{L_{-\alpha}^{p'}(G)} \leq c \|u\|_{L_\alpha^p(G)} n_j(\varphi)$$

for a certain j to be chosen accordingly. Thus u is a linear continuous form on $\mathcal{S}(G)$ and the latter continuous injection is proved. \square

Corollary .1. *Let $(m, p, \alpha) \in \mathbb{N} \times (1, \infty) \times \mathbb{R}$ then*

$$\mathcal{S}(G) \hookrightarrow W_\alpha^{m,p}(G) \hookrightarrow \mathcal{S}'(G).$$

Let $\alpha \in \mathbb{R}$, m be a negative integer, and $p \in (1, \infty)$, then the Sobolev space $W_\alpha^{m,p}(G)$ is defined as the dual space of $W_{-\alpha}^{-m,p'}(G)$, where p' is the conjugate of p .

Corollary .2. *Let $\alpha \in \mathbb{R}$, m be a negative integer, and $p \in (1, \infty)$, then elements of Sobolev spaces $W_\alpha^{m,p}(G)$ are tempered distributions.*

Theorem 3.1. *Let $m \geq 1$ be an integer, $p \in (1, \infty)$ and α be real numbers such that $1/p + \alpha \notin \{1, \dots, m\}$. Then there exists a constant $C > 0$, depending only on m, p and n such that*

$$\forall u \in W_\alpha^{m,p}(G), \quad \|u\|_{W_\alpha^{m,p}(G) / \mathcal{D}_{q'(m,p,\alpha)}} \leq C \|u\|_{W_\alpha^{m,p}(G)}$$

where $q'(m, p, \alpha) = \min(q(m, p, \alpha), m-1)$. In other words, the semi-norm $|\cdot|_{W_\alpha^{m,p}(G)}$ defines a norm on $W_\alpha^{m,p}(G) / \mathcal{D}_{q'(m,p,\alpha)}$ that is equivalent to the quotient norm.

The proof follows the same lines as in [3, Theorem 8.3 p 598]. As a direct consequence of Theorem 3.1, we can prove isomorphism results on the gradient and the divergence operator. To that end, let us define the spaces

$$\mathcal{V} := \{\boldsymbol{\varphi} \in \mathcal{D}(G)^n, \operatorname{div} \boldsymbol{\varphi} = 0\} \quad \text{and} \quad H_{p,\alpha} := \{\mathbf{v} \in L_\alpha^p(G)^n, \operatorname{div} \mathbf{v} = 0\}.$$

Given a Banach space B and a closed subspace X of B , we denote by $B' \perp X$ (or more simply X^\perp , if there is no ambiguity as to the duality product), the subspace of B' orthogonal to X , i.e.

$$B' \perp X = X^\perp = \{f \in B' : \forall v \in X, \langle f, v \rangle = 0\} = (B/X)'$$

the space X^\perp being also called the polar space of X in B' and possibly denoted X^0 . Let $p \in (1, \infty)$ and α be real numbers such that $1/p + \alpha \neq 1$. Then the operators defined by

$$\nabla : W_\alpha^{1,p}(G) / \mathcal{P}_{q'(1,p,\alpha)} \rightarrow L_\alpha^p(G)^n \perp H_{p',-\alpha} \quad (15)$$

and

$$\operatorname{div} : L_\alpha^p(G)^n / H_{p,\alpha} \rightarrow W_\alpha^{-1,p}(G) \perp \mathcal{P}_{q'(1,p',-\alpha)} \quad (16)$$

are isomorphisms.

Proof. The gradient operator defined by (15) is linear, continuous and its kernel is reduced to $\mathcal{P}_{q'(1,p,\alpha)}$. Actually, thanks to Proposition 3, this means that if $\alpha < 1/p'$ then $q'(1,p,\alpha) = 0$ otherwise $q'(1,p,\alpha) < 0$ which implies that $\mathcal{P}_{q'(1,p,\alpha)} = \{0\}$. Since $1/p + \alpha \neq 1$, then thanks to Theorem 3.1, the semi-norm $|\cdot|_{W_\alpha^{1,p}(G)}$ is a norm on the quotient space $W_\alpha^{1,p}(G) / \mathcal{P}_{q'(1,p,\alpha)}$. Thus the gradient is an isomorphism from $W_\alpha^{1,p}(G) / \mathcal{P}_{q'(1,p,\alpha)}$ onto its range R_g and R_g is a closed subspace of $L_\alpha^p(G)$. Hence by the Closed Range Theorem $R_g = (\ker(\operatorname{div}))^0$, where div is the operator defined by

$$\operatorname{div} : L^{p'}(G) \rightarrow (W_\alpha^{1,p}(G) / \mathcal{P}_{q'(1,p,\alpha)})'.$$

We deduce that $R_g = (H_{p',-\alpha})^0$, so that the gradient operator defined by (15) is an isomorphism and through duality and transposition the divergence operator defined by (16) is also an isomorphism. \square

In the remaining part of this section, our aim is to characterize tempered distributions by means of their gradients.

Let $p \in [1, \infty)$ and $\alpha \in \mathbb{R}$. If $\mathbf{f} \in L_{\operatorname{loc}}^p(G)^n$ such that

$$\langle \mathbf{f}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in \mathcal{V},$$

then there exists $p \in L_{\operatorname{loc}}^p(G)$ such that $\mathbf{f} = \nabla p$.

Proof. For every \mathbf{y} in \mathbb{R}^n we associate the periodic extension of \mathbf{f} reading :

$$\tilde{\mathbf{f}}(\mathbf{y}) := \mathbf{f}(\Xi_G(\mathbf{y})), \quad \hat{\text{A}}\hat{\text{a}}.\text{e. } \mathbf{y} \in \mathbb{R}^n.$$

where the restriction operator Ξ_G was introduced in (9). As $\mathbf{f} \in L_{\operatorname{loc}}^p(G)^n$, then for any $K \subset \mathbb{R}^n$ compact set, one can write:

$$\int_K |\tilde{\mathbf{f}}(\mathbf{y})|^p d\mathbf{y} = \sum_{\mathbf{k} \in \Pi_{\ell=1}^{n-1} L_\ell \mathbb{Z}} \int_{(G + \sum_{\ell=1}^{n-1} k_\ell \mathbf{e}_\ell) \cap K} |\tilde{\mathbf{f}}(\mathbf{y})|^p d\mathbf{y} \leq C(|K|) \int_{K'} |\mathbf{f}(\mathbf{y})|^p d\mathbf{y}$$

where $K' \subset G$ is a compact set. Then $\tilde{\mathbf{f}}$ belongs to $L_{\operatorname{loc}}^p(\mathbb{R}^n)^n$ and as a consequence $\tilde{\mathbf{f}}$ also belongs to $L_{\operatorname{loc}}^1(\mathbb{R}^n)^n$. Let now $\tilde{\varphi}$ be in $\mathcal{D}(\mathbb{R}^n)^n$ and satisfy $\operatorname{div} \tilde{\varphi} = 0$ in \mathbb{R}^n . Recalling that ω is the \mathbf{y}' -periodic transform defined in Lemma 3, then $\omega \tilde{\varphi} \in \mathcal{V}$ and is a \mathbf{y}' -periodic function. Moreover one has

$$\int_{\mathbb{R}^n} \tilde{\mathbf{f}}(\mathbf{y}) \tilde{\varphi}(\mathbf{y}) d\mathbf{y} = \sum_{\mathbf{k} \in \Pi_{\ell=1}^{n-1} L_\ell \mathbb{Z}} \int_{G + \sum_{\ell=1}^{n-1} k_\ell \mathbf{e}_\ell} \tilde{\mathbf{f}}(\mathbf{y}) \tilde{\varphi}(\mathbf{y}) d\mathbf{y} = \int_G \mathbf{f}(\mathbf{y}) \omega \tilde{\varphi}(\mathbf{y}) d\mathbf{y} = 0.$$

Thanks to [14, Lemma III.1.1. p. 144] : there exists $\tilde{p} \in W_{\operatorname{loc}}^{1,1}(\mathbb{R}^n)$ s.t. $\nabla \tilde{p} = \tilde{\mathbf{f}}$ almost everywhere in \mathbb{R}^n .

Thanks to [14, Lemma II.6.1. p. 81], $\tilde{p} \in W_{\operatorname{loc}}^{1,p}(\mathbb{R}^n)$. Then defining $p := \tilde{p}|_{\Pi_{\ell=1}^{n-1}(0, L_\ell) \times \mathbb{R}}$ proves that p belongs to $L_{\operatorname{loc}}^p(G)$. \square

Hereafter we adapt to our case arguments from [26]. We define $X_{p,\alpha}$ the closure of \mathcal{V} for the $L_\alpha^p(G)$ norm. We also define $A_{p,\alpha} := X_{p',-\alpha}^\perp$, the annihilator of $X_{p',-\alpha}$. For any $p \in [1, \infty)$, and $\alpha \in \mathbb{R}$, $A_{p,\alpha} \equiv \{\nabla p \in L_\alpha^p(G)^n, p \in L_{\operatorname{loc}}^p(G)\}$

Proof. Since $\mathbf{f} \in L^p_\alpha(G)^n$, integrations by part are well defined so that if $\mathbf{f} = \nabla p$, then we can write

$$\langle \mathbf{f}, \mathbf{u} \rangle = \langle \nabla p, \mathbf{u} \rangle = -\langle p, \operatorname{div} \mathbf{u} \rangle = 0, \quad \forall \mathbf{u} \in \mathcal{V}.$$

this proves the inclusion of the gradients in the annihilator set. Thanks to the previous Lemma one has the converse claim. \square

The tempered distribution $u \in \mathcal{S}'(G)$ satisfies $\Delta u = 0$ in G if and only if $u \in \mathcal{D}_1^\Delta$.

Proof. It is clear that if $u \in \mathcal{D}_1^\Delta$, then $\Delta u = 0$. Conversely, let $u \in \mathcal{S}'(G)$ satisfy $\Delta u = 0$. Applying the Fourier transform, one has

$$|\boldsymbol{\eta}|^2 \mathcal{F}_G(u)(\boldsymbol{\eta}) = 0, \quad \forall \boldsymbol{\eta} \in \hat{G},$$

which implies that the transform is compactly supported in $\{0\} \subset \hat{G}$. Then by Theorem A.2 (see Appendix) there exists $q \in \mathbb{N}$ such that

$$\langle \mathcal{F}_G(u), \psi \rangle_{\mathcal{E}'(\hat{G}) \times \mathcal{E}(\hat{G})} = \sum_{r=0}^q c_r D_{\xi^r}^r \psi(0), \quad \forall \psi \in \mathcal{E}(\hat{G}).$$

Now we prove that $\mathcal{F}_G^{-1}(\mathcal{F}_G(u))$ is a polynomial. Indeed

$$\begin{aligned} \langle \mathcal{F}_G^{-1}(\mathcal{F}_G(u)), \varphi \rangle_{\mathcal{S}'(G) \times \mathcal{S}(G)} &= \langle \mathcal{F}_G(u), \mathcal{F}_G^{-1}(\varphi) \rangle_{\mathcal{S}'(\hat{G}) \times \mathcal{S}(\hat{G})} \\ &= \sum_{r=0}^p c_r \left(D_{\xi^r}^r (\mathcal{F}_G(\varphi)(-\boldsymbol{\eta})) \right) \Big|_{\boldsymbol{\eta}=0} = \frac{1}{|\Sigma|} \sum_{r=0}^p c_r \left(D_{\xi^r}^r \int_G \varphi(\mathbf{y}) \exp(i\boldsymbol{\eta} \cdot \mathbf{y}) d\mathbf{y} \right) \Big|_{\boldsymbol{\eta}=0} \\ &= \frac{1}{|\Sigma|} \sum_{r=0}^p c_r \int_G \varphi(\mathbf{y}) (i y_n)^r \exp(i\boldsymbol{\eta} \cdot \mathbf{y}) d\mathbf{y} \Big|_{\boldsymbol{\eta}=0} = \int_G \left\{ \frac{1}{|\Sigma|} \sum_{r=0}^p c_r (i y_n)^r \right\} \varphi(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

Since the polynomial should be harmonic and in $\mathcal{S}'(G)$ it might belong by definition to \mathcal{D}_1^Δ . \square

Let $p \in (1, \infty)$ and $\alpha \in (-1/p, 1/p')$ be real numbers. Moreover, assume that $q \in L^p_{\text{loc}}(G)$ is such that $\Delta q = 0$ and $\nabla q \in L^p_\alpha(G)^n$. Then $\nabla q = 0$.

Proof. Since $\nabla q \in L^p_\alpha(G)$ then $\nabla q \in \mathcal{S}'(G)$. As q is harmonic, ∇q is harmonic thus by Prop. 3, ∇q is either constant or proportional to y_n . Since neither y_n nor constants belong to $L^p_\alpha(G)^n$, the gradient is zero. \square

Corollary .3. *If $p \in (1, \infty)$ and $\alpha \in (-1/p, 1/p')$, then $H_{p,\alpha} \cap A_{p,\alpha} = \{0\}$.*

If $p \in (1, \infty)$ and $\alpha \in (-1/p, 1/p')$, there exists a bounded operator $\Theta_{p,\alpha}$ from $L^p_\alpha(G)^n$ onto $H_{p,\alpha}$ such that $\Theta_{p,\alpha} \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in H_{p,\alpha}$.

Proof. In a first step we give a formal picture of the rigorous proof below : inspired by Proposition 1.5 [26], we aim at solving, in the sense of distributions $-\Delta q = \operatorname{div} \mathbf{u}$, so that we use the Fourier transform and write that $\mathcal{F}_G(q) = -i\boldsymbol{\eta} \cdot \mathcal{F}_G(\mathbf{u}) / |\boldsymbol{\eta}|^2$. Formally, the Fourier transform of ∇q can be computed as :

$$\mathcal{F}_G(\nabla q) = \underline{m}(\boldsymbol{\eta}) \mathcal{F}_G(\mathbf{u}), \quad \text{with } \underline{m}(\boldsymbol{\eta}) := \frac{\boldsymbol{\eta} \otimes \boldsymbol{\eta}}{|\boldsymbol{\eta}|^2}$$

where \otimes denotes the tensorial product. But as we want to avoid circular arguments about the resolution of the Laplace problem and since we are interested, for Helmholtz decomposition issues, into the gradient of q , in what follows, \mathbf{g} will denote a rigorous computation of the formal quantity ∇q . As \underline{m} is a bounded function but is not continuous near the origin, the transference principle applies [34, 8], but after a suitable regularization.

We define the matrix :

$$\underline{m}_\dagger : \hat{G} \rightarrow \mathcal{M}_n(\mathbb{C}), \quad \underline{m}_\dagger := (1 - \underline{m}_p(\boldsymbol{\eta}))(\boldsymbol{\eta} \otimes \boldsymbol{\eta})/|\boldsymbol{\eta}|^2,$$

where $\underline{m}_p(\boldsymbol{\eta}) := \mathbb{1}_{\{0\}}(\mathbf{k})$ for all $\boldsymbol{\eta} := (\mathbf{k}, \xi) \in \hat{G}$ and we define the corresponding continuous equivalent to \underline{m}_\dagger :

$$\underline{M}_\dagger : \mathbb{R}^{n-1} \times \mathbb{R} \rightarrow \mathcal{M}_n(\mathbb{C}), \quad \underline{M}_\dagger(\boldsymbol{\zeta}) = \underline{M}_\dagger(\boldsymbol{\kappa}, \xi) := (1 - \varphi(|\boldsymbol{\kappa}|))(\boldsymbol{\zeta} \otimes \boldsymbol{\zeta})/|\boldsymbol{\zeta}|^2$$

where $\varphi \in C_0^\infty(\mathbb{R})$ and $\varphi(0) = 1$ and $\text{supp } \varphi \in (-\frac{1}{2}, \frac{1}{2})$. The symbol \underline{M}_\dagger vanishes in a neighborhood of $\boldsymbol{\kappa} = 0$, and since it is bounded and continuous, it is a Mihklin multiplier. We define P , the mean value operator wrt \mathbf{y}' and its complement :

$$(P\psi)(y_n) := \frac{1}{|\Sigma|} \int_\Sigma \psi(\mathbf{y}', y_n) d\mathbf{y}', \quad P_\dagger \psi = \psi - P\psi, \quad \forall \psi \in \mathcal{D}(G).$$

Then we set

$$\mathbf{g} = Pu_n \mathbf{e}_n + \mathcal{F}_G^{-1} \left(\underline{m}_\dagger \mathcal{F}_G(P_\dagger \mathbf{u}) \right) =: \bar{\mathbf{g}} + \mathbf{g}_\dagger$$

one has trivially that $\|\bar{\mathbf{g}}\|_{L_\alpha^p(G)} \leq \|\mathbf{u}\|_{L_\alpha^p(G)}$, and since \underline{m}_\dagger and \underline{M}_\dagger are well defined, Proposition 5.3 [34] shall be used giving :

$$\|\mathbf{g}_\dagger\|_{L_\alpha^p(G)} \leq c \|P_\dagger \mathbf{u}\|_{L_\alpha^p(G)} \leq c' \|\mathbf{u}\|_{L_\alpha^p(G)},$$

where we used the decomposition of $L_\alpha^p(G)$ provided in Lemma 5.6 [34]. At this stage, using the Fourier transform, it is easy to check that $\text{div } \mathbf{g} = \text{div } \mathbf{u}$ in $\mathcal{S}'(G)$. Indeed, one has :

$$\begin{aligned} \langle \text{div } \mathbf{g}, \psi \rangle &= - \langle \mathbf{g}, \nabla \psi \rangle = - \langle \bar{\mathbf{g}} + \mathbf{g}_\dagger, \nabla \psi \rangle \\ &= - \langle Pu_n, \partial_n \psi \rangle - \langle \underline{m}_\dagger \mathcal{F}_G(P_\dagger \mathbf{u}), \mathcal{F}_G^{-1}(\nabla \psi) \rangle \\ &= \langle \partial_n Pu_n, \psi \rangle + \langle \underline{m}_\dagger \mathcal{F}_G(P_\dagger \mathbf{u}), i \boldsymbol{\eta} \mathcal{F}_G^{-1}(\psi) \rangle \end{aligned}$$

Now since $\underline{m}_\dagger \in C^\infty(\hat{G})$, it is trivial to prove that $\langle \underline{m}_\dagger \mathcal{F}_G(P_\dagger \mathbf{u}), i \boldsymbol{\eta} \mathcal{F}_G^{-1}(\psi) \rangle = \langle \mathcal{F}_G(P_\dagger \mathbf{u}), i \underline{m}_\dagger \boldsymbol{\eta} \mathcal{F}_G^{-1}(\psi) \rangle$ and whence as $\underline{m}_\dagger \boldsymbol{\eta} = (1 - \underline{m}_p) \boldsymbol{\eta}$, which gives in turn

$$\begin{aligned} \langle \text{div } \mathbf{g}, \psi \rangle &= \langle \partial_n Pu_n, \psi \rangle + \langle (1 - \underline{m}_p) \mathcal{F}_G(P_\dagger \mathbf{u}), i \boldsymbol{\eta} \mathcal{F}_G^{-1}(\psi) \rangle \\ &= \langle \partial_n Pu_n, \psi \rangle - \langle (1 - \underline{m}_p) \mathcal{F}_G(P_\dagger \mathbf{u}), \mathcal{F}_G^{-1}(\nabla \psi) \rangle \\ &= \langle \partial_n Pu_n, \psi \rangle - \langle \mathcal{F}_G(P_\dagger \mathbf{u}), \mathcal{F}_G^{-1}(\nabla \psi) \rangle = \langle \partial_n Pu_n, \psi \rangle - \langle P_\dagger \mathbf{u}, \nabla \psi \rangle, \end{aligned}$$

where the brackets are intended in the $\mathcal{S}'(G)$ or the $\mathcal{S}'(\hat{G})$ senses, and we have used that $(1 - \underline{m}_p) \mathcal{F}_G(P_\dagger \mathbf{u}) = \mathcal{F}_G(P_\dagger \mathbf{u})$ since $\mathcal{F}_G(P_\dagger \mathbf{u})(\mathbf{k} = 0, \xi) = 0$, which is indeed the case since $\mathcal{F}_G(P_\dagger \mathbf{u}) = \mathcal{F}_G(\mathbf{u}) - \mathcal{F}_G(P\mathbf{u}) = (1 - \underline{m}_p) \mathcal{F}_G(\mathbf{u})$ (see (19) p.14 [34]). Now one defines $\Theta_{p,\alpha} \mathbf{u} = \mathbf{u} - \mathbf{g}$ and the claim follows. \square

Theorem 3.2. *If $p \in (1, \infty)$ and $\alpha \in (-1/p, 1/p')$ then*

- i) *there is a Helmholtz decomposition of $L_\alpha^p(G)^n = H_{p,\alpha} \oplus A_{p,\alpha}$*
- ii) *the space \mathcal{V} is dense in $H_{p,\alpha}$.*

Proof. i) By our construction of $\Theta_{p,\alpha}$, one has that $\mathcal{D}(G)^n \subset H_{p,\alpha} \oplus A_{p,\alpha} \subset L_\alpha^p(G)^n$, which proves that $H_{p,\alpha} \oplus A_{p,\alpha}$ is dense in $L_\alpha^p(G)^n$. As the projector $\Theta_{p,\alpha} : L_\alpha^p(G)^n \rightarrow H_{p,\alpha}(G)$ is a bounded linear operator and $H_{p,\alpha}, A_{p,\alpha}$ are closed, $H_{p,\alpha} \oplus A_{p,\alpha}$ is closed. Thus i) holds.

ii) By i), $H_{p,\alpha} = L_\alpha^p(G)^n / A_{p,\alpha}$. Hence

$$H'_{p,\alpha} = (L_\alpha^p(G)^n / A_{p,\alpha})' = A_{p,\alpha}^\perp \subset H_{p',-\alpha}.$$

The latter inclusion holds. Indeed, suppose that $\mathbf{u} \in A_{p,\alpha}^\perp$ then it belongs to $L_{-\alpha}^{p'}(G)$ and one writes :

$$\langle \operatorname{div} \mathbf{u}, \varphi \rangle_{\mathcal{S}'(G) \times \mathcal{S}(G)} = - \langle \mathbf{u}, \nabla \varphi \rangle_{L_{-\alpha}^{p'}(G) \times L_\alpha^p(G)} = 0, \quad \forall \varphi \in \mathcal{S}(G),$$

since $\varphi \in \hat{\mathcal{A}} \mathcal{S}(G)$, $\nabla \varphi \in L_\alpha^p(G)$ and thus $\nabla \varphi \in A_{p,\alpha}$ by Lemma 3. Thus $\operatorname{div} \mathbf{u} = 0$ and since $\mathbf{u} \in L_{-\alpha}^{p'}(G)$ one concludes that $\mathbf{u} \in H_{p',-\alpha}$.

The subset $X_{p',-\alpha}$ is total in $H_{p',-\alpha}$ if and only if the only continuous linear functional that vanishes on $X_{p',-\alpha}$ is the 0 functional. Thus suppose that there exists $\mathbf{w} \in (H_{p',-\alpha})'$ s.t. for any $\mathbf{v} \in X_{p',-\alpha}$, $\langle \mathbf{w}, \mathbf{v} \rangle = 0$ (where the brackets are understood in the $L_\alpha^p(G), L_{-\alpha}^{p'}(G)$ duality), this means that $\mathbf{w} \in X_{p',-\alpha}^\perp = A_{p,\alpha}$, but since $(H_{p',-\alpha})' \subset H_{p,\alpha}$ then $\mathbf{w} \in H_{p,\alpha} \cap A_{p,\alpha} = \{0\}$, which ends the proof. \square

Assume that $p \in (1, \infty)$ and $\alpha \in (-1/p, 1/p')$ are real numbers. Let $u \in \mathcal{S}'(G)$ be such that $\nabla u \in L_\alpha^p(G)^n$. Then $u \in W_\alpha^{1,p}(G)$ and there exists $c > 0$ independent on u such that

$$\|u\|_{W_\alpha^{1,p}(G)/\mathbb{R}} \leq c \|\nabla u\|_{L_\alpha^p(G)^n}.$$

Proof. If $u \in \mathcal{S}'(G)$ and $\nabla u \in L_\alpha^p(G)^n$, then for all $\varphi \in \mathcal{V}$ one has

$$\langle \nabla u, \varphi \rangle = - \langle u, \operatorname{div} \varphi \rangle = 0$$

where the brackets are to be understood in the $\mathcal{S}'(G), \mathcal{S}(G)$ -duality. The left hand side of the previous equality is nevertheless an integral as well since ∇u is in $L_\alpha^p(G)$. Since $\alpha \in (-1/p, 1/p')$, then thanks to Theorem 3.2, \mathcal{V} is dense in $H_{p',-\alpha}$ and we see that $\nabla u \in L_\alpha^p(G) \perp H_{p',-\alpha}$. Moreover, under the assumptions on α , we have $q'(1, p, \alpha) = 0$. Thus by Prop. 3, there exists $w \in W_\alpha^{1,p}(G)/\mathbb{R}$ such that $\nabla w = \nabla u$. Since these functions differ at most by a constant ([35], Theorem VI, Chap. 2) and, under the assumptions on α , constants are in $W_\alpha^{1,p}(G)$, the u belongs to the latter space and the quotient norm provides the estimates. \square

4 The Laplace operator

In this section, we solve the Laplace problem

$$-\Delta u = f \quad \text{in } G.$$

We begin by characterizing the kernel.

Let $m \geq 0$ be an integer, α be a real number. A function $u \in W_\alpha^{m,p}(G)$ satisfies $\Delta u = 0$ in G if and only if $u \in \mathcal{P}_{q(m,p,\alpha)}^\Delta$.

Proof. In view of Remark 3, it is clear that if $u \in \mathcal{P}_{q(m,p,\alpha)}^\Delta$, then $\Delta u = 0$. Conversely, if $u \in W_\alpha^{m,p}(G)$, by Corollary .1, $u \in \mathcal{S}'(G)$. Proposition 3 implies then that u is a polynomial in \mathcal{P}_1^Δ , as it is also in $W_\alpha^{m,p}(G)$, it is necessarily in $\mathcal{P}_{q(m,p,\alpha)}^\Delta$. \square

In order to establish the first isomorphism result for the Laplace operator, we compute the fundamental solution of the Laplace equation in G .

The fundamental solution of the Laplace equation in G reads:

$$\Phi(\mathbf{y}) = \sum_{\mathbf{k} \in \Pi_{\ell=1}^{n-1} \frac{2\pi}{L_\ell} \mathbb{Z}} \frac{\exp(-|\mathbf{k}||y_n| + i(\mathbf{k}, \mathbf{y}'))}{2|\mathbf{k}|} - \frac{1}{2}|y_n|,$$

Proof. The Fourier transform of the fundamental solution is: $\mathcal{F}_{\hat{G}}(\Phi)(\boldsymbol{\eta}) = 1/|\boldsymbol{\eta}|^2$ for all $\boldsymbol{\eta} \in \hat{G}$. Inverting this expression yields formally:

$$\Phi(\mathbf{y}) = \sum_{\mathbf{k} \in \Pi_{\ell=1}^{n-1} \frac{2\pi}{L_\ell} \mathbb{Z}} \int_{\mathbb{R}} \frac{\exp(i\xi y_n)}{(|\mathbf{k}|^2 + \xi^2)} d\xi \exp(i(\mathbf{k}, \mathbf{y}')).$$

Then separating the case $\mathbf{k} \equiv 0$ from the rest provides the result using residue calculus as in [31, example 1, Chap. IX, p. 58]. \square

The fundamental solution Φ is in $\mathcal{S}'(G)$.

Proof. A simple computation gives that $|y_n| \in L_{\text{loc}}^1(G)$ is a tempered distribution since for any $\varphi \in \mathcal{S}(G)$, one has

$$|\langle |y_n|, \varphi \rangle| \leq c_1 \int_G \frac{|y_n|}{(1 + |y_n|^2)^{j/2}} d\mathbf{y} n_j(\varphi)$$

the integral being bounded for $j \geq 3$. We define $\Phi_{\tilde{\mathbf{k}}}(\mathbf{y}) := \exp(-|\tilde{\mathbf{k}}||y_n| + i(\tilde{\mathbf{k}}, \mathbf{y}')) / (2|\tilde{\mathbf{k}}|)$ for any $\tilde{\mathbf{k}} \in \Pi_{\ell=1}^{n-1} \frac{2\pi}{L_\ell} \mathbb{Z}$: it is a tempered distribution. Moreover, one has that

$$\widehat{\Phi}_{\tilde{\mathbf{k}}}(\boldsymbol{\eta}) := \mathcal{F}_G(\Phi_{\tilde{\mathbf{k}}})(\mathbf{k}, \xi) = \frac{\mathbb{1}_{\{\mathbf{k}=\tilde{\mathbf{k}}\}}(\mathbf{k}) \hat{\Delta} \check{\Delta}}{|\mathbf{k}|^2 + \xi^2}.$$

A simple computation shows that $\widehat{\Phi}_{\tilde{\mathbf{k}}}$ is indeed in $L^1(\hat{G})$, and thus, below, the duality bracket in $(\mathcal{S}'(\hat{G}), \mathcal{S}(\hat{G}))$ becomes an integral with respect to the Haar measure associated to \hat{G} . Let now M be a positive real number large enough, define $Z_M := \{\tilde{\mathbf{k}} \in \Pi_{\ell=1}^{n-1} \frac{2\pi}{L_\ell} \mathbb{Z} \text{ s.t. } \tilde{\mathbf{k}} \neq 0 \text{ and } |\tilde{\mathbf{k}}| < M\}$ and

$$\Phi^M(\mathbf{y}) := \sum_{\tilde{\mathbf{k}} \in Z_M} \Phi_{\tilde{\mathbf{k}}}(\mathbf{y}).$$

Then applying the Fourier transform,

$$\widehat{\Phi}_M(\boldsymbol{\eta}) := \mathcal{F}_G(\Phi^M)(\boldsymbol{\eta}) = \sum_{\tilde{\mathbf{k}} \in Z_M} \frac{\mathbb{1}_{\{\mathbf{k}=\tilde{\mathbf{k}}\}}(\mathbf{k}) \hat{\Delta} \check{\Delta}}{|\mathbf{k}|^2 + \xi^2}.$$

The previous arguments show that $\widehat{\Phi}_M \in L^1(\hat{G})$, but the bound is not uniform with respect to M , since it behaves as $\sum_{\mathbf{k} \in Z_M} |\mathbf{k}|^{-1}$. Nevertheless for any $\tilde{\varphi} \in \mathcal{S}(\hat{G})$, we have

$$\begin{aligned} \langle \widehat{\Phi}_M, \tilde{\varphi} \rangle_{\mathcal{S}'(\hat{G}), \mathcal{S}(\hat{G})} &= \sum_{\tilde{\mathbf{k}} \in Z_M} \int_{\mathbb{R}_+} \frac{\tilde{\varphi}(\mathbf{k}, \xi) \hat{\Delta} \check{\Delta}}{|\mathbf{k}|^2 + \xi^2} d\xi \\ &\leq \hat{n}_j(\tilde{\varphi}) \sum_{\tilde{\mathbf{k}} \in Z_M} (1 + |\tilde{\mathbf{k}}|)^{-j} \int_{\mathbb{R}_+} \frac{1 \hat{\Delta} \check{\Delta}}{|\mathbf{k}|^2 + \xi^2} d\xi \leq c_2 \hat{n}_j(\tilde{\varphi}). \end{aligned}$$

For j sufficiently large with respect to n , the constant c_2 does not depend on M . This ends the proof since the Fourier transform is an homeomorphism between $\mathcal{S}'(G)$ and $\mathcal{S}'(\hat{G})$. \square

If $f \in \mathcal{S}(G)$, then $\Phi * f \in \mathcal{S}'(G)$. Moreover if $f \in \mathcal{D}(G)$ then

$$\langle \Phi * f, \varphi \rangle_{\mathcal{S}'(G) \times \mathcal{S}(G)} = \langle f, \Phi * \varphi \rangle_{\mathcal{E}'(G) \times \mathcal{E}(G)}, \quad \forall \varphi \in \mathcal{S}(G).$$

Proof. The proof of the first part is obvious and relies on the Fourier transform: indeed $\widehat{\Phi * f} = \widehat{\Phi} \widehat{f}$ and as the product $\widehat{\Phi} \widehat{f}$ is again in $\mathcal{S}'(\widehat{G})$, then so it is for the inverse transform giving the result. Moreover, one has :

$$\langle \Phi * f, \varphi \rangle = \langle \Phi * \varphi, f \rangle, \quad \forall \varphi \in \mathcal{S}(G), \quad \forall f \in \mathcal{S}(G).$$

Indeed, Φ^M , since it belongs to $L^1(G)$, this allows to convert the brackets into sums and integrals and use Fubini : one obtains the second result for any fixed M . Then passing to the limit necessitates the same arguments as in the previous Lemma. Then since f and φ are both in $\mathcal{E}(G)$ so are the respective convolutions with Φ . If now we suppose moreover that $f \in \mathcal{D}(G)$ then it is also in $\mathcal{E}'(G)$ and thanks to these two latter arguments, the result follows, and the bracket can be transformed :

$$\langle \Phi * \varphi, f \rangle_{\mathcal{S}'(G) \times \mathcal{S}(G)} = \langle f, \Phi * \varphi \rangle_{\mathcal{E}'(G) \times \mathcal{E}(G)}.$$

□

We can now state the first isomorphism result.

Theorem 4.1. *Let $p \in (1, \infty)$ and $\alpha \in (-1/p, 1/p')$ be real numbers. Then the Laplace operator defined by*

$$W_\alpha^{1,p}(G)/\mathbb{R} \xrightarrow{\Delta} W_\alpha^{-1,p}(G) \perp \mathbb{R}$$

is an isomorphism.

Proof. We adapt the ideas from [3, Theorem 5.1, p. 586–587]. The operator is clearly linear and continuous, thanks to Proposition 4, it is also injective. Hereafter we prove surjectivity. Given $f \in W_\alpha^{-1,p}(G) \perp \mathbb{R}$ an obvious candidate could be $\Phi * f$, the problem is that neither Φ nor f are not compactly supported : the latter convolution does not make sense *a priori*.

i) Thanks to Prop. 3, there exists $\mathbf{v} \in L_\alpha^p(G)$ s.t. $\operatorname{div} \mathbf{v} = f$ and

$$\|\mathbf{v}\|_{L_\alpha^p(G)} \leq c \|f\|_{W_\alpha^{-1,p}(G)}$$

where the constant is independent on \mathbf{v} . Now since $\mathcal{D}(G)$ is dense in $L_\alpha^p(G)$, there exists a sequence $\mathbf{v}_j \in \mathcal{D}(G)$ s.t. $\mathbf{v}_j \rightarrow \mathbf{v}$ in $L_\alpha^p(G)$.

ii) Set $f_j = \operatorname{div} \mathbf{v}_j$, and $\psi_j = \Phi * f_j$, for all $i = 1, \dots, n$ and $\varphi \in \mathcal{D}(G)$ we have :

$$\begin{aligned} \langle \partial_i \psi_j, \varphi \rangle_{\mathcal{S}'(G) \times \mathcal{S}(G)} &= - \langle \Phi * f_j, \partial_i \varphi \rangle_{\mathcal{S}'(G) \times \mathcal{S}(G)} \\ &= - \langle f_j, \partial_i (\Phi * \varphi) \rangle_{\mathcal{E}'(G) \times \mathcal{E}(G)} = \langle \mathbf{v}_j, \nabla \partial_i (\Phi * \varphi) \rangle_{\mathcal{E}'(G) \times \mathcal{E}(G)}, \end{aligned}$$

the change in the sense of duality brackets is justified by Lemma 4. Then thanks to Prop. 3, $\rho^{-p'\alpha}(y_2)$ is a Muckenhoupt weight, we apply the weighted Calderon-Zygmund inequality from the proof of Theorem 1.1 p. 15-16 [34], and one has

$$\begin{aligned} |\langle \partial_i \psi_j, \varphi \rangle| &\leq \|\mathbf{v}_j\|_{L_\alpha^p(G)} \|\nabla \partial_i (\Phi * \varphi)\|_{L_{-\alpha}^{p'}(G)} \\ &\leq c \|\mathbf{v}_j\|_{L_\alpha^p(G)} \|\Delta (\Phi * \varphi)\|_{L_{-\alpha}^{p'}(G)} \leq c' \|f\|_{W_\alpha^{-1,p}(G)} \|\varphi\|_{L_{-\alpha}^{p'}(G)} \end{aligned}$$

So that $\nabla \psi_j$ is uniformly bounded wrt j in $L_\alpha^p(G)$.

iii) We can apply Proposition 3, so there exists constants $(c_j(\psi_j))_{j \in \mathbb{N}}$ and $c > 0$ s.t.

$$\psi_j + c_j \in W_\alpha^{1,p}(G), \quad \|\psi_j + p_j\|_{W_\alpha^{1,p}(G)} \leq c \|f\|_{W_\alpha^{-1,p}(G)}.$$

From this it follows that $\psi_j + p_j$ converges weakly to some function u in $W_\alpha^{1,p}(G)$ and it solves (8) so that the mapping is indeed surjective. □

In order to extend Theorem 4.1 to other values of α , we first prove a regularity result.

Let ℓ be a non-negative integer, $p \in (1, \infty)$ and $\alpha \in (-1/p, 1/p')$ be real numbers. Then the Laplace operator defined by

$$W_{\alpha+\ell}^{1+\ell,p}(G)/\mathbb{R} \xrightarrow{\Delta} W_{\alpha+\ell}^{-1+\ell,p}(G) \perp \mathbb{R} \quad (17)$$

is an isomorphism.

Proof. Owing to Theorem 4.1, the claim is true for $\ell = 0$. Assume that it is true for $\ell = k$ and let us prove that it is still true for $\ell = k + 1$. The Laplace operator defined by (17) is clearly linear and continuous. It is also injective : if $u \in W_{\alpha+k+1}^{2+k,p}(G)$ and $\Delta u = 0$ then $u \in \mathcal{P}_{[1-1/p-\alpha]}^\Delta \equiv \mathbb{R}$ since $\alpha \in (-1/p, 1/p')$. To prove that it is onto, let f be given in $W_{\alpha+k+1}^{k,p}(G) \perp \mathbb{R}$. According to (10), f belongs to $W_{\alpha+k}^{-1+k,p}(G) \perp \mathbb{R}$. Then the induction assumption implies that there exists $u \in W_{\alpha+k}^{1+k,p}(G)$ such that $\Delta u = f$. Next, we have

$$\Delta(\rho \partial_i u) = \rho \partial_i f + \partial_i u \Delta \rho + 2\nabla \rho \cdot \nabla(\partial_i u). \quad (18)$$

Using (10), (11), (12), and (13), all the terms of the right-hand side belong to $W_{\alpha+k}^{-1+k,p}(G)$. This implies that $\Delta(\rho \partial_i u)$ also belongs to $W_{\alpha+k}^{-1+k,p}(G)$. In order to apply again the induction argument, it remains to prove that $\Delta(\rho \partial_i u)$ is orthogonal to constants. Let us first note that $\Delta(\rho \partial_i u)$ also belongs to $W_{\alpha+k-1}^{-2+k,p}(G)$. Moreover, since u belongs to $W_{\alpha+k}^{1+k,p}(G)$ implies that $\rho \partial_i u$ belongs $W_{\alpha+k-1}^{k,p}(G)$, then, for any $\varphi \in W_{-\alpha-k+1}^{2-k,p'}(G)$, we can write

$$\langle \Delta(\rho \partial_i u), \varphi \rangle_{W_{\alpha+k-1}^{-2+k,p}(G) \times W_{-\alpha-k+1}^{2-k,p'}(G)} = \langle \rho \partial_i u, \Delta \varphi \rangle_{W_{\alpha+k-1}^{k,p}(G) \times W_{-\alpha-k+1}^{-k,p'}(G)}.$$

Since $\mathbb{R} \subset W_{-\alpha-k+1}^{2-k,p'}(G)$, we can take $\varphi \in \mathbb{R}$ which implies that $\Delta \varphi = 0$. It follows that $\Delta(\rho \partial_i u)$ is indeed orthogonal to constants. Thanks to the induction assumption, there exists v in $W_{\alpha+k}^{1+k,p}(G)$ such that

$$\Delta v = \Delta(\rho \partial_i u).$$

Hence, the difference $v - \rho \partial_i u$ is a constant. Since the constants are in $W_{\alpha+k}^{1+k,p}(G)$, we deduce that $\rho \partial_i u$ belongs to $W_{\alpha+k}^{1+k,p}(G)$ which in turn implies that u belongs to $W_{\alpha+k+1}^{2+k,p}(G)$. □

We continue by establishing a new family of isomorphism using Calderón-Zygmund's inequality and Theorem 3.1.

Let $p \in (1, \infty)$, $\alpha \in (-1/p, 1/p')$ be real numbers and $m \geq 2$ be an integer. Then the Laplace operator defined by

$$W_\alpha^{m,p}(G)/\mathcal{P}_{m-1} \xrightarrow{\Delta} W_\alpha^{m-2,p}(G)/\mathcal{P}_{m-3}$$

is an isomorphism.

Proof. The mapping defined in the claim is bounded linear and injective. Again as in the proof of Theorem 4.1, as $\alpha \in (-1/p, 1/p')$, the weights are of Muckenhoupt type and one can apply the weighted Calderón-Zygmund estimates giving :

$$\begin{aligned} \forall u \in W_\alpha^{m,p}(G), \quad \forall \lambda \in \mathbb{N}^n : |\lambda| = m-2, \forall i, j = 1, \dots, n \\ \left\| \partial_i \partial_j D^\lambda u \right\|_{L_\alpha^p(G)} \leq c \left\| \Delta D^\lambda u \right\|_{L_\alpha^p(G)} \leq c \|\Delta u\|_{W_\alpha^{m-2,p}(G)/\mathcal{D}_{m-3}}. \end{aligned}$$

Moreover, using again the assumption on α , we have $m-1 < m-1/p - \alpha < m$. Then thanks to Theorem 3.1, the semi-norm $|\cdot|_{W_\alpha^{m,p}(G)}$ is a norm on the quotient space $W_\alpha^{m,p}(G)/\mathcal{D}_{m-1}$. This yields

$$\|u\|_{W_\alpha^{m,p}(G)/\mathcal{D}_{m-1}} \leq c \|\Delta u\|_{W_\alpha^{m-2,p}(G)/\mathcal{D}_{m-3}}$$

which shows that the range of the operator is a closed subspace of the target quotient space. The Closed Range Theorem and the injectivity of the adjoint operator imply that the range is the whole target space. \square

A direct consequence of this result is the Let $p \in (1, \infty)$, $\alpha \in (-1/p, 1/p')$ be real numbers and $m \geq 2$ be an integer. Then the Laplace operator defined by

$$W_\alpha^{m,p}(G)/\mathcal{D}_1 \xrightarrow{\Delta} W_\alpha^{m-2,p}(G)$$

is an isomorphism.

We can now extend Theorem 4.1 to other values of α .

Theorem 4.2. *Let $p \in (1, \infty)$, $\alpha \in (-1/p, 1/p')$ be real numbers and $\ell \geq 1$ be an integer. Then the Laplace operators defined by*

$$W_{\alpha-\ell}^{1,p}(G)/\mathcal{D}_1 \xrightarrow{\Delta} W_{\alpha-\ell}^{-1,p}(G) \tag{19}$$

and

$$W_{-\alpha+\ell}^{1,p'}(G) \xrightarrow{\Delta} W_{-\alpha+\ell}^{-1,p'}(G) \perp \mathcal{D}_1 \tag{20}$$

are isomorphisms.

Proof. Let us recall that due to Proposition 4, the mapping

$$W_\alpha^{m,p}(G)/\mathcal{D}_1 \xrightarrow{\Delta} W_\alpha^{m-2,p}(G)$$

is an isomorphism. Now through duality and transposition, the mapping

$$W_{-\alpha}^{-m+2,p'}(G) \xrightarrow{\Delta} W_{-\alpha}^{-m,p'}(G) \perp \mathcal{D}_1$$

is an isomorphism. Next, using the same arguments as in the proof of Proposition 4, we are able to show that for any integer $\ell \geq 1$, the mapping

$$W_{-\alpha+\ell}^{-m+2+\ell,p'}(G) \xrightarrow{\Delta} W_{-\alpha+\ell}^{-m+\ell,p'}(G) \perp \mathcal{D}_1$$

is an isomorphism. Choosing $\ell = m-1$ yields the isomorphism result defined by (20) which through duality and transposition also enables to obtain the one defined by (19). \square

Summarizing Theorems 4.1 and 4.2, we deduce

Theorem 4.3. For $p \in (1, \infty)$ and any $\alpha \in \mathbb{R}$ satisfying

$$\alpha + 1/p \notin \mathbb{Z} \quad \text{and} \quad \alpha - 1/p' \notin \mathbb{Z}, \quad (21)$$

the mapping

$$W_\alpha^{1,p}(G)/\mathcal{D}_{[1-1/p-\alpha]}^\Delta \xrightarrow{\Delta} W_\alpha^{-1,p}(G) \perp \mathcal{D}_{[1-1/p'+\alpha]}^\Delta \quad (22)$$

is an isomorphism. As a consequence, for any $m \in \mathbb{Z}$ and for any $\alpha \in \mathbb{R}$ satisfying (21), the mapping

$$W_\alpha^{m+2,p}(G)/\mathcal{D}_{[m+2-1/p-\alpha]}^\Delta \xrightarrow{\Delta} W_\alpha^{m,p}(G) \perp \mathcal{D}_{[-m-1/p'+\alpha]}^\Delta \quad (23)$$

is an isomorphism.

5 The biharmonic operator

Here we consider the polyharmonic problem: given f , we look for u solution of

$$\Delta^2 u = f \quad \text{in } G.$$

Proceeding as in Proposition 4, we can prove the following characterization of the kernel of the biharmonic operator.

Let α a real number. A function $u \in W_\alpha^{m,p}(G)$ satisfies $\Delta^2 u = 0$ in G if and only if $u \in \mathcal{D}_{[m-1/p-\alpha]}^{\Delta^2}$.

We first have the following isomorphism result.

Theorem 5.1. Let $p \in (1, \infty)$ and $\alpha \in (-1/p, 1/p')$ be real numbers. Then the biharmonic operator defined by

$$\Delta^2 : W_\alpha^{2,p}(G)/\mathcal{P}_1 \mapsto W_\alpha^{-2,p}(G) \perp \mathcal{P}_1$$

is an isomorphism.

Proof. Since $\alpha \in (-1/p, 1/p')$, applying Proposition 4 for $m = 2$, we see that the Laplace operator defined by

$$W_\alpha^{2,p}(G)/\mathcal{P}_1 \rightarrow L_\alpha^p(G) \quad (24)$$

is an isomorphism. Then using duality and transposition we also see that

$$L_{-\alpha}^{p'}(G) \rightarrow W_{-\alpha}^{-2,p'}(G) \perp \mathcal{P}_1$$

is an isomorphism. Since $\alpha \in (-1/p, 1/p')$ implies that $-\alpha \in (-1/p', 1/p)$, it follows that the Laplace operator defined by

$$L_\alpha^p(G) \rightarrow W_\alpha^{-2,p}(G) \perp \mathcal{P}_1 \quad (25)$$

is also an isomorphism. Composing (24) and (25) proves the statement. \square

Let $p \in (1, \infty)$ and α be real numbers satisfying (21). Then the biharmonic operator defined by

$$\Delta^2 : W_\alpha^{2,p}(G)/\mathcal{D}_{[2-1/p-\alpha]}^{\Delta^2} \mapsto W_\alpha^{-2,p}(G) \perp \mathcal{D}_{[2-1/p'+\alpha]}^{\Delta^2} \quad (26)$$

is an isomorphism.

Proof. Thanks to Theorem 5.1, the statement is already proved for $\alpha \in (-1/p, 1/p')$.

Assume now that $\alpha < -1/p$.

The biharmonic operator defined above is clearly linear and continuous. Its injectivity follows from Proposition 5.

Let us prove that the compatibility condition is necessary. For any $u \in W_\alpha^{2,p}(G)$ and $\varphi \in \mathcal{S}(G)$, we can write

$$\begin{aligned} |\langle \Delta^2 u, \varphi \rangle_{\mathcal{S}'(G) \times \mathcal{S}(G)}| &\leq \int_G |\Delta u| |\Delta \varphi| dy \\ &\leq \|\Delta u\|_{L_\alpha^p(G)} \|\Delta \varphi\|_{L_\alpha^{p'}(G)} \\ &\leq \|\Delta u\|_{L_\alpha^p(G)} |\varphi|_{W_\alpha^{2,p'}(G)}. \end{aligned}$$

But if $\alpha < -1/p$, then the semi-norm $|\cdot|_{W_\alpha^{2,p'}(G)}$ is a norm on $W_\alpha^{2,p'}(G)/\mathcal{P}_{[2-1/p'+\alpha]}$. Moreover if $\alpha < -1/p$ the spaces $\mathcal{P}_{[2-1/p'+\alpha]}$, $\mathcal{P}_{[2-1/p'+\alpha]}^\Delta$ and $\mathcal{P}_{[2-1/p'+\alpha]}^{\Delta^2}$ coincide. Hence Δ^2 belongs to $W_\alpha^{-2,p}(G) \perp \mathcal{P}_{[2-1/p'+\alpha]}^{\Delta^2}$.

Let now prove that the operator is onto. Take f in $W_\alpha^{-2,p}(G) \perp \mathcal{P}_{[2-1/p'+\alpha]}$. Then thanks to the isomorphism result of the Laplace operator defined by (23), there exists $v \in L_\alpha^p(G)$ such that $\Delta v = f$ in G . Since $\alpha < -1/p$, there are no polynomials in the space $\mathcal{P}_{[-1/p'+\alpha]}$, then using again (23), there exists $u \in W_\alpha^{2,p}(G)$ such that $\Delta u = v$ in G which implies that $\Delta^2 u = f$ in G . As a consequence the operator defined by (26) is an isomorphism if $\alpha < -1/p$. Then using duality and transposition it is also an isomorphism if $\alpha > 1/p'$ which ends the proof. \square

As a consequence we have the below general result.

Theorem 5.2. *Let $p \in (1, \infty)$ and α be real numbers satisfying (21) and $\ell \in \mathbb{Z}$. Then the biharmonic operator defined by*

$$\Delta^2 : W_\alpha^{2+\ell,p}(G)/\mathcal{P}_{[2+\ell-1/p-\alpha]}^{\Delta^2} \mapsto W_\alpha^{-2+\ell,p}(G) \perp \mathcal{P}_{[2-\ell-1/p'+\alpha]}^{\Delta^2}$$

is an isomorphism.

6 The Stokes equation

This section is devoted to the Stokes problem: given f and g , we look for a pair (\mathbf{u}, π) satisfying

$$\begin{cases} -\Delta \mathbf{u} + \nabla \pi = \mathbf{f} & \text{in } G, \\ \operatorname{div} \mathbf{u} = g & \text{in } G. \end{cases} \quad (27)$$

For any integer k , we introduce the space of polynomials

$$\mathcal{N}_k = \{(\boldsymbol{\theta}, \lambda) \in \mathcal{P}_k^{\Delta^2} \times \mathcal{P}_{k-1}^\Delta, \operatorname{div} \boldsymbol{\theta} = 0, -\Delta \boldsymbol{\theta} + \nabla \lambda = 0\}.$$

Let $m \geq 0$ be an integer and α a real number. A pair $(\mathbf{u}, \pi) \in W_\alpha^{m+1,p}(G) \times W_\alpha^{m,p}(G)$ satisfies

$$-\Delta \mathbf{u} + \nabla \pi = 0 \quad \text{and} \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } G, \quad (28)$$

if and only if $(\mathbf{u}, \pi) \in \mathcal{N}_{[m+1-1/p-\alpha]}$.

Proof. Let $(\mathbf{u}, \pi) \in W_\alpha^{m+1,p}(G) \times W_\alpha^{m,p}(G)$ satisfies (28). Then taking the divergence of the first equation of (28) implies that $\Delta \pi = 0$ and therefore π is a polynomial of $\mathcal{P}_{[m-1/p-\alpha]}^\Delta$. As a consequence, we have $\Delta^2 \mathbf{u} = \Delta(\nabla \pi) = 0$ which implies that \mathbf{u} belongs to $\mathcal{P}_{[m+1-1/p-\alpha]}^{\Delta^2}$. \square

We first look for solutions in $W_\alpha^{1,p}(G) \times L_\alpha^p(G)$.

Theorem 6.1. *Let α be a real number satisfying (21). Let $\mathbf{f} \in W_\alpha^{-1,p}(G)$, $g \in L_\alpha^p(G)$ satisfy the compatibility condition*

$$\forall (\boldsymbol{\theta}, \lambda) \in \mathcal{N}_{[1-1/p'+\alpha]}, \quad \langle \mathbf{f}, \boldsymbol{\theta} \rangle_{W_\alpha^{-1,p}(G) \times W_\alpha^{-1,p'}(G)} + \langle g, \lambda \rangle_{L_\alpha^p(G) \times L_\alpha^{p'}(G)} = 0. \quad (29)$$

Then the Stokes equations (27) has a unique solution $(\mathbf{u}, \pi) \in (W_\alpha^{1,p}(G) \times L_\alpha^p(G)) / \mathcal{N}_{[1-1/p-\alpha]}$.

Proof. Let us introduce the Stokes operator

$$T : (\mathbf{u}, \pi) \mapsto (-\Delta \mathbf{u} + \nabla \pi, -\operatorname{div} \mathbf{u}).$$

Then the statement amounts to prove that the operator

$$T : (W_\alpha^{1,p}(G) \times L_\alpha^p(G)) / \mathcal{N}_{[1-1/p-\alpha]} \mapsto (W_\alpha^{-1,p}(G) \times L_\alpha^p(G)) \perp \mathcal{N}_{[1-1/p'+\alpha]}$$

is an isomorphism.

We assume first $\alpha < 1/p'$.

The operator T defined above is clearly linear and continuous. Its injectivity follows from Proposition 6.

Let us show that the compatibility condition is necessary. Note that under the assumption $\alpha < 1/p'$, there are no polynomials in $L_\alpha^{p'}(G)$ and the polynomials of $W_\alpha^{-1,p'}(G)$ are at most constants. Then (29) is reduced to show that $\mathbf{f} \in W_\alpha^{-1,p}(G) \perp \mathcal{P}_{[1-1/p'+\alpha]}$. Consider now $(\mathbf{u}, \pi) \in W_\alpha^{1,p}(G) \times L_\alpha^p(G)$. Then $-\Delta \mathbf{u} + \nabla \pi \in W_\alpha^{-1,p}(G)$ and for any $\boldsymbol{\theta} \in \mathcal{P}_{[1-1/p'+\alpha]}$, we have

$$\begin{aligned} & \langle -\Delta \mathbf{u} + \nabla \pi, \boldsymbol{\theta} \rangle_{W_\alpha^{-1,p}(G) \times W_\alpha^{-1,p'}(G)} \\ &= \langle \mathbf{u}, -\Delta \boldsymbol{\theta} \rangle_{W_\alpha^{1,p}(G) \times W_\alpha^{-1,p'}(G)} - \langle \pi, \operatorname{div} \boldsymbol{\theta} \rangle_{L_\alpha^p(G) \times L_\alpha^{p'}(G)} \\ &= 0 \end{aligned}$$

since $\boldsymbol{\theta}$ is at most a constant vector. This shows that the compatibility condition is necessary.

Let us now show that the operator T is onto. Let $\mathbf{f} \in W_\alpha^{-1,p}(G)$ and $g \in L_\alpha^p(G)$. Then this implies that $\operatorname{div} \mathbf{f} + \Delta g$ belongs to $W_\alpha^{-2,p}(G)$. Since $\alpha < 1/p'$, the degree of the polynomials of $\mathcal{P}_{[2-1/p'+\alpha]}$ is at most one. Hence Δg belongs to $W_\alpha^{-2,p}(G) \perp \mathcal{P}_{[2-1/p'+\alpha]}$. Then since $\mathbf{f} \in W_\alpha^{-1,p}(G) \perp \mathcal{P}_{[1-1/p'+\alpha]}$, it follows that $\operatorname{div} \mathbf{f} + \Delta g \in W_\alpha^{-2,p}(G) \perp \mathcal{P}_{[2-1/p'+\alpha]}$. Thanks to the isomorphism result (23) of the Laplace operator, there exists $\pi \in L_\alpha^p(G)$ such that

$$\Delta \pi = \operatorname{div} \mathbf{f} + \Delta g \quad \text{in } G.$$

It follows that $\mathbf{f} - \nabla \pi \in W_\alpha^{-1,p}(G) \perp \mathcal{P}_{[1-1/p'+\alpha]}$. Thanks again to (23), there exists $\mathbf{u} \in W_\alpha^{1,p}(G)$ such that

$$-\Delta \mathbf{u} + \nabla \pi = \mathbf{f} \quad \text{in } G.$$

As a consequence $\operatorname{div} \mathbf{u} - g \in L_\alpha^p(G)$ and satisfies

$$\Delta(\operatorname{div} \mathbf{u} - g) = 0 \quad \text{in } G.$$

Therefore $\operatorname{div} \mathbf{u} - g = \lambda \in \mathcal{P}_{[-1/p-\alpha]}^\Delta$. Let $\gamma(y_n) = \int_0^{y_n} \lambda(s) ds$, then γ belongs to $\mathcal{P}_{\min(2, [1-1/p-\alpha])}^\Delta \subset \mathcal{P}_{[1-1/p-\alpha]}^{\Delta^2}$.

Let now $\boldsymbol{\theta}$ be a polynomial such that $\theta_j = 0$ for any $j = 1, \dots, n-1$ and $\theta_n = \gamma$. Then $\boldsymbol{\theta}$ belongs to $\mathcal{P}_{[1-1/p-\alpha]}^{\Delta^2}$ and satisfies $\operatorname{div} \boldsymbol{\theta} = \lambda$. Thus $(\mathbf{u} - \boldsymbol{\theta}, \pi)$ satisfies (27) and as a consequence, the operator T is onto.

Through duality the operator T is also an isomorphism if $\alpha > -1/p$ and this ends the proof. \square

Using again regularity and duality arguments, we have the following statement.

Theorem 6.2. *Let α be a real number satisfying (21) and m be an integer. Let $\mathbf{f} \in W_\alpha^{m,p}(G)$, $g \in W_\alpha^{m+1,p}(G)$ satisfy the compatibility condition*

$$\forall (\boldsymbol{\theta}, \lambda) \in \mathcal{N}_{[-m-1/p'+\alpha]}, \quad \langle \mathbf{f}, \boldsymbol{\theta} \rangle_{W_\alpha^{m,p}(G) \times W_\alpha^{-m,p'}(G)} + \langle g, \lambda \rangle_{W_\alpha^{m+1,p}(G) \times W_\alpha^{-m-1,p'}(G)} = 0.$$

Then the Stokes equations (27) has a unique solution $(\mathbf{u}, \pi) \in (W_\alpha^{m+2,p}(G) \times W_\alpha^{m+1,p}(G)) / \mathcal{N}_{[m+2-1/p-\alpha]}$.

A Compactly supported distributions, $\mathcal{E}'(\hat{G})$

Theorem A.1. *If $u \in \mathcal{E}'(\hat{G})$ of order p , then for any test function $\psi \in \mathcal{E}(\hat{G})$ s.t. $D_{\xi_j}^j \psi(\eta) = 0$ for all points η in the support of u and all $j \in \{0, \dots, p\}$, one has $\langle u, \psi \rangle = 0$, where the brackets denote the duality in $\mathcal{E}'(\hat{G}), \mathcal{E}(\hat{G})$.*

Proof. As the support of u is compact in \hat{G} there exists a finite collection of integers $J \subset \mathbb{N}$ and finite sequence of compact intervals $(K(j))_{j \in J}$ s.t. $\text{supp } u \subset \{(\mathbf{k}_j, K(j))_{j \in J}\}$, we denote the set of all $S := \{(\mathbf{k}_j)_{j \in J}\}$, a finite subset of $\Pi_{\ell=1}^{n-1}(2\pi \mathbb{Z}/L_\ell)$. Let $K_\epsilon(j) := \{\zeta \in \mathbb{R} \text{ s.t. } d(\zeta, K(j)) \leq \epsilon\}$. We define the smooth cut-off function

$$\chi_\epsilon(\mathbf{k}, \xi) := \begin{cases} \frac{1}{\epsilon} \int_{K_{2\epsilon}(j)} \varphi(\xi - \zeta) d\zeta & \text{if } \mathbf{k} \in S \\ 0 & \text{otherwise} \end{cases}$$

where $\varphi \in C_0^\infty(\mathbb{R})$ s.t. $\varphi \geq 0$, $\int_{\mathbb{R}} \varphi(\xi) d\xi = 1$ and $\text{supp } \varphi \subset B(0, 1)$. Then it is clear that $\chi_\epsilon(\mathbf{k}_j, \cdot) = 1$ in $K_\epsilon(j)$ and that $\text{supp } \chi_\epsilon(\mathbf{k}_j, \cdot) \subset K_{3\epsilon}(j)$. One has then $\langle u, \psi \rangle = \langle u, \psi \chi_\epsilon \rangle$ since $\psi(1 - \chi_\epsilon) = 0$ in a neighborhood of $\text{supp } u$. Hence

$$|\langle u, \psi \rangle| \leq C \sup_{\eta \in \hat{G}, j \leq p} \left| D_{\xi_j}^j (\psi \chi_\epsilon) \right|$$

which by arguments similar to Theorem 1.5.4 [19] is bounded uniformly by ϵ , and thus tends to 0, as $\epsilon \rightarrow 0$. Indeed, for any fixed \mathbf{k}_j using

$$|\psi(\mathbf{k}_j, \xi')| \leq \frac{1}{(p+1)!} \sup_{t \in (0, 1)} \left| D_{\xi^{p+1}}^{p+1} \psi(\mathbf{k}_j, \xi + t(\xi' - \xi)) \right| |\xi' - \xi|^{(p+1)}$$

for all $\xi \in \mathbb{R}$ and similar estimates for all derivatives of higher order, it implies that $\sup_{\xi' \in K_{3\epsilon}} |D^q \psi(\mathbf{k}_j, \xi')| \leq c_{j,q} e^{p+1-q}$. On the other hand $\sup_{\xi \in K_{3\epsilon}(j)} |D^q \chi_\epsilon(\mathbf{k}_j, \xi)| \leq c'_{j,q} \epsilon^{-q}$, which together with the Leibnitz formula gives that

$$|\langle u, \psi \rangle| \leq c \sup_{j \in J} \sup_{\xi \in K_{3\epsilon}(j)} \binom{p}{q} c_{j,q} e^{p+1-q} c'_{j,p-q} \epsilon^{-(p-q)} \leq c\epsilon.$$

□

Theorem A.2. *If $u \in \mathcal{E}'(\hat{G})$ s.t. $\text{supp } u \subset \{0\}$ then for any $\varphi \in \mathcal{E}(\hat{G})$, there exists a $p \in \mathbb{N}$ s.t. $\langle u, \varphi \rangle = \sum_{r=0}^p c_r \partial_{\xi^r}^r \varphi(0)$.*

Proof. Let p be the order of u . Set $\psi(\mathbf{k}, \xi) = \varphi(\mathbf{k}, \xi) - \sum_{r=0}^p \partial_{\xi^r}^r \varphi(\mathbf{k}, 0) \frac{\xi^r}{r!}$ it is a smooth function s.t. any derivative in direction ξ vanishes up to order p . Since u is compactly supported, one defines $\chi(\mathbf{k}, \xi) = \mathbb{1}_{\mathbf{k}=0}$ which is in $C^\infty(\hat{G})$. Applying the previous result it comes $\langle u, \psi \rangle = \langle u, \chi \psi \rangle = 0$. Then using that u is linear, one recovers :

$$\langle u, \varphi \rangle = \langle u, \chi_{\mathbf{k}=0} \sum_{r=0}^p \partial_{\xi^r}^r \varphi(\mathbf{k}, 0) \frac{\xi^r}{r!} \rangle = \sum_{r=0}^p \langle u, \chi_{\mathbf{k}=0} \frac{\xi^r}{r!} \rangle \partial_{\xi^r}^r \varphi(0),$$

which gives the definition of the constants $(c_r)_{r \in \{0, \dots, p\}}$. □

References

- [1] Y. Achdou, P. Le Tallec, F. Valentin, and O. Pironneau. Constructing wall laws with domain decomposition or asymptotic expansion techniques. *Comput. Methods Appl. Mech. Eng.*, 151(1-2):215–232, 1998.
- [2] F. Alliot. *Étude des équations stationnaires de Stokes et Navier-Stokes dans des domaines extérieurs*. PhD thesis, CERMICS, École des Ponts ParisTech, 1998.

- [3] C. Amrouche, V. Girault, and J. Giroire. Weighted sobolev spaces and laplace's equation in \mathbb{R}^n . *Journal des Mathematiques Pures et Appliquees*, 73:579–606, January 1994.
- [4] C. Amrouche, V. Girault, and J. Giroire. Dirichlet and neumann exterior problems for the n-dimensional Laplace operator an approach in weighted sobolev spaces. *Journal des Mathematiques Pures et Appliquees*, 76:55–81(27), January 1997.
- [5] Ch. Amrouche and U. Razafison. The stationary Oseen equations in \mathbb{R}^3 . An approach in weighted Sobolev spaces. *J. Math. Fluid Mech.*, 9(2):211–225, 2007.
- [6] Ch. Amrouche and U. Razafison. Weighted Sobolev spaces for a scalar model of the stationary Oseen equations in \mathbb{R}^3 . *J. Math. Fluid Mech.*, 9(2):181–210, 2007.
- [7] Ch. Amrouche and U. Razafison. Isotropically and anisotropically weighted Sobolev spaces for the Oseen equation. In *Advances in mathematical fluid mechanics*, pages 1–24. Springer, Berlin, 2010.
- [8] K. F. Andersen and P. Mohanty. Restriction and extension of Fourier multipliers between weighted L^p spaces on \mathbb{R}^n and \mathbb{T}^n . *Proc. Amer. Math. Soc.*, 137(5):1689–1697, 2009.
- [9] E. Bonnetier, D. Bresch, and V. Milisic. High order multi-scale wall laws : part ii, the non periodic case. 2009. *Advances in Mathematical Fluid Dynamics*.
- [10] T. Z. Boulmezaoud. On the Stokes system and on the biharmonic equation in the half-space: an approach via weighted Sobolev spaces. *Math. Methods Appl. Sci.*, 25(5):373–398, 2002.
- [11] T. Z. Boulmezaoud. On the Laplace operator and on the vector potential problems in the half-space: an approach using weighted spaces. *Math. Methods Appl. Sci.*, 26(8):633–669, 2003.
- [12] D. Bresch and V. Milisic. High order multi-scale wall-laws, Part I: the periodic case. *Quart. Appl. Math.*, 68(2):229–253, 2010.
- [13] F. Bruhat. Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes p -adiques. *Bulletin de la Société Mathématique de France*, 89:43–75, 1961.
- [14] G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations*. Springer Monographs in Mathematics. Springer, New York, second edition, 2011. Steady-state problems.
- [15] V. Girault. The gradient, divergence, curl and Stokes operators in weighted Sobolev spaces of \mathbb{R}^3 . *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 39(2):279–307, 1992.
- [16] V. Girault. The Stokes problem and vector potential operator in three-dimensional exterior domains: an approach in weighted Sobolev spaces. *Differential Integral Equations*, 7(2):535–570, 1994.
- [17] J. Giroire. Étude de quelques problèmes aux limites extérieurs et résolution par équations intégrales. *Thèse de Doctorat d'État, Université Pierre et Marie Curie-ParisVI*, 1987.
- [18] B. Hanouzet. Espaces de Sobolev avec poids application au problème de Dirichlet dans un demi espace. *Rend. Sem. Mat. Univ. Padova*, 46:227–272, 1971.
- [19] L. Hörmander. *Linear partial differential operators*. Springer Verlag, Berlin-New York, 1976.
- [20] W. Jäger and A. Mikelić. On the boundary conditions at the contact interface between a porous medium and a free fluid. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 23(3):403–465, 1996.

- [21] W. Jäger and A. Mikelić. Couette flows over a rough boundary and drag reduction. *Commun. Math. Phys.*, 232(3):429–455, 2003.
- [22] M. Kyed. Maximal regularity of the time-periodic linearized Navier-Stokes system. *J. Math. Fluid Mech.*, 16(3):523–538, 2014.
- [23] A. L. Madureira and F. Valentin. Asymptotics of the Poisson problem in domains with curved rough boundaries. *SIAM J. Math. Anal.*, 38(5):1450–1473 (electronic), 2006/07.
- [24] V. Milisic. Very weak estimates for a rough poisson-dirichlet problem with natural vertical boundary conditions. *Methods and Applications of Analysis*, 2009. to appear.
- [25] V. Milišić. Blood-flow modelling along and trough a braided multi-layer metallic stent. **submitted**.
- [26] T. Miyakawa. On nonstationary solutions of the Navier-Stokes equations in an exterior domain. *Hiroshima Math. J.*, 12(1):115–140, 1982.
- [27] N. Neuss, M. Neuss-Radu, and A. Mikelić. Effective laws for the poisson equation on domains with curved oscillating boundaries. *Applicable Analysis*, 85:479–502, 2006.
- [28] G. Nguetseng. Espaces de distributions sur des ouverts périodiques et applications. *Inria report RR-0172*, pages available at <https://hal.inria.fr/inria-00076386/en/>, 1982.
- [29] U. Razafison. *Théorie L^p avec poids pour les équations d’Oseen dans des domaines non bornés*. PhD thesis, Université de Pau et des Pays de l’Adour, 2004.
- [30] U. Razafison. The stationary Navier-Stokes equations in 3D exterior domains. An approach in anisotropically weighted L^q spaces. *J. Differential Equations*, 245(10):2785–2801, 2008.
- [31] M. Reed and B. Simon. *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.
- [32] W. Rudin. *Fourier analysis on groups*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1990. Reprint of the 1962 original, A Wiley-Interscience Publication.
- [33] J. Sauer. Extrapolation theorem on locally compact abelian groups. preprint.
- [34] J. Sauer. Weighted resolvent estimates for the spatially periodic stokes equations. *Annali dell’universita’ di ferrara*, pages 1–22, 2014.
- [35] L. Schwartz. *Théorie des distributions*. Publications de l’Institut de Mathématique de l’Université de Strasbourg, No. IX-X. Nouvelle édition, entièrement corrigée, refondue et augmentée. Hermann, Paris, 1966.
- [36] A. Weil. Sur certains groupes d’opérateurs unitaires. *Acta Math.*, 111:143–211, 1964.