Numerical Zoom for localized Multiscales

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Summary. We investigate the Schwarz' domain decomposition algorithm for numerical zooms. Error estimates are given for non-matching meshes for elliptic and parabolic partial differential equations. The method is applied to the security assessment of the burial of nuclear waste which is a typical multiscale problem for porous media flow.

Key words: Multiscale, Finite Elements, Domain Decomposition, Numerical Zoom, Porous Media Flow.

1 Introduction

Security assessment of nuclear waste repository sites requires the numerical simulations of elliptic and parabolic partial differential equations, namely Darcy's law for the hydrostatic potential and the convection diffusion equation of the radio-nucleides. These problems have multiscales because the simulations are done over half a million years while the canister leaks over less than 5000 years; also the repository sites could pollute as much as 10 square km of area while the length scales of a canister is of the order of the meter.

Strictly speaking the computations should be done with billions of mesh points on supercomputers; however often enough engineers prototype their applications with a coarse calculation and then a finer one on a subset (zoom) D of the whole domain Ω.

We wish here to justify this approach, i.e. to study convergence and errors when the strategy is implemented in the time loop of the convection diffusion equations and when the meshes used for the zooms are not divided sub-meshes of the coarse mesh. One may argue that it is unnecessarily complex to do so; however many engineers use the shelves solvers which do not have a numerical zoom capacity built in and so it is quite complex – if even possible – to use them with a zoom mesh made of divided triangles of the coarse mesh.

For Darcy's law the situation is as follows: a coarse calculation is done in Ω, then another one in a zoom $D \subset \Omega$; a correction to the coarse problem must be found which takes into account the calculation in the zoom. In [6] and [11] two algorithms have been shown to converge: a subspace correction method (or Hilbert space Decomposition, equivalent to Schwarz domain decomposition on the continuous level) and the Harmonic Patch Iterator.

It is very natural to work with composite meshes, a coarse one for most of the domain and a fine one in the region where the data are irregular; engineers do it spontaneously without even bothering about convergence issues; this idea seems to be as old as the finite element method itself and bears many names such as Chimera [16] in fluid mechanics, Global-local [12], composite grids [5], Arlequin [7] in structure mechanics, domain decomposition [10], Hilbert Space Decomposition [9], Mortars [2], etc. Therefore it is the analysis, more than the method, which is possibly new in this article.

For non matching meshes the error estimations started with [9] and [6] for Hilbert space Decomposition but stumbled on the problem of quadrature errors for integrals involving products of functions on coarse and fine meshes [3]. The problem was solved in [8] by working with the sup-norm. Here we present an extension of the same argument for parabolic problems. For the heat equation and the convection diffusion problems we will analyze the error for Schwarz' when only one iteration is done at each time step.

2 Convergence of Schwarz' Algorithm for Darcy's law on Arbitrary Non-Matching Meshes

In simple fully saturated cases the hydrostatic pressure H of the flow of water underground is given by

$$-\nabla \cdot (K \nabla H) = f \text{ in } \Omega, \quad H \text{ or } (K\nabla H) \cdot n \text{ given on } \Gamma \tag{1}$$

where K is the permeability tensor, n the normal to $\Gamma := \partial\Omega$. Assume that K has multiple scales or is irregular in $D \subset \Omega$ and consider the following Schwarz algorithm to zoom in D numerically.

The Schwarz-Zoom Method

Let Ω_H be a triangulated subdomain of Ω such that $\partial\Omega_H = \Gamma_H \cup S_H$ with the property that S_H is strictly inside D and non empty. Let Ω_h be a triangulated approximation of D and call $S_h = \partial\Omega_h$ (see Fig. 1). The triangumation of Ω_H (resp Ω_h) will be denoted $\mathcal{T}_H$ (resp $\mathcal{T}_h$).

Let

$$V_H = \{v \in C^0(\Omega_H) \ : \ v|_K \in P^1, \ \forall K \in \mathcal{T}_H\}, \quad V_{0H} = \{v \in V_H \ : \ v|_{\partial\Omega_H} = 0\},$$

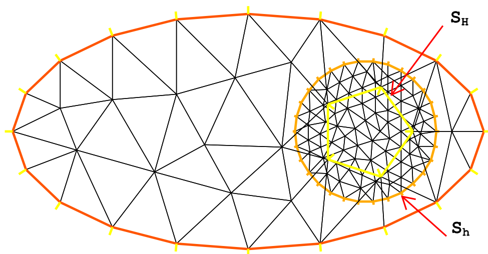

Fig. 1. The drawing shows the fine mesh of the zoom region Ω_h overlapping the coarse mesh in of Ω_H and its hole of boundary S_H.

and similarly with h. Denote by γ_H (resp γ_h) the interpolation operator on V_H (resp V_h).

To simplify the notation, we choose the variant of (1) with the boundary condition $H|_\Gamma = g$ and set K to the identity matrix so that (1) becomes the Dirichlet problem for the Poisson equation. The solution of this problem will be denoted by u from now on. Let $g_H = \gamma_H g$ (to avoid further technical difficulties we suppose here that all the boundary nodes of $\mathcal{T}_H$ lying on Γ_H also belong to Γ so that $\gamma_H g$ is well defined). Starting from any $u_h^0 \in V_h$, the Schwarz-Zoom algorithm proceeds as follows: find $u_H^m \in V_H,\ u_h^m \in V_h$, $m = 1, 2, \ldots$, such that $\forall w_H \in V_{0H},\ \ \forall w_h \in V_{0h}$

$$\begin{aligned} &a_H(u_H^m, w_H) = (f, w_H), \ \ u_H^m|_{S_H} = \gamma_H u_h^{m-1}, \ \ u_H^m|_{\Gamma_H} = g_H, \\ &a_h(u_h^m, w_h) = (f, w_h), \ \ u_h^m|_{S_h} = \gamma_h u_H^m. \end{aligned} \tag{2}$$

Hypothesis H1:
Assume that all the off-diagonal elements in the matrices obtained on the left hand side of the two problems in (2) are non-negative so that the maximum principle holds for these problems. Moreover, assume that $\nu_H \in V_H$ solution of

$$a_H(\nu_H, w_H) = 0, \ \ \forall w_H \in V_{0H}, \ \ \nu_H|_{S_H} = 1, \ \ \nu_H|_{\Gamma_H} = 0 \tag{3}$$

satisfies $\lambda := |\nu_H|_{\infty, S_h} < 1$.

Notice that the maximum principle is known to be true when all the angles of the triangulation are acute [4]. Error estimates in maximum norm of order $h^2 \log \frac{1}{h}$ with respect to the mesh edge size h for linear elements have been obtained in [15].

Theorem 1. *Under Hypothesis H1 the Schwarz-Zoom algorithm converges towards u_H^*, u_h^*, which satisfy*

$$a_H(u^*_H, w_H) = (f, w_H), \quad \forall w_H \in V_{0H}, \quad u^*_H|_{S_H} = \gamma_H u^*_h, \quad u^*_H|_{\Gamma_H} = g_H$$
$$a_h(u^*_h, w_h) = (f, w_h), \quad \forall w_h \in V_{0h}, \quad u^*_h|_{S_h} = \gamma_h u^*_H. \tag{4}$$

Provided the exact solution u of the Dirichlet problem for the Poisson equation is sufficiently smooth and the distance between Γ and Γ_H is of order H^2, one has

$$\|u - u_H\|_{\infty,\Omega_H} + \|u - u_h\|_{\infty,\Omega_h} \leq C(H^2 \log \frac{1}{H} \|u\|_{2,\infty,\Omega_H} + h^2 \log \frac{1}{h} \|u\|_{2,\infty,\Omega_h}) \tag{5}$$

The proof can be found in [8]. It shows also that the Schwarz algorithm converges linearly at speed proportional to λ.

Remark 1. This results gives a rule to choose h/H because it is logical to have both pieces in (5) of the same size, giving

$$\frac{h^2 \log \frac{1}{h}}{H^2 \log \frac{1}{H}} \sim \frac{\|u\|_{2,\infty,\Omega_H}}{\|u\|_{2,\infty,\Omega_h}} \tag{6}$$

and since D has been chosen so that $\|u\|_{2,\infty,\Omega_h}$ is large compared with $\|u\|_{2,\infty,\Omega_H}$, it gives an h smaller than H.

3 Convergence of the Schwarz-zoom Algorithm for Convection-Diffusion on Non-Matching Meshes

To solve

$$\frac{\partial u}{\partial t} + b \cdot \nabla u - \Delta u = f \text{ in } \Omega \text{ with } u = g \text{ on } \Gamma = \partial\Omega,$$
$$u|_{t=0} = u_0, \tag{7}$$

we proceed to analyze first the case $b = 0$.

***Schwarz method for the heat equation** ($b \neq 0$)*

To avoid further technical difficulties, we suppose that Ω is a polygonal domain. As before we choose two sub-domains of Ω, Ω_H and Ω_h, and two triangulations $\mathcal{T}_H$ of Ω_H, $\mathcal{T}_h$ of Ω_h, such that

$$\Omega_H \cup \Omega_h = \Omega, \quad \partial\Omega_h \subset \Omega_H, \quad \partial\Omega_H \backslash \Gamma \subset \Omega_h,$$

and work with the P^1 finite element spaces V_H and V_h on $\mathcal{T}_H$ and $\mathcal{T}_h$. As in Fig. 1 we denote by S_h the boundary of Ω_h and by S_H the part of the boundary of Ω_H different from Γ.

Euler-Schwarz algorithm starts from $u^0_H = \gamma_H u_0$, $u^0_h = \gamma_h u_0$ and finds $u^n_H \in V_H$ with $u^n_H|_\Gamma = g^n_H$ and $u^n_h \in V_h$, $n = 1, \ldots, N$ such that $\forall w_H \in V_{0H}, \ \ \forall w_h \in V_{0h}$:

$$
\begin{aligned}
&(\frac{u_H^n - u_H^{n-1}}{\Delta t}, w_H)_H + a_H(u_H^n, w_H) = (f^n, w_H), \quad u_H^n|_{S_H} = \gamma_H u_h^{n-1}, \\
&(\frac{u_h^n - u_h^{n-1}}{\Delta t}, w_h)_h + a_h(u_h^n, w_h) = (f^n, w_h), \quad u_h^n|_{S_h} = \gamma_h u_H^{n-1},
\end{aligned} \tag{8}
$$

where $f^n = f(t^n, \cdot)$, $g_H^n = g_H(t^n, \cdot)$, $g_H = \gamma_H g$, $a_{H,h}(u,v) = \int_{\Omega_{H,h}} \nabla u \cdot \nabla v$ and γ_H (resp γ_h) is the interpolation operator on V_H (resp V_h). The parentheses $(\cdot,\cdot)$ stand for the L^2 inner product and $(\cdot,\cdot)_H$ (resp $(\cdot,\cdot)_h$) denotes mass lumping on $\mathcal{T}_H$ (resp $\mathcal{T}_h$).

Theorem 2. *Under Hypothesis H1 and provided that u, solution to (7) with $b = 0$, is sufficiently smooth, (u_h^n, u_H^n) in (8) solves approximately (7) with the following L^∞ error estimate*

$$
\begin{aligned}
\max_{1\le n\le N} &(||u_H^n - u(t^n,\cdot)||_{\infty,\Omega_H}, ||u_h^n - u(t^n,\cdot)||_{\infty,\Omega_h}) \\
&\le CT((\Delta t + H^2 \log\frac{1}{H})||u||_{C^2([0,T],C(\Omega_H))\cap C^1([0,T],H^{2,\infty}(\Omega_H))} \\
&\quad +(\Delta t + h^2 \log\frac{1}{h})||u||_{C^2([0,T],C(\Omega_h))\cap C^1([0,T],H^{2,\infty}(\Omega_h))}).
\end{aligned} \tag{9}
$$

with $T = N\Delta t$ and a constant C depending only on the domains Ω_H and Ω_h.

Proof. Let $u_H^* = u_H^*(t,\cdot) \in V_H$ and $u_h^* = u_h^*(t,\cdot) \in V_h$ for any $t \in [0,T]$ be the solutions of (4) with $u = u(t,\cdot)$ and $g_H = g_H(t,\cdot)$. By Theorem 1, the solution of (4) exists for all t and the following estimate holds

$$
\begin{aligned}
&||u_H^* - u||_{\infty,\Omega_H\times[0,T]} + ||u_h^* - u||_{\infty,\Omega_h\times[0,T]} \\
&\quad \le C(H^2\log\frac{1}{H}||u||_{C([0,T],H^{2,\infty}(\Omega_H))} + h^2\log\frac{1}{h}||u||_{C([0,T],H^{2,\infty}(\Omega_h))})
\end{aligned} \tag{10}
$$

and similarly for $||u_h^* - u||_{\infty,\Omega_h\times[0,T]}$. Moreover, we can differentiate (4) with respect to time to obtain the estimates for the derivatives $\partial u_{H,h}^*/\partial t$:

$$
\begin{aligned}
&\left\|\frac{\partial u_H^*}{\partial t} - \frac{\partial u}{\partial t}\right\|_{\infty,\Omega_H\times[0,T]} + \left\|\frac{\partial u_h^*}{\partial t} - \frac{\partial u}{\partial t}\right\|_{\infty,\Omega_h\times[0,T]} \\
&\quad \le C(H^2\log\frac{1}{H}||u||_{C^1([0,T],H^{2,\infty}(\Omega_H))} + h^2\log\frac{1}{h}||u||_{C^1([0,T],H^{2,\infty}(\Omega_h))})
\end{aligned} \tag{11}
$$

It is now easy to obtain an *a priori* estimates for a fully implicit version of the Euler-Schwarz algorithm (8): find $\bar{u}_H^n \in V_H$ with $\bar{u}_H^n|_\Gamma = g_H^n$ and $\bar{u}_h^n \in V_h$ such that $\forall w_H \in V_{0H}, \quad \forall w_h \in V_{0h}$:

$$
\begin{aligned}
&(\frac{\bar{u}_H^n - \bar{u}_H^{n-1}}{\Delta t}, w_H)_H + a_H(\bar{u}_H^n, w_H) = (f^n, w_H), \quad \bar{u}_H^n|_{S_H} = \gamma_H \bar{u}_h^n, \\
&(\frac{\bar{u}_h^n - \bar{u}_h^{n-1}}{\Delta t}, w_h)_h + a_h(\bar{u}_h^n, w_h) = (f^n, w_h), \quad \bar{u}_h^n|_{S_h} = \gamma_h \bar{u}_H^n.
\end{aligned} \tag{12}
$$

Introduce the errors $\varepsilon_H^n \in V_H$, $\varepsilon_h^n \in V_h$ as

$$\varepsilon_H^n = \bar{u}_H^n - u_H^{*n}, \quad \varepsilon_h^n = \bar{u}_h^n - u_h^{*n},$$

where $u_H^{*n} = u_H^*(t^n,\cdot)$, $u_h^{*n} = u_h^*(t^n,\cdot)$. After some simplification, the finite element problem for $\varepsilon_{H,h}^n$ reads: $\forall w_H \in V_{0H}, \quad \forall w_h \in V_{0h}$

$$\begin{aligned}
&(\frac{\varepsilon_H^n - \varepsilon_H^{n-1}}{\Delta t}, w_H)_H + a_H(\varepsilon_H^n, w_H) = (r_H^n, w_H), \quad \varepsilon_H^n|_{S_H} = \gamma_H \varepsilon_h^n, \quad \varepsilon_H^n|_{\Gamma_H} = 0,\\
&(\frac{\varepsilon_h^n - \varepsilon_h^{n-1}}{\Delta t}, w_h)_h + a_h(\varepsilon_h^n, w_h) = (r_h^n, w_h), \quad \varepsilon_h^n|_{S_h} = \gamma_h \varepsilon_H^n, \qquad (13)
\end{aligned}$$

with

$$r_H^n = \frac{\partial u}{\partial t}(t^n,\cdot) - \frac{u_H^{*n} - u_H^{*(n-1)}}{\Delta t}, \quad r_h^n = \frac{\partial u}{\partial t}(t^n,\cdot) - \frac{u_h^{*n} - u_h^{*(n-1)}}{\Delta t}.$$

Estimating r_H^n with the help of the tringle inequality and (11) gives

$$\begin{aligned}
||r_H^n||_{\infty,\Omega_H} &\le ||\frac{\partial u}{\partial t}(t^n,\cdot) - \frac{u(t^n,\cdot) - u(t^{n-1},\cdot)}{\Delta t}||_{\infty,\Omega_H}\\
&\quad + \frac{1}{\Delta t}||\int_{t^{n-1}}^{t^n} \frac{\partial}{\partial t}(u(s,\cdot) - u^*(s,\cdot))ds||_{\infty,\Omega_H}\\
&\le C\Delta t||\frac{\partial^2 u}{\partial t^2}||_{\infty,\Omega_H} + ||\frac{\partial}{\partial t}(u-u^*)||_{\infty,\Omega_H}\\
&\le C(\Delta t||u||_{C^2([0,T],C(\Omega_H))} + H^2 \log\frac{1}{H}||u||_{C^1([0,T],H^{2,\infty}(\Omega_H))}\\
&\qquad + h^2 \log\frac{1}{h}||u||_{C^1([0,T],H^{2,\infty}(\Omega_h))}). \qquad (14)
\end{aligned}$$

A similar bound holds for r_h^n. We observe also that the joint maximum of $||\varepsilon_H^n||_{\infty,\Omega_H}$ and $||\varepsilon_h^n||_{\infty,\Omega_h}$ is always attained in an iternal node of $\mathcal{T}_H$ or $\mathcal{T}_h$. Indeed, the property $\varepsilon_h^n|_{S_h} = \gamma_h \varepsilon_H^n$ entails $\|\varepsilon_h^n\|_{\infty,S_h} \le \|\varepsilon_H^n\|_{\infty,S_h}$. Hence there is always an internal node of $\mathcal{T}_H$ such that the absolute value of u_H^n at this node is greater than $||\varepsilon_h^n||_{S_h}$, provided H is sufficiently small in comparison to the distance between S_H and S_h. Without loss of generality, we can now assume that the joint maximum of $|\varepsilon_H^n|$ and $|\varepsilon_h^n|$ is attained at an internal node x_n of $\mathcal{T}_h$ and $\varepsilon_h^n(x_n) \ge 0$. Taking the hat function ϕ_n associated to the node x_n in the standard basis of V_h as the test function w_h in (13) and observing that $a_h(u_n^n, \phi_n) \ge 0$ by hypothesis *H1*, we see that

$$\frac{u_h^n(x_n) - u_h^{n-1}(x_n)}{\Delta t}(1,\phi_n) \le (r_h^n, \phi_n)$$

so that

$$\begin{aligned}
&\max(||\varepsilon_H^n||_{\infty,\Omega_H}, ||\varepsilon_h^n||_{\infty,\Omega_h}) = u_h^n(x_n) \le u_h^{n-1}(x_n) + \Delta t||r_H^n||_{\infty,\Omega_H}\\
&\le \max(||\varepsilon_H^{n-1}||_{\infty,\Omega_H}, ||\varepsilon_h^{n-1}||_{\infty,\Omega_h}) + \Delta t \max(||r_H^n||_{\infty,\Omega_H}, ||r_h^n||_{\infty,\Omega_h}) \qquad (15)
\end{aligned}$$

Summing up these inequalities on $n = 1,\dots,N$ and combining with (10) and (14), we obtain the error estimates for the implicit Schwarz algorithm

$$\max_{1\le n\le N}(||\bar{u}_H^n - u(t^n,\cdot)||_{\infty,\Omega_H}, ||\bar{u}_h^n - u(t^n,\cdot)||_{\infty,\Omega_h})$$
$$\le CT((\Delta t + H^2\log\frac{1}{H})||u||_{C^2([0,T],C(\Omega_H))\cap C^1([0,T],H^{2,\infty}(\Omega_H))}$$
$$+(\Delta t + h^2\log\frac{1}{h})||u||_{C^2([0,T],C(\Omega_h))\cap C^1([0,T],H^{2,\infty}(\Omega_h))}). \quad (16)$$

Let us now return to our original Schwarz algorithm (8). Introduce the finite differences $d_H^n = (u_H^n - u_H^{n-1})/\Delta t$, $d_h^n = (u_h^n - u_h^{n-1})/\Delta t$. The equations for them are obtained by subtracting (8) at time t^{n-1} from the same equation at time t^n: $d_H^n|_{\Gamma_H} = g_H^n - g_H^{n-1}$, $\forall w_H \in V_{0H}$, $\forall w_h \in V_{0h}$

$$(\frac{d_H^n - d_H^{n-1}}{\Delta t}, w_H)_H + a_H(d_H^n, w_H) = (\frac{f^n - f^{n-1}}{\Delta t}, w_H), \quad d_H^n|_{S_H} = \gamma_H d_h^{n-1},$$
$$(\frac{d_h^n - d_h^{n-1}}{\Delta t}, w_h)_h + a_h(d_h^n, w_h) = (\frac{f^n - f^{n-1}}{\Delta t}, w_h), \quad d_h^n|_{S_h} = \gamma_h d_H^{n-1}, \quad (17)$$

for $n \ge 2$. By an application of the maximum principle, similar as above, we observe that

$$\max(||d_H^n||_{\infty,\Omega_H}, ||d_h^n||_{\infty,\Omega_h}) \le \max(||g_H^n - g_H^{n-1}||_{\infty,\Gamma},$$
$$\max(||d_H^{n-1}||_{\infty,\Omega_H}, ||d_h^{n-1}||_{\infty,\Omega_h}) + \Delta t||\tfrac{f^n - f^{n-1}}{\Delta t}||_{\infty,\Omega}). \quad (18)$$

The norm of the finite difference in time for f in the last line can be bounded by the maximum of $\frac{\partial f}{\partial t}$ which, in turn, is bounded by $||u||_{C^2([0,T],C(\Omega))\cap C^1([0,T],H^{2,\infty}(\Omega))}$ since $\frac{\partial f}{\partial t} = \frac{\partial^2 u}{\partial t^2} - \Delta\frac{\partial u}{\partial t}$. It is also easy to obtain the appropriate estimates for $||g_H^n - g_H^{n-1}||_{\infty,\Gamma}$, $||d_H^1||_{\infty,\Omega_H}$ and $||d_h^1||_{\infty,\Omega_h}$, so that we have finally

$$\max_{1\le n\le N}(||d_H^n||_{\infty,\Omega_H}, ||d_h^n||_{\infty,\Omega_h})$$
$$\le C_d := CT||u||_{C^2([0,T],C(\Omega))\cap C^1([0,T],H^{2,\infty}(\Omega))}. \quad (19)$$

The last step in the proof is to estimate the difference between the original Schwarz algorithm (8) and the implicit one (12). We thus introduce $e_H^n = \bar{u}_H^n - u_H^n$, $e_h^n = \bar{u}_h^n - u_h^n$ and observe that they are solutions to the finite element problems: $\forall w_H \in V_{0H}$, $\forall w_h \in V_{0h}$

$$(\frac{e_H^n - e_H^{n-1}}{\Delta t}, w_H)_H + a_H(e_H^n, w_H) = 0, \quad e_H^n|_{S_H} = \gamma_H e_h^n + \Delta t\gamma_H d_h^n, \quad e_H^n|_{\Gamma_H} = 0,$$
$$(\frac{e_h^n - e_h^{n-1}}{\Delta t}, w_h)_h + a_h(e_h^n, w_h) = 0, \quad e_h^n|_{S_h} = \gamma_h e_H^n + \Delta t\gamma_h d_H^n, \quad (20)$$

with initial conditions $u_H^0 = 0$, $u_h^0 = 0$. We remind that all d_H^n and d_h^n are bounded in the maximum norm by C_d, take any number $A > C_d\Delta t$ and consider $s_H^n = e_H^n - A\alpha_H$, $s_h^n = e_h^n - A\alpha_h$ where the auxiliary functions $\alpha_H \in V_H$, $\alpha_h \in V_h$ satisfy

$$a_H(\alpha_H, w_H) = 0 \ \ \forall w_H \in V_H, \quad \alpha_H|_{S_H} = \gamma_H\alpha_h + 1, \quad \alpha_H|_\Gamma = 0,$$

$$a_h(\alpha_h, w_h) = 0 \ \ \forall w_h \in V_h, \quad \alpha_h|_{S_h} = \gamma_h \alpha_H + 1. \tag{21}$$

We now want to prove that s_H^n and s_h^n are non-positive everywhere and for all n. If this is not the case, we can suppose without loss of generality that the global maximum of $\{s_H^n\}$, $\{s_h^n\}$ is attained at the time step number $k \geq 1$ on $s_h^k(x_k)$ where x_k is a node of $\mathcal{T}_h$. We have

$$\begin{aligned} &(\frac{s_h^k - s_h^{k-1}}{\Delta t}, w_h)_h + a_h(s_h^k, w_h) = 0 \ \ \forall w_h \in V_h, \\ &s_h^k|_{S_h} = \gamma_h s_H^k + \Delta t \gamma_h d_H^k - A. \end{aligned} \tag{22}$$

The first line in (22) tells us that x_k cannot be an internal node of $\mathcal{T}_h$, while the second line implies that it cannot be a boundary node either since $\Delta t d_H^k - A < 0$ on S_h. We obtain thus $\forall n \geq 1$

$$\sup_{x\in\Omega} \max(e_H^n(x), e_h^n(x)) \leq AC_\alpha := A \max(||\alpha_H||_{\infty,\Omega_H}, ||\alpha_h||_{\infty,\Omega_h}).$$

Note that the constant C_α can be bounded independently of H and h. Repeating the same derivation with e_H^n, e_h^n replaced by $-e_H^n$, $-e_h^n$ and tending A to $C_d \Delta t$ from above, we see that

$$\max(||e_H^n||_{\infty,\Omega_H}, ||e_h^n||_{\infty,\Omega_h}) \leq C_\alpha C_d \Delta t, \quad \forall n \geq 1. \tag{23}$$

The desired result (9) is obtained from (16) and (23) by the triangle inequality.

Remark 2. The fully implicit Schwarz algorithm (12) can provide an interesting alternative to our basic algorithm (8) if several iterations of a standard Schwarz method like (2) are used on every time step to find an approximation for the couple (u_H^n, u_h^n). The convergence proof above can be easily adapted for such an algorithm. It can be preferable to (8) in terms of the cost to accuracy ratio.

Remark 3. One can associate two different time steps for the two sub-domains, say Δt for the problems with u_H^n and δt for the problems with u_h^n. A variant of the Schwarz algorithm can be then constructed provided $\Delta t = m \Delta t$ with an integer m. The convergence proof is extensible to this case without difficulties.

Remark 4. Extension of our results to the operator $\nabla \cdot (K\nabla)$ in place of Δ is straightforward.

***Extension to Convection Diffusion** ($b \neq 0$)*

The convection term in (7) can be discretized by the characteristic-Galerkin scheme. Combination with the Schwarz method gives the following algorithm: find $u_H^n \in V_H$, $u_h^n \in V_h$, such that $\forall w_H \in V_{0H}$, $\forall w_h \in V_{0h}$

$$\frac{1}{\Delta t}(u_H^n - u_H^{n-1} o X^{n-1}, w_H)_H + a_H(u_H^n, w_H) = (f, w_H),$$

$$u_H^n|_{S_H} = \gamma_H u_h^{n-1}, \quad u_H^n|_\Gamma = g_H^n,$$
$$\frac{1}{\Delta t}(u_h^n - u_h^{n-1} o X^{n-1}, w_h)_h + a_h(u_h^n, w_h) = (f, w_h),$$
$$u_h^n|_{S_h} = \gamma_h u_H^{n-1}, \tag{24}$$

where $X^{n-1}(x) = X(t^{n-1})$ and X is the solution of $\dot{X} = b(X(t), t)$, $X(t^n) = x$. We consider the case where X^{n-1} can be computed exactly (b piecewise constant for instance).

As shown in [13] and [14], the accuracy of the characteristic-Galerkin method deteriorates when the time step Δt becomes too small before the mesh size. One cannot expect therefore to extend Theorem 2 to the general case $b \neq 0$. We have, however, a weaker result:

Theorem 3. *Under Hypothesis H1 and provided that u, solution to (7), is sufficiently smooth, (u_h^n, u_H^n) in (24) solves approximately (7) with the following L^∞ error estimate*

$$\max_{1\le n\le N}(||u_H^n - u(t^n)||_{\infty,\Omega_H}, ||u_h^n - u(t^n)||_{\infty,\Omega_h})$$
$$\le CT((\Delta t + \frac{H^2 \log\frac{1}{H}}{\Delta t})||u||_{H^{2,\infty}(\Omega_H\times[t^{n-1},t^n])}$$
$$+(\Delta t + \frac{h^2 \log\frac{1}{h}}{\delta t})||u||_{H^{2,\infty}(\Omega_h\times[t^{n-1},t^n])}, \tag{25}$$

with $T = N\Delta t$ and a constant C depending only on the domains Ω_H and Ω_h.

Proof. Denote $u^n = u(t^n, \cdot)$ and consider $\tilde{u}_H^n \in V_H$ such that $\forall w_H \in V_{0H}$

$$a_H(\tilde{u}_H^n, w_H) = a_H(u^n, w_H), \quad \tilde{u}_H^n|_{S_H} = \gamma_H u(t^n), \quad \tilde{u}_H^n|_\Gamma = g_H(t^n, \cdot).$$

By the results in [15], we have

$$||\tilde{u}_H^n - u^n||_{\infty,\Omega_H} \le CH^2 \log\frac{1}{H}||u^n||_{2,\infty,\Omega_H}. \tag{26}$$

We also introduce $\tilde{u}_h^n \in V_h$ in a similar way and observe that the same estimate as (26) holds for $\tilde{u}_h^n$ replacing H by h.

Denote

$$\varepsilon_H^n = u_H^n - \tilde{u}_H^n, \quad \varepsilon_h^n = u_h^n - \tilde{u}_h^n.$$

Then $\varepsilon_H^n \in V_H$ solves $\forall w_H \in V_{0H}$

$$(\frac{\varepsilon_H^n - \varepsilon_H^{n-1} o X^{n-1}}{\Delta t}, w_H)_H + a_H(\varepsilon_H^n, w_H) = (r_H^n, w_H),$$
$$\varepsilon_H^n|_{S_H} = \gamma_H \varepsilon_h^{n-1}, \quad u_H^n|_{\Gamma_H} = 0 \tag{27}$$

with

$$r_H^n = \frac{Du}{Dt}(t^n, \cdot) - \frac{\tilde{u}_H^n - \tilde{u}_H^{n-1} o X^{n-1}}{\Delta t},$$

where $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + b\nabla u$ is the material derivative. By the triangle inequality and (26) we have

$$\begin{aligned}
&||r_H^n||_{\infty,\Omega_H} \\
&\quad \le ||\frac{Du}{Dt}(t^n,\cdot) - \frac{u^n - u^{n-1}oX^{n-1}}{\Delta t}||_{\infty,\Omega_H} \\
&\qquad + ||\frac{u^n - u^{n-1}oX^{n-1}}{\Delta t} - \frac{\tilde{u}_H^n - \tilde{u}_H^{n-1}oX^{n-1}}{\Delta t}||_{\infty,\Omega_H} \\
&\quad \le C\Delta t||u||_{H^{2,\infty}(\Omega_H\times[t^{n-1},t^n])} \\
&\qquad + \frac{1}{\Delta t}||u^n - \tilde{u}_H^n||_{\infty,\Omega_H} + \frac{1}{\Delta t}||(u^{n-1} - \tilde{u}_H^{n-1})oX^{n-1}||_{\infty,\Omega_H} \\
&\quad \le C\left[(\Delta t + \frac{H^2\log\frac{1}{H}}{\Delta t})||u||_{H^{2,\infty}(\Omega_H\times[0,T])} + \frac{h^2\log\frac{1}{h}}{\Delta t}||u||_{H^{2,\infty}(\Omega_h\times[0,T])}\right].
\end{aligned}$$

Note here the use of the inequality

$$||(u^{n-1} - \tilde{u}_H^{n-1})oX^{n-1}||_{\infty,\Omega_H} \le ||u^{n-1} - \tilde{u}_H^{n-1}||_{\infty,\Omega_H} + ||u^{n-1} - \tilde{u}_h^{n-1}||_{\infty,\Omega_h}.$$

One cannot leave only the norm in Ω_H in the right hand side of this inequality since a characteristic starting in Ω_H at time t^n can go back to Ω_h at time t^{n-1}.

Writing down an estimate for r_h^n similar to what is done above for r_H^n and noting that

$$\begin{aligned}
\max(||\varepsilon_H^n||_{\infty,\Omega_H}, ||\varepsilon_h^n||_{\infty,\Omega_h}) \le\ & \max(||\varepsilon_H^{n-1}||_{\infty,\Omega_H}, ||\varepsilon_h^{n-1}||_{\infty,\Omega_h}) \\
& + \Delta t\max(||r_H^n||_{\infty,\Omega_H}, ||r_h^n||_{\infty,\Omega_h}),
\end{aligned}$$

we conclude the proof in the same way as that of the estimate (16) for the implicit Schwarz method in the proof of Theorem 2.

4 Numerical Tests

We shall apply the method to solve one of Andra's COUPLEX test cases (see [1]). but before that let us verify the error estimate on a simpler example. A closed form solution $u = tr^{-2}$, $r^2 = x^2 + y^2$ is computed in a square $(-1,1)^2$ minus the circle of radius 0.1 by solving the heat equation with an appropriate source term $f = (r^2 - 4t)r^{-4}$. The subdomains are the square minus the disk of radius 0.25 and a disk of radius 0.3 minus the disk of radius 0.05. With $h = H/8$, $\Delta t = \Delta t = H/2$ the left part of Table 4 displays the L^4 errors E and e in each subdomain. The right part of the table displays the same errors but functions of $\Delta t \to 0$. The COUPLEX test case is made first of an elliptic problem $\nabla\cdot(K\nabla H) = 0$ to obtain the hydrostatic pressure H from the porosity K and convection velocity as $-K\nabla H$ and then of a convection diffusion equation for the concentration $c(x,y,t)$ of the radionucleide in the underground. The difficulties come from the large differences of K in each

H	0.2	0.1	0.05	0.025	Δt	0.25	0.0625	0.0156	0.004	0.001
E	4.57	2.86	1.88	1.02		7.97	3.66	0.26	1.50	1.83
e	10.1	4.86	2.50	1.30		23.7	6.1	1.5	1.5	1.8

Table 1. The left part shows the errors as a function of H when $h = H/8$ and $\Delta t = H/2$. The right part shows the behavior of the errors when $H = 0.1, h = 0.1/8$ versus Δt.The errors first decrease then grow when $\Delta t \to 0$, but not like $1/\Delta t$ which indicates that the bounds in Theorem 3 are not tight.

geological layer and from the different length scales in the problem. The details of the calculation are not very important but Fig. 2 shows that the Schwarz-zoom algorithm works for this problem.

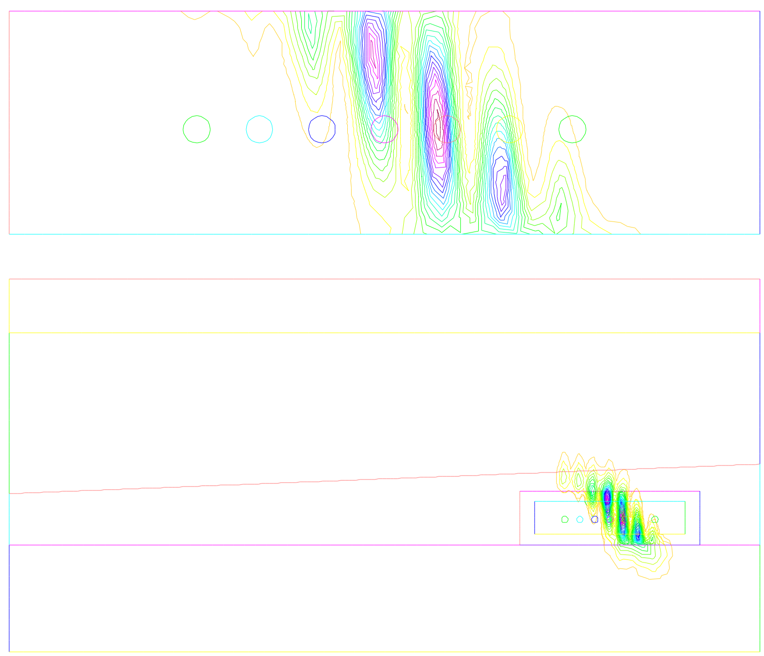

Fig. 2. The top pictures shows detailed level lines of the concentration of radionucleide nearby the canisters (the 7 circles) after 600 time steps. A concentration is given at initial time in the canister which are all supposed to leak due to rust and spread their content by convection and diffusion in the clay. The calculation is obtained from a Schwarz-zoom refinement . The bottom picture shows the full domain, its decomposition into 2 subdomains and a superposition of both fields of level lines of c_H and c_h; both fields match fairly well. The domain is physically a rectangle (0,10km)×(0,500m) but it is rescaled for the simulation; the long horizontal lines are the separations between (from top to bottom), marl, limestone, clay and dogger layers.

5 Conclusion

The paper has explained why the standard Schwarz algorithm can be used for a numerical zoom procedure, with standard interpolations at the boundaries. The method works for time dependent problems as well and it is not necessary to iterate the Schwarz algorithm because the time stepping procedure has the same effect. A compatibility condition between the time step and the mesh size may be necessary in the case of convection diffusion problems. However, the same feature is present at the standard finite element characteristic-Galerkin simulations on a single mesh so that it cannot be attributed to the Schwarz method.

Application to flows through porous media is covered by the two theorems of this paper; it would be nice to extend the results to other fluid flows such as Navier-Stokes flow but the proofs here rely heavily on the maximum principle which is not valid for the generalized Stokes operator.

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