# Lorenzen's theory of divisibility in monoid-preordered sets

## 1 "Teilbarkeitstheorie in Bereichen" (1952), § 1.

**Definition 1.1** ([Lor52, Definition 1, p. 270])**.** A *G-ordered set B* is a set with a preorder $\preccurlyeq_B$ and a monoid $G$ of order-preserving operators on $B$:

$$a \preccurlyeq_B a, \quad \frac{a \preccurlyeq_B b \quad b \preccurlyeq_B c}{a \preccurlyeq_B c}, \quad \frac{a \preccurlyeq_B b}{xa \preccurlyeq_B xb}.$$

If $a \preccurlyeq_B b$ and $b \preccurlyeq_B a$ hold, then we say that $a$ and $b$ are equal w.r.t. $\preccurlyeq_B$ and write $a \equiv_B b$.

*Remark* 1.2. We shall use latin letters $a$, $b$, $c$ for elements of a $G$-ordered set and latin letters $x$, $y$, $z$ for elements of a monoid.

**Definition 1.3** ([Lor52, Definition 2, p. 270])**.** A *G-semilattice H* is a $G$-ordered set with a preorder $\preccurlyeq_H$ and a meet $\wedge_H$ that make it a semilattice, and with a monoid $G$ of meet-preserving operators on $H$:

$$\genfrac{}{}{1.2pt}{}{\mathfrak{c} \preccurlyeq_H \mathfrak{a} \quad \mathfrak{c} \preccurlyeq_H \mathfrak{b}}{\mathfrak{c} \preccurlyeq_H \mathfrak{a} \wedge_H \mathfrak{b}}, \quad x(\mathfrak{a} \wedge_H \mathfrak{b}) \equiv_H x\mathfrak{a} \wedge_H x\mathfrak{b}.$$

*Remark* 1.4. We shall use Kurrent letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ for elements of a $G$-semilattice.

There may be finer meet-preserving preorders on a $G$-semilattice $H$ for which the operators in $G$ are still meet-preserving. To make this precise, let us state the following definition and proposition.

**Definition 1.5** ([Lor52, Definition 3, p. 270])**.** Let $(H, \preccurlyeq_H, \wedge_H)$ be a $G$-semilattice. A preorder $\preccurlyeq$ on $H$ is *admissible for* $\preccurlyeq_H$ if it is finer than $\preccurlyeq_H$ and if $(H, \preccurlyeq, \wedge_H)$ is a $G$-semilattice.

**Proposition 1.6.** *A preorder $\preccurlyeq$ on a G-semilattice $(H, \preccurlyeq_H, \wedge_H)$ is admissible for $\preccurlyeq_H$ if and only if*

$$\frac{\mathfrak{a} \preccurlyeq_H \mathfrak{b}}{\mathfrak{a} \preccurlyeq \mathfrak{b}}, \quad \frac{\mathfrak{a} \preccurlyeq \mathfrak{b}}{x\mathfrak{a} \preccurlyeq x\mathfrak{b}}, \quad \frac{\mathfrak{c} \preccurlyeq \mathfrak{a} \quad \mathfrak{c} \preccurlyeq \mathfrak{b}}{\mathfrak{c} \preccurlyeq \mathfrak{a} \wedge_H \mathfrak{b}}. \tag{1}$$

*Proof.* [Argument in Lor50, p. 499.] As $\mathfrak{a} \wedge_H \mathfrak{b} \preccurlyeq_H \mathfrak{a}$ and $\mathfrak{a} \wedge_H \mathfrak{b} \preccurlyeq_H \mathfrak{b}$, the first condition implies that $\mathfrak{a} \wedge_H \mathfrak{b} \preccurlyeq \mathfrak{a}$ and $\mathfrak{a} \wedge_H \mathfrak{b} \preccurlyeq \mathfrak{b}$, so that with the third condition $\mathfrak{a} \wedge_H \mathfrak{b}$ is equal w.r.t. $\preccurlyeq$ to the meet of $\mathfrak{a}$ and $\mathfrak{b}$ w.r.t. $\preccurlyeq$:

$$\genfrac{}{}{1.2pt}{}{\mathfrak{c} \preccurlyeq \mathfrak{a} \quad \mathfrak{c} \preccurlyeq \mathfrak{b}}{\mathfrak{c} \preccurlyeq \mathfrak{a} \wedge_H \mathfrak{b}}.$$

As any $x$ satisfies $x(\mathfrak{a} \wedge_H \mathfrak{b}) \equiv_H x\mathfrak{a} \wedge_H x\mathfrak{b}$, the first condition shows that $x(\mathfrak{a} \wedge_H \mathfrak{b})$ and $x\mathfrak{a} \wedge_H x\mathfrak{b}$ are also equal w.r.t. $\preccurlyeq$. ☐

**Proposition 1.7.** *A conjunction of admissible preorders is admissible.*

## 2 "Teilbarkeitstheorie in Bereichen" (1952), § 2.

**Definition 2.1** ([Lor52, Definition 4, p. 270])**.** An *ideal $G$-semilattice* for a $G$-ordered set $(B, \preccurlyeq_B)$ is a minimal $G$-supersemilattice for $B$: this is the set $\widehat{B}$ of formal meets (that is, finite lists) $\mathscr{a} = a_1 \wedge \cdots \wedge a_m$ of elements of $B$ endowed with a preorder $\preccurlyeq_H$ that extends $\preccurlyeq_B$ and makes it a semilattice, and with a monoid of operators that extend those on $B$:

$$\text{for all } a, b \in B, \; \frac{a \preccurlyeq_B b}{\overline{a \preccurlyeq_H b}}.$$

We say that $\preccurlyeq_H$ is an *ideal $G$-semilattice preorder* for $(B, \preccurlyeq_B)$.

*Remark* 2.2. It is a question of taste whether to define formal meets as lists or as multisets (lists up to permutation) or as sets (multisets with contraction).

*Remark* 2.3. Lorenzen [Lor50, pp. 503–505] proves the equivalence of this definition with the one proposed by Prüfer [Prü32]. In fact, if one considers the finite meets $\mathscr{a}$ as sets, an ideal $G$-semilattice preorder $\preccurlyeq_H$ is given by an application $\mathscr{a} \mapsto \mathscr{a}_r$ into the subsets of $B$ endowed with the relation of containment, where the application satisfies

1. $\mathscr{a} \subseteq \mathscr{a}_r$;
2. if $\mathscr{a} \subseteq \mathscr{b}_r$, then $\mathscr{a}_r \subseteq \mathscr{b}_r$;
3. if $a \in B$, then $\{ a \}_r = \{ b \mid a \preccurlyeq_B b \}$;
4. $x\mathscr{a}_r = (x\mathscr{a})_r$.

Jaffard [Jaf60, p. 120] states condition 2 with the weaker hypothesis $\mathscr{a} \subseteq \mathscr{b}$, but this is probably a typo. Then $\mathscr{a}_r = \{ a \in B : \mathscr{a} \preccurlyeq_H a \}$.

*Remark* 2.4. Ideal $G$-semilattices correspond exactly to single-statement entailment relations defined from $(B, \preccurlyeq_B)$ with an additional structure given by $G$: see [Lor51, Satz 3].

The following constructions will be important.

**Definition 2.5.** Let $\preccurlyeq_S$ be any preorder that makes $(B, \preccurlyeq_S)$ a $G$-ordered set. The preorder $\preccurlyeq_{\widehat{S}_s}$ on $\widehat{B}$ is defined by

$$a_1 \wedge \cdots \wedge a_m \preccurlyeq_{\widehat{S}_s} a \quad \text{if} \quad a_1 \preccurlyeq_S a \text{ or } \ldots \text{ or } a_m \preccurlyeq_S a.$$

The preorder $\preccurlyeq_{\widehat{S}_v}$ on $\widehat{B}$ is defined by

$$a_1 \wedge \cdots \wedge a_m \preccurlyeq_{\widehat{S}_v} a \quad \text{if} \quad \forall x \in G \, \frac{c \preccurlyeq_S xa_1 \quad \ldots \quad c \preccurlyeq_S xa_m}{c \preccurlyeq_S xa}.$$

**Proposition 2.6.** *Let $(B, \preccurlyeq_S)$ be a $G$-ordered set. The $G$-ordered sets $(\widehat{B}, \preccurlyeq_{\widehat{S}_s})$ and $(\widehat{B}, \preccurlyeq_{\widehat{S}_v})$ endowed with the operation*

$$x(a_1 \wedge \cdots \wedge a_m) = xa_1 \wedge \cdots \wedge xa_m \tag{2}$$

*are respectively the minimal and the maximal ideal $G$-semilattice for $(B, \preccurlyeq_S)$: every preorder $\preccurlyeq$ that makes $\widehat{B}$ an ideal $G$-semilattice for $(B, \preccurlyeq_S)$ satisfies*

$$\frac{\mathscr{a} \preccurlyeq_{\widehat{S}_s} \mathscr{b}}{\mathscr{a} \preccurlyeq \mathscr{b}} \quad \text{and} \quad \frac{\mathscr{a} \preccurlyeq \mathscr{b}}{\mathscr{a} \preccurlyeq_{\widehat{S}_v} \mathscr{b}}.$$

*Proof.* [Argument in Lor50, Satz 14 and Satz 15, pp. 507–508.] The two preorders make $\widehat{B}$ a semilattice.

The preorder $\preccurlyeq_{\widehat{S}_s}$ clearly induces $\preccurlyeq_S$ on $B$ and is clearly preserved by the operation (2). If $\preccurlyeq$ is a preorder that makes $\widehat{B}$ an ideal $G$-semilattice for $B$ and $a_1 \wedge \cdots \wedge a_m \preccurlyeq_{\widehat{S}_s} a$, then $a_\mu \preccurlyeq_S a$ for some $\mu$ and therefore $a_1 \wedge \cdots \wedge a_m \preccurlyeq a$.

Let us show that $\preccurlyeq_{\widehat{S}_v}$ induces $\preccurlyeq_S$ on $B$: if $a_1 \preccurlyeq_S a$, then $xa_1 \preccurlyeq_S xa$ and therefore $c \preccurlyeq_S xa_1$ entails $c \preccurlyeq_S xa$. Conversely, this yields for $x$ the unit and $c$ the element $a_1$ that $a_1 \preccurlyeq_S a_1$ entails $a_1 \preccurlyeq_S a$, so that $a_1 \preccurlyeq_S a$.

Let us show that the operation (2) is $\preccurlyeq_{\widehat{S}_v}$-preserving. Suppose that $a_1 \wedge \cdots \wedge a_m \preccurlyeq_{\widehat{S}_v} a$ and let us show that $ya_1 \wedge \cdots \wedge ya_m \preccurlyeq_{\widehat{S}_v} ya$: but if $c \preccurlyeq_S (xy)a_1, , \ldots, c \preccurlyeq_S (xy)a_m$, then $c \preccurlyeq_S (xy)a$.

If $\preccurlyeq$ is a preorder that makes $\widehat{B}$ an ideal $G$-semilattice for $B$, then

$$\dfrac{\dfrac{c \preccurlyeq_S xa_1 \quad \ldots \quad c \preccurlyeq_S xa_m}{c \preccurlyeq xa_1 \wedge \cdots \wedge xa_m} \qquad \dfrac{a_1 \wedge \cdots \wedge a_m \preccurlyeq a}{xa_1 \wedge \cdots \wedge xa_m \preccurlyeq xa}}{c \preccurlyeq_S xa},$$

so that $\preccurlyeq_{\widehat{S}_v}$ is finer than $\preccurlyeq$. □

*Remark* 2.7. The ideal semilattice $(\widehat{B}, \preccurlyeq_{\widehat{B}_s})$ has been introduced in [Lor39, p. 537], while the ideal semilattice $(\widehat{B}, \preccurlyeq_{\widehat{B}_v})$ dates back to van der Waerden and Prüfer [see Kru35, § 43].

**Proposition 2.8.** *A preorder $\preccurlyeq$ on $\widehat{B}$ is admissible for $\preccurlyeq_{\widehat{B}_s}$ if and only if $(\widehat{B}, \preccurlyeq)$ is a $G$-semilattice and the preorder induced by $\preccurlyeq$ on $B$ is finer than $\preccurlyeq_B$.*

There may be finer preorders on $B$ that give rise to the same ideal $G$-semilattice. To make this precise, let us state the following definition and proposition.

**Definition 2.9** ([Lor52, Definition 5, p. 271])**.** Let $\preccurlyeq_H$ be an ideal $G$-semilattice preorder for $(B, \preccurlyeq_B)$. A preorder $\preccurlyeq_S$ on $B$ is $\preccurlyeq_H$*-admissible* if $\preccurlyeq_S$ is induced by a preorder $\preccurlyeq$ that is admissible for $\preccurlyeq_H$: rules (1) hold and

$$\frac{a \preccurlyeq b}{\overline{\overline{a \preccurlyeq_S b}}}.$$

**Proposition 2.10.** *A preorder $\preccurlyeq_S$ on $B$ is $\preccurlyeq_{\widehat{B}_s}$-admissible if and only if it is finer than $B$.*

**Proposition 2.11.** *Let $\preccurlyeq_H$ be an ideal $G$-semilattice preorder for $(B, \preccurlyeq_B)$. A preorder $\preccurlyeq_S$ on $B$ is $\preccurlyeq_H$-admissible if and only if the preorder $\preccurlyeq_{\widehat{S}_v}$ given in definition 2.5 is admissible for $\preccurlyeq_H$.*

*Proof.* [Argument in Lor50, Satz 17, p. 511.] By proposition 2.6, $(\widehat{B}, \preccurlyeq_{\widehat{S}_v})$ is the finest ideal $G$-semilattice for $(B, \preccurlyeq_S)$: in particular, $\preccurlyeq_{\widehat{S}_v}$ induces $\preccurlyeq_S$ on $B$.

If $\preccurlyeq_{\widehat{S}_v}$ is a preorder that is admissible for $\preccurlyeq_H$, then $\preccurlyeq_S$ is $\preccurlyeq_H$-admissible by definition.

Conversely, if $\preccurlyeq$ is a preorder that is admissible for $\preccurlyeq_H$ and induces $\preccurlyeq_S$ on $B$, then $\preccurlyeq$ is finer than $\preccurlyeq_H$, and $\preccurlyeq_{\widehat{S}_v}$ is finer than $\preccurlyeq$ because $\preccurlyeq$ makes $\widehat{B}$ an ideal $G$-semilattice for $(B, \preccurlyeq_S)$: therefore $\preccurlyeq_{\widehat{S}_v}$ is finer than $\preccurlyeq_H$ and thus admissible for $\preccurlyeq_H$. □

**Definition 2.12.** Let $\preccurlyeq_H$ be an ideal $G$-semilattice preorder for $(B, \preccurlyeq_B)$ and consider an $\preccurlyeq_H$-admissible preorder $\preccurlyeq_S$. The *minimal ideal $G$-semilattice extension* $\preccurlyeq'_S$ is the conjunction of all ideal $G$-semilattice preorders for $(B, \preccurlyeq_S)$ that are admissible for $\preccurlyeq_H$.

*Remark* 2.13. Strangely, Lorenzen did not impose in his definition that the extensions must be admissible! Is this an error or an omission?

**Proposition 2.14.** *Consider the ideal $G$-semilattice preorder $\preccurlyeq_{\widehat{B}_s}$ for $(B, \preccurlyeq_B)$ and a $\preccurlyeq_{\widehat{B}_s}$-admissible preorder $\preccurlyeq_S$. The minimal ideal semilattice extension of $\preccurlyeq_S$ is $\preccurlyeq_{\widehat{S}_s}$.*

**Lemma 2.15.** *Let $\preccurlyeq_H$ be an ideal $G$-semilattice preorder for $(B, \preccurlyeq_B)$ and consider $\preccurlyeq_H$-admissible preorders $\preccurlyeq_S$ and $\preccurlyeq_T$ such that $\preccurlyeq_T$ is finer than $\preccurlyeq_S$. Then $\preccurlyeq'_T$ is finer than $\preccurlyeq'_S$.*

*Proof.* Let $\preccurlyeq$ be an preorder that is admissible for $\preccurlyeq_H$ and induces $\preccurlyeq_S$ on $B$. For every preorder $\preccurlyeq_{\widehat{T}}$ that is admissible for $\preccurlyeq_H$ and induces $\preccurlyeq_T$ on $B$, the conjunction of $\preccurlyeq$ and $\preccurlyeq_{\widehat{T}}$ is also an extension of $\preccurlyeq_S$ that is admissible for $\preccurlyeq_H$. □

**Lemma 2.16.** *Let $\preccurlyeq_H$ be an ideal G-semilattice preorder for $(B, \preccurlyeq_B)$. If $\leqslant_S$ is an $\preccurlyeq_H$-admissible total preorder on $B$, then its unique extension to $\widehat{B}$ is the admissible total preorder $\leqslant_{\widehat{S}}$ on $\widehat{B}$ given by*

$$a_1 \wedge \cdots \wedge a_m \leqslant_{\widehat{S}} a \quad \text{iff} \quad \min(a_1, \ldots, a_m) \leqslant_S a. \tag{3}$$

*Proof.* [Argument in Lor50, Satz 18, p. 512.] It suffices to prove that the preorders $\leqslant_{\widehat{S}_s}$ and $\leqslant_{\widehat{S}_v}$ in definition 2.5 coincide with the definition of $\leqslant_{\widehat{S}}$ in (3). This follows at once for $\leqslant_{\widehat{S}_s}$. For $\leqslant_{\widehat{S}_v}$, note that $c \leqslant_S xa_1, \ldots, c \leqslant_S xa_m$ hold simultaneously if and only if $c \leqslant_S x \min(a_1, \ldots, a_m)$. □

**Definition 2.17** ([Lor52, Definition 6, p. 271])**.** A $G$-ordered set $B$ is $\preccurlyeq_H$-*principal*, where $\preccurlyeq_H$ is an ideal $G$-semilattice preorder for $(B, \preccurlyeq_B)$, if its preorder $\preccurlyeq_B$ is a conjunction of $\preccurlyeq_H$-admissible total preorders.

**Definition 2.18.** Let $\preccurlyeq_H$ be an ideal $G$-semilattice preorder for $(B, \preccurlyeq_B)$. We define the preorder $\preccurlyeq_{H_a}$ on $\widehat{B}$ by

$$a_1 \wedge \cdots \wedge a_m \preccurlyeq_{H_a} a \text{ iff for all } \preccurlyeq_H\text{-admissible total preorders} \leqslant \text{ on } B \text{ holds } \min(a_1, \ldots, a_m) \leqslant a,$$

and the preorder $\preccurlyeq_{S,H_a}$ for an $\preccurlyeq_H$-admissible preorder $\preccurlyeq_S$ by

$$a_1 \wedge \cdots \wedge a_m \preccurlyeq_{S,H_a} a \text{ iff for all } \preccurlyeq_H\text{-admissible total preorders} \leqslant \text{ on } B \text{ that refine } \preccurlyeq_S \text{ holds } \min(a_1, \ldots, a_m) \leqslant a.$$

**Proposition 2.19.** *Let $\preccurlyeq_H$ be an ideal G-semilattice preorder for $(B, \preccurlyeq_B)$. The preorder $\preccurlyeq_{H_a}$ on $\widehat{B}$ is admissible for $\preccurlyeq_H$.*

*Proof.* The preorder $\preccurlyeq_{H_a}$ is a conjunction of admissible preorders by lemma 2.16. □

**Corollary 2.20.** *Let $\preccurlyeq_H$ be an ideal G-semilattice preorder for $(B, \preccurlyeq_B)$. T.f.a.e.*

1. *$B$ is $\preccurlyeq_H$-principal.*
2. *$(\widehat{B}, \preccurlyeq_{H_a})$ is an ideal G-semilattice for $B$.*
3. *The preorder $\preccurlyeq_{H_a}$ induces $\preccurlyeq_B$ on $B$.*
4. *The preorder $\preccurlyeq_B$ is finer than the preorder induced by $\preccurlyeq_{H_a}$ on $B$.*

**Proposition 2.21** ([Lor50, Satz 19, p. 512])**.** *Let $\preccurlyeq_H$ and $\preccurlyeq_K$ be ideal G-semilattice preorders for $(B, \preccurlyeq_B)$. Then $\preccurlyeq_K$ is a conjunction of total preorders that are admissible for $\preccurlyeq_H$ if and only if $\preccurlyeq_K$ is a conjunction of total preorders that are admissible for $\preccurlyeq_K$ and every total preorder that is admissible for $\preccurlyeq_K$ induces an $\preccurlyeq_H$-admissible total preorder on G.*

*Proof.* The condition is clearly sufficient. Conversely, let $\preccurlyeq$ be a preorder that is admissible for $\preccurlyeq_K$ and let $\preccurlyeq_S$ be the preorder induced by $\preccurlyeq$ on $B$. Then

$$\cfrac{\cfrac{a_1 \wedge \cdots \wedge a_m \preccurlyeq_K b}{a_1 \wedge \cdots \wedge a_m \preccurlyeq b}}{a_1 \wedge \cdots \wedge a_m \preccurlyeq_{\widehat{S}_v} b} \quad \text{and} \quad \cfrac{\cfrac{a_1 \wedge \cdots \wedge a_m \preccurlyeq_H b}{a_1 \wedge \cdots \wedge a_m \preccurlyeq_{H_a} b.}}{a_1 \wedge \cdots \wedge a_m \preccurlyeq_K b}$$

Therefore $\preccurlyeq_{\widehat{S}_v}$ is a preorder that is admissible for $\preccurlyeq_H$ and $\preccurlyeq_S$ is $\preccurlyeq_H$-admissible by proposition 2.11. □

The proof shows the following.

**Proposition 2.22.** *Let $\preccurlyeq_H$ and $\preccurlyeq_K$ be ideal G-semilattice preorders for $(B, \preccurlyeq_B)$. If $\preccurlyeq_K$ is a conjunction of total preorders that are admissible for $\preccurlyeq_H$, then every preorder that is admissible for $\preccurlyeq_K$ induces an $\preccurlyeq_H$-admissible preorder on G.*

**Definition 2.23** ([Lor52, Definition 7, p. 271]). Let $\preccurlyeq_H$ be an ideal $G$-semilattice preorder for $(B, \preccurlyeq_B)$. An element $a$ of $B$ is $\preccurlyeq_{H_a}$*-dependent* from the elements $a_1, \ldots, a_m \in B$ if

$$a_1 \wedge \cdots \wedge a_m \preccurlyeq_{H_a} a.$$

An element $a$ of $B$ is $\preccurlyeq_{S,H_a}$*-dependent* from the elements $a_1, \ldots, a_m \in B$ for an $\preccurlyeq_H$-admissible preorder $\preccurlyeq_S$ on $B$ if

$$a_1 \wedge \cdots \wedge a_m \preccurlyeq_{S,H_a} a.$$

**Definition 2.24.** Given a pair $\alpha = (a_1, a_2)$ out of $B$, we define the $\preccurlyeq_H$*-extension* $\preccurlyeq_{S[\alpha]_H}$ as the conjunction of all $\preccurlyeq_H$-admissible preorders $\preccurlyeq$ on $B$ that refine $\preccurlyeq_S$ and such that $a_1 \preccurlyeq a_2$:

$$\frac{a \preccurlyeq_S b}{a \preccurlyeq b} \quad \text{and} \quad a_1 \preccurlyeq a_2.$$

We let $\alpha^{+1} = (a_1, a_2)$ and $\alpha^{-1} = (a_2, a_1)$.

**Lemma 2.25** ([Lor52, Lemma, p. 272]). *An element $a$ is $\preccurlyeq_{S,H_a}$-dependent from the elements $a_1, \ldots, a_m$ of $B$ if and only if*

$$\begin{array}{l}\textit{there are pairs } \alpha_1, \ldots, \alpha_e \textit{ out of } B \textit{ such that for all choices of} \\ \textit{signs } \varepsilon_1, \ldots, \varepsilon_e \in \{+1, -1\} \textit{ holds } a_1 \wedge \cdots \wedge a_m \preccurlyeq'_{S[\alpha_1^{\varepsilon_1}, \ldots, \alpha_e^{\varepsilon_e}]_H} a.\end{array} \tag{4}$$

*Proof.* If condition (4) holds, consider an $\preccurlyeq_H$-admissible total preorder $\leqslant$ on $B$ that refines $\preccurlyeq_S$. For every pair $\alpha = (a_1, a_2)$ out of $B$, either $a_1 \leqslant a_2$ (set $\varepsilon = +1$) or $a_2 \leqslant a_1$ (set $\varepsilon = -1$): therefore $\leqslant$ also refines $\preccurlyeq_{S[\alpha^\varepsilon]_H}$. By reiterating this, one proves that $\leqslant$ also refines $\preccurlyeq_{S[\alpha_1^{\varepsilon_1}, \ldots, \alpha_e^{\varepsilon_e}]_H}$ for some choice of signs $\varepsilon_1, \ldots, \varepsilon_e \in \{+1, -1\}$: therefore $\leqslant'$ refines $\preccurlyeq'_{S[\alpha_1^{\varepsilon_1}, \ldots, \alpha_e^{\varepsilon_e}]_H}$ and $\min(a_1, \ldots, a_m) \leqslant a$.

Conversely, suppose that condition (4) does not hold and consider a maximal $\preccurlyeq_H$-admissible preorder $\preccurlyeq_T$ refining $\preccurlyeq_S$ for which condition (4) fails. Let us prove by contradiction that $\preccurlyeq_T$ is a total preorder: let $\alpha = (a_1, a_2)$ be a pair out of $B$; if neither $a_1 \leqslant a_2$ nor $a_2 \leqslant a_1$ did hold, then $\preccurlyeq_{T[\alpha^{+1}]_H}$ and $\preccurlyeq_{T[\alpha^{-1}]_H}$ would be strictly finer $\preccurlyeq_H$-admissible preorders and thus satisfy condition (4). But this would correspond to condition (4) for $\preccurlyeq_T$ itself. Therefore $\preccurlyeq_T$ is a total preorder $\leqslant$ refining $\preccurlyeq_S$ for which $\min(a_1, \ldots, a_m) \leqslant a$ does not hold. □

*Remark* 2.26. One should be able here to check directly that

$$a_1, \ldots, a_m \vdash a \quad \text{if} \quad \exists_{\alpha_1, \ldots, \alpha_e} a_1 \wedge \cdots \wedge a_m \preccurlyeq'_{S[\alpha_1^{\pm 1}, \ldots, \alpha_e^{\pm 1}]_H} a$$

defines a single-conclusion entailment relation in the sense of [Lor51, 1.–4. in Satz 1, p. 84].

**Theorem 2.27** ([Lor52, Satz 1, p. 272]). *The $G$-ordered set $B$ is $\preccurlyeq_H$-principal, where $\preccurlyeq_H$ is an ideal $G$-semilattice preorder for $(B, \preccurlyeq_B)$, if and only if*

$$\frac{a_1 \preccurlyeq_{B[\alpha_1^{\varepsilon_1}, \ldots, \alpha_e^{\varepsilon_e}]_H} a \quad \text{for all } \varepsilon_1, \ldots, \varepsilon_e \in \{+1, -1\}}{a_1 \preccurlyeq_B a}. \tag{5}$$

*Proof.* By corollary 2.20, $B$ is $\preccurlyeq_H$-principal if and only

$$\frac{a \text{ is } \preccurlyeq_{H_a}\text{-dependent from } a_1}{a_1 \preccurlyeq_B a}. \tag{6}$$

By lemma 2.25, if $a$ is $\preccurlyeq_{H_a}$-dependent from $a_1$, then there are pairs $\alpha_1, \ldots, \alpha_e$ out of $B$ such that $a_1 \preccurlyeq_{B[\alpha_1^{\varepsilon_1}, \ldots, \alpha_e^{\varepsilon_e}]_H} a$ for all choices of signs $\varepsilon_1, \ldots, \varepsilon_e$. This shows that rule (6) may be derived from rule (5), and therefore sufficiency.

Conversely, lemma 2.25 tells also that

$$\frac{a_1 \preccurlyeq_{B[\alpha_1^{\varepsilon_1}, \ldots, \alpha_e^{\varepsilon_e}]_H} a \quad \text{for all } \varepsilon_1, \ldots, \varepsilon_e \in \{+1, -1\}}{a \text{ is } \preccurlyeq_{H_a}\text{-dependent from } a_1}.$$

If $B$ is $\preccurlyeq_H$-principal, rule (6) holds and we derive rule (5). □

# 3 "Teilbarkeitstheorie in Bereichen" (1952), § 3.

**Definition 3.1** ([Lor52, Definition 8, p. 272])**.** A *G-lattice* $V$ is a $G$-semilattice with a preorder $\preccurlyeq_V$, a meet $\wedge_V$ and a join $\vee_V$ that make it a lattice, and with a monoid $G$ of join-meet-preserving operators on $V$:

$$\frac{\mathfrak{c} \preccurlyeq_V \mathfrak{a} \quad \mathfrak{c} \preccurlyeq_V \mathfrak{b}}{\mathfrak{c} \preccurlyeq_V \mathfrak{a} \wedge_V \mathfrak{b}}, \quad \frac{\mathfrak{a} \preccurlyeq_V \mathfrak{c} \quad \mathfrak{b} \preccurlyeq_V \mathfrak{c}}{\mathfrak{a} \vee_V \mathfrak{b} \preccurlyeq_V \mathfrak{c}},$$
$$x(\mathfrak{a} \wedge_V \mathfrak{b}) \equiv x\mathfrak{a} \wedge_V x\mathfrak{b}, \quad x(\mathfrak{a} \vee_V \mathfrak{b}) \equiv x\mathfrak{a} \vee_V x\mathfrak{b}.$$

*Remark* 3.2. We shall use Fraktur letters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ for elements of a $G$-lattice.

**Definition 3.3.** Let $(V, \preccurlyeq_V, \wedge_V, \vee_V)$ be a $G$-lattice. A preorder $\preccurlyeq$ on $V$ is *admissible for* $\preccurlyeq_V$ if it is finer than $\preccurlyeq_V$ and if $(V, \preccurlyeq, \wedge_V, \vee_V)$ is a $G$-lattice.

**Proposition 3.4.** *A preorder* $\preccurlyeq$ *on a G-lattice* $(V, \preccurlyeq_V, \wedge_V, \vee_V)$ *is admissible for* $\preccurlyeq_V$ *if and only if*

$$\frac{\mathfrak{a} \preccurlyeq_V \mathfrak{b}}{\mathfrak{a} \preccurlyeq \mathfrak{b}}, \quad \frac{\mathfrak{a} \preccurlyeq \mathfrak{b}}{x\mathfrak{a} \preccurlyeq x\mathfrak{b}}, \quad \frac{\mathfrak{c} \preccurlyeq \mathfrak{a} \quad \mathfrak{c} \preccurlyeq \mathfrak{b}}{\mathfrak{c} \preccurlyeq \mathfrak{a} \wedge_V \mathfrak{b}}, \quad \frac{\mathfrak{a} \preccurlyeq \mathfrak{c} \quad \mathfrak{b} \preccurlyeq \mathfrak{c}}{\mathfrak{a} \vee_V \mathfrak{b} \preccurlyeq \mathfrak{c}}. \tag{7}$$

**Definition 3.5.** An *ideal G-lattice* for a $G$-semilattice $(H, \preccurlyeq_H, \wedge_H)$ is a minimal distributive $G$-superlattice for $H$: this is the set $\check{H}$ of formal joins $\mathfrak{a} = \mathscr{a}_1 \vee \cdots \vee \mathscr{a}_m$ of elements of $H$ endowed with a preorder $\preccurlyeq_V$ that extends $\preccurlyeq_V$ and makes it a distributive lattice, and with a monoid of operators that extend those on $H$:

$$\text{for all } \mathscr{a}, \mathscr{b} \in H, \frac{\mathscr{a} \preccurlyeq_H \mathscr{b}}{\mathscr{a} \preccurlyeq_V \mathscr{b}}.$$

We say that $\preccurlyeq_V$ is an *ideal G-lattice preorder* for $H$.

**Definition 3.6.** An *ideal G-lattice* for a $G$-ordered set $B$ is a minimal distributive $G$-superlattice for $B$: this is the ideal $G$-lattice $\widetilde{B}$ of formal joins of formal meets of elements of $B$ endowed with a preorder $\preccurlyeq_V$ that extends $\preccurlyeq_H$ and makes it a distributive lattice, and with a monoid of operators that extend those on $B$. We say that $\preccurlyeq_V$ is an *ideal G-lattice preorder* for $B$.

*Remark* 3.7. Ideal $G$-lattices correspond exactly to entailment relations defined from $(B, \preccurlyeq_B)$ with an additional structure given by $G$: see [Lor51, Satz 7].

**Proposition 3.8.** *An ideal G-lattice for a G-ordered set B is the ideal G-lattice for an ideal G-semilattice for B.*

*Proof.* It suffices to consider the restriction $\preccurlyeq_H$ of the preorder $\preccurlyeq_V$ to $\widehat{B}$. ☐

The following constructions will be important.

**Definition 3.9.** Let $(H, \preccurlyeq_{\widehat{S}}, \wedge_H)$ be a $G$-semilattice. The preorder $\preccurlyeq_{\check{S}_s}$ on $\check{H}$ is defined by

$$\mathscr{a} \preccurlyeq_{\check{S}_s} \mathscr{a}_1 \vee \cdots \vee \mathscr{a}_m \quad \text{iff} \quad \mathscr{a} \preccurlyeq_{\widehat{S}} \mathscr{a}_1 \text{ or } \ldots \text{ or } \mathscr{a} \preccurlyeq_{\widehat{S}} \mathscr{a}_m. \tag{8}$$

The preorder $\preccurlyeq_{\check{H}_v}$ on $\check{H}$ is defined by

$$\mathscr{a} \preccurlyeq_{\check{S}_v} \mathscr{a}_1 \vee \cdots \vee \mathscr{a}_m \quad \text{iff} \quad \frac{x\mathscr{a}_1 \wedge \mathscr{b} \preccurlyeq_{\widehat{S}} \mathscr{v} \quad \ldots \quad x\mathscr{a}_m \wedge \mathscr{b} \preccurlyeq_{\widehat{S}} \mathscr{v}}{x\mathscr{a} \wedge \mathscr{b} \preccurlyeq_{\widehat{S}} \mathscr{v}}. \tag{9}$$

**Proposition 3.10.** *Let* $(H, \preccurlyeq_{\widehat{S}}, \wedge_H)$ *be a G-semilattice. The G-ordered sets* $(\check{H}, \preccurlyeq_{\check{S}_s})$ *and* $(\check{H}, \preccurlyeq_{\check{S}_v})$ *endowed with the operation*

$$x(\mathscr{a}_1 \vee \cdots \vee \mathscr{a}_m) = x\mathscr{a}_1 \vee \cdots \vee x\mathscr{a}_m \tag{10}$$

*are ideal G-lattices for H: they will be called respectively* $\check{H}_s$ *and* $\check{H}_v$ *and are the coarsest and the finest ideal G-lattice for H: every preorder* $\preccurlyeq$ *on* $\check{H}$ *that extends* $(H, \preccurlyeq_{\widehat{S}})$ *and makes* $\check{H}$ *an ideal G-lattice for B satisfies*

$$\frac{\mathscr{a} \preccurlyeq_{\check{S}_s} \mathscr{b}}{\mathscr{a} \preccurlyeq \mathscr{b}} \quad \text{and} \quad \frac{\mathscr{a} \preccurlyeq \mathscr{b}}{\mathscr{a} \preccurlyeq_{\check{S}_v} \mathscr{b}}.$$

There may be finer preorders on $B$ that give rise to the same ideal $G$-lattice. To make this precise, let us state the following definition.

**Definition 3.11.** Let $\preccurlyeq_V$ be an ideal $G$-lattice preorder for $B$. A preorder $\preccurlyeq_S$ on $B$ is $\preccurlyeq_V$*-admissible* if $\preccurlyeq_S$ is induced by a preorder $\preccurlyeq$ that is admissible for $\preccurlyeq_V$: rules (7) hold and

$$\frac{a \preccurlyeq b}{\overline{a \preccurlyeq_S b}}.$$

*Remark* 3.12. One should state and prove here a counterpart to proposition 2.11.

**Definition 3.13.** Let $\preccurlyeq_V$ be an ideal $G$-lattice preorder for $B$ and consider a $\preccurlyeq_V$-admissible preorder $\preccurlyeq_S$. The *minimal ideal lattice extension* $\preccurlyeq'_S$ is the conjunction of all extensions of $\preccurlyeq_S$ to $\widetilde{B}$ that are admissible for $\preccurlyeq_V$.

**Lemma 3.14.** *Let $\preccurlyeq_V$ be an ideal $G$-lattice preorder for $B$ and consider $\preccurlyeq_V$-admissible preorders $\preccurlyeq_S$ and $\preccurlyeq_T$ such that $\preccurlyeq_T$ is finer than $\preccurlyeq_S$. Then $\preccurlyeq'_T$ is finer than $\preccurlyeq'_S$.*

**Lemma 3.15.** *Let $\preccurlyeq_V$ be an ideal $G$-lattice preorder for $B$. If $\leqslant_S$ is a $\preccurlyeq_V$-admissible total preorder on $B$, then its unique extension to $\widetilde{B}$ is the admissible total preorder $\leqslant_{\widetilde{S}}$ on $\widetilde{B}$ given by*

$$a_1 \wedge \cdots \wedge a_m \leqslant_{\widetilde{S}} b_1 \vee \cdots \vee b_n \textit{ iff } \min(a_1, \ldots, a_m) \leqslant_S \max(b_1, \ldots, b_n). \tag{11}$$

*Proof.* [Argument in Lor50, Satz 20, p. 513.] By lemma 2.16, the unique extension of $\leqslant_S$ to $\widehat{B}$ is the total preorder $\leqslant_{\widehat{S}}$ defined in (3). Let us check that the preorders $\leqslant_{\breve{S}_s}$ and $\leqslant_{\breve{S}_v}$ defined in definition 3.9 coincide with the definition of $\leqslant_{\widetilde{S}}$ in (11). This follows at once for $\leqslant_{\breve{S}_s}$. For $\leqslant_{\breve{S}_v}$, note that $x\mathscr{a}_1 \wedge \mathscr{b} \leqslant_{\widehat{S}} \mathscr{v}$, $\ldots$, $x\mathscr{a}_m \wedge \mathscr{b} \leqslant_{\widehat{S}} \mathscr{v}$ hold simultaneously if and only if $x \max(\mathscr{a}_1, \ldots, a_m) \wedge \mathscr{b} \leqslant_{\widehat{S}} \mathscr{v}$. ☐

**Definition 3.16.** A $G$-ordered set $B$ is $\preccurlyeq_V$*-principal*, where $\preccurlyeq_V$ is an ideal $G$-lattice preorder, if its preorder $\preccurlyeq_B$ is a conjunction of $\preccurlyeq_V$-admissible total preorders.

**Definition 3.17.** Let $\preccurlyeq_V$ be an ideal $G$-lattice preorder for $B$. We define the preorder $\preccurlyeq_{V_a}$ on $\widetilde{B}$ by

$$a_1 \wedge \cdots \wedge a_m \preccurlyeq_{V_a} b_1 \vee \cdots \vee b_n \text{ iff for all } \preccurlyeq_V\text{-admissible total preorders } \leqslant \text{ on } B \text{ holds } \min(a_1, \ldots, a_m) \leqslant \max(b_1, \ldots, b_n). \tag{12}$$

and the preorder $\preccurlyeq_{S,V_a}$ for a $\preccurlyeq_H$-admissible preorder $\preccurlyeq_S$ by

$$a_1 \wedge \cdots \wedge a_m \preccurlyeq_{S,V_a} b_1 \vee \cdots \vee b_n \text{ iff for all } \preccurlyeq_V\text{-admissible total preorders } \leqslant \text{ on } B \text{ refining } \preccurlyeq_S \text{ holds } \min(a_1, \ldots, a_m) \leqslant \max(b_1, \ldots, b_n). \tag{13}$$

**Proposition 3.18.** *Let $\preccurlyeq_V$ be an ideal $G$-lattice preorder for $B$. The preorder $\preccurlyeq_{V_a}$ on $\widetilde{B}$ is admissible for $\preccurlyeq_V$.*

*Proof.* The preorder $\preccurlyeq_{V_a}$ is a conjunction of admissible preorders by lemma 3.15. ☐

**Corollary 3.19.** *Let $\preccurlyeq_V$ be an ideal $G$-lattice preorder for $B$. T.f.a.e.*

1. *$B$ is* $\preccurlyeq_V$-principal.
2. *$(\widehat{B}, \preccurlyeq_{V_a})$ is an ideal $G$-lattice for $B$.*
3. *The preorder $\preccurlyeq_{V_a}$ induces $\preccurlyeq_B$ on $B$.*
4. *The preorder $\preccurlyeq_B$ is finer than the preorder induced by $\preccurlyeq_{V_a}$ on $B$.*

*Remark* 3.20. Compare with [Lor50, Satz 21, p. 514] which at first view seems to state the opposite.

**Definition 3.21.** Given a pair $\alpha = (a_1, a_2)$ out of $B$, we define the $\preccurlyeq_V$*-extension* $\preccurlyeq_{S[\alpha]_V}$ as the conjunction of all $\preccurlyeq_V$-admissible preorders $\preccurlyeq$ on $B$ that refine $\preccurlyeq_S$ and such that $a_1 \preccurlyeq a_2$:

$$\frac{a \preccurlyeq_S b}{a \preccurlyeq b} \quad \text{and} \quad a_1 \preccurlyeq a_2.$$

**Lemma 3.22** ([Lor52, Satz 2, p. 273])**.** *The inequality $a_1 \wedge \cdots \wedge a_m \preccurlyeq_{S,V_a} b_1 \vee \cdots \vee b_n$ holds for elements $a_1, \ldots, a_m, b_1, \ldots, b_n$ of $B$ and a $\preccurlyeq_V$-admissible preorder $\preccurlyeq_S$ on $B$ if and only if there are pairs $\alpha_1, \ldots, \alpha_e$ out of $B$ such that for all choices of signs $\varepsilon_1, \ldots, \varepsilon_e \in \{+1, -1\}$ we have $a_1 \wedge \cdots \wedge a_m \preccurlyeq'_{S[\alpha_1^{\varepsilon_1}, \ldots, \alpha_e^{\varepsilon_e}]_V} b_1 \vee \cdots \vee b_n$.*

*Proof.* The same proof as for lemma 2.25. ☐

*Remark* 3.23. Strangely, Lorenzen states this lemma only under the hypothesis that $B$ is $\preccurlyeq_V$-principal.

**Theorem 3.24.** *A $G$-ordered set $B$ is $\preccurlyeq_V$-principal, where $\preccurlyeq_V$ is an ideal $G$-lattice preorder for $B$, if and only if*

$$\frac{a \preccurlyeq_{B[\alpha_1^{\varepsilon_1}, \ldots, \alpha_e^{\varepsilon_e}]_V} b \quad \text{for all } \varepsilon_1, \ldots, \varepsilon_e \in \{+1, -1\}}{a \preccurlyeq_B b}.$$

*Proof.* The same proof as for theorem 2.27. ☐

# 4 “Teilbarkeitstheorie in Bereichen” (1952), § 4.

**Definition 4.1** ([Lor52, Definition 9, p. 273])**.** A $G$-lattice $(V, \preccurlyeq_V, \wedge_V, \vee_V)$ is *regular* if it is distributive and if for all $a, b$ in $V$ and $x, y$ in $G$ holds

$$xa \wedge_V yb \preccurlyeq_V xb \vee_V ya. \tag{14}$$

**Proposition 4.2.** *If $\preccurlyeq_V$ is a total preorder $\leqslant$, then $V$ is regular.*

*Proof.* Let $a, b$ in $V$. If $a \leqslant b$, then $xa \leqslant xb$; if $b \leqslant a$, then $yb \leqslant ya$. In both cases, $\min(xa, yb) \leqslant \max(xb, ya)$. ☐

**Proposition 4.3.** *If $(V, \preccurlyeq_V, \wedge_V, \vee_V)$ is regular and $\preccurlyeq$ is admissible for $\preccurlyeq_V$, then $(V, \preccurlyeq, \wedge_V, \vee_V)$ is also regular.*

*Proof.* This follows from the fact that the meets and joins for $\preccurlyeq_V$ are equal w.r.t. $\preccurlyeq$ to the meets and joins for $\preccurlyeq$, and that $\preccurlyeq$ is finer than $\preccurlyeq_V$. ☐

**Corollary 4.4.** *If the preorder $\preccurlyeq_V$ of a $G$-lattice $V$ is a conjunction of admissible total preorders of $(V, \preccurlyeq_V)$, then $V$ is regular.*

**Theorem 4.5** ([Lor52, Satz 3, p. 274])**.** *The preorder $\preccurlyeq_V$ of a $G$-lattice $V$ is a conjunction of admissible total preorders of $(V, \preccurlyeq_V)$ if and only if $V$ is regular.*

*Proof.* Only sufficiency remains to be proved. For every pair $\gamma = (c_1, c_2)$ such that $c_1 \not\preccurlyeq_V c_2$, we need to find an admissible total preorder $\leqslant$ such that $c_1 \not\leqslant c_2$. Consider a maximal admissible preorder $\preccurlyeq_S$ on $(V, \preccurlyeq_V)$ with $c_1 \not\preccurlyeq_S c_2$ and let us consider the orders $\preccurlyeq_{S_\beta}$ defined for every pair $\beta = (b_1, b_2)$ by

$$a \preccurlyeq_{S_\beta} b \quad \text{if} \quad xa \wedge yb_1 \preccurlyeq_S xb \vee yb_2 \text{ for all } x \text{ and } y. \tag{15}$$

Then $\preccurlyeq_{S_\beta}$ is admissible. In fact,

- $\preccurlyeq_{S_\beta}$ refines $\preccurlyeq_S$: if $a \preccurlyeq_S b$, then $xa \preccurlyeq_S xb$ and $xa \wedge yb_1 \preccurlyeq_S xb \vee yb_2$;
- $\preccurlyeq_{S_\beta}$ is transitive: if $xa \wedge yb_1 \preccurlyeq_S xb \vee yb_2$, then $xa \wedge yb_1 \preccurlyeq_S (xb \vee yb_2) \wedge yb_1 \equiv_S (xb \wedge yb_1) \vee (yb_1 \wedge yb_2)$; if furthermore $xb \wedge yb_1 \preccurlyeq_S xc \vee yb_2$, then $xa \wedge yb_1 \preccurlyeq_S (xc \vee yb_2) \vee (yb_1 \wedge yb_2) \equiv_S xc \vee yb_2$;

- every $z$ preserves $\preccurlyeq_{S_\beta}$: if $a \preccurlyeq_{S_\beta} b$, then $(xz)a \wedge yb_1 \preccurlyeq_S (xz)b \vee yb_2$ for all $x$ and $y$, that is, $za \preccurlyeq_{S_\beta} zb$;
- $\preccurlyeq_{S_\beta}$ preserves meets: if $xc \wedge yb_1 \preccurlyeq_S xa \vee yb_2$ and $xc \wedge yb_1 \preccurlyeq_S xb \vee yb_2$, then $xc \wedge yb_1 \preccurlyeq_S (xa \vee yb_2) \wedge (xb \vee yb_2) \equiv_S (xa \wedge xb) \vee (xa \wedge yb_2) \vee (yb_2 \wedge xb) \vee yb_2 \equiv_S x(a \wedge b) \vee yb_2$;
- dually, $\preccurlyeq_{S_\beta}$ also preserves joins.

Furthermore, $\preccurlyeq_{S_\gamma}$ is $\preccurlyeq_S$ by maximality, for if we had $c_1 \preccurlyeq_{S_\gamma} c_2$, then $c_1 \preccurlyeq_S c_2$ would hold by letting $x$ and $y$ be the identical operator in the definition of $S_\gamma$.

Let us prove that $\preccurlyeq_S$ is a total preorder and suppose that $b_2 \not\preccurlyeq_S b_1$: note that $b_2 \preccurlyeq_{S_\beta} b_1$ by regularity, so that $\preccurlyeq_{S_\beta}$ is strictly finer than $\preccurlyeq_S$; by maximality holds $c_1 \preccurlyeq_{S_\beta} c_2$. But then, by the symmetry of definition (15), $b_1 \preccurlyeq_{S_\gamma} b_2$. □

*Remark* 4.6. This proof is still not too involved because the lattice structure of $V$ and the semigroup structure of $G$ do not interfere too much.

# 5 "Die Erweiterung halbgeordneter Gruppen zu Verbandsgruppen" (1953), § 1.

Lemma 3.22 shows that if one starts by letting $\preccurlyeq_{V_a}$ be the preorder on $\widetilde{B}$ given by

$$a_1 \wedge \cdots \wedge a_m \preccurlyeq_{V_a} b_1 \vee \cdots \vee b_n \text{ iff } \exists_{\alpha_1,\ldots,\alpha_e}\ a_1 \wedge \cdots \wedge a_m \preccurlyeq'_{B[\alpha_1^{\pm 1},\ldots,\alpha_e^{\pm 1}]_V} b_1 \vee \cdots \vee b_n \tag{16}$$

one defines a regular distributive $G$-lattice. This distributivity may be proved directly by showing that (16) defines an entailment relation [argument in Lor53, pp. 16–17]. Regularity may also be proved directly [argument in Lor53, pp. 17–18].

# 6 "Die Erweiterung halbgeordneter Gruppen zu Verbandsgruppen" (1953), § 2.

In the case of a preordered group $G$, its ideal lattice $(\widetilde{G}, \preccurlyeq_V)$ is already determined by its ideal semilattice $(\widehat{G}, \preccurlyeq_H)$: as $(b_1 \vee \cdots \vee b_n)(b_1^{-1} \wedge \cdots \wedge b_n^{-1}) = 1$, one has

$$a_1 \wedge \cdots \wedge a_m \preccurlyeq_V b_1 \vee \cdots \vee b_n \quad \text{iff} \quad a_1 b_1^{-1} \wedge \cdots \wedge a_m b_n^{-1} \preccurlyeq_H 1.$$

One can therefore resort to lemma 2.25 and try to start by letting $\preccurlyeq_{H_a}$ be the preorder given on $\widetilde{G}$ by

$$a_1 \wedge \cdots \wedge a_m \preccurlyeq_{H_a} b_1 \vee \cdots \vee b_n \text{ iff } \exists_{\gamma_1,\ldots,\gamma_e}\ a_1 b_1^{-1} \wedge \cdots \wedge a_m b_n^{-1} \preccurlyeq'_{B[\gamma_1^{\pm 1},\ldots,\gamma_e^{\pm 1}]_H} 1 \tag{17}$$

and prove that this defines a distributive lattice domain $(\widetilde{G}, \preccurlyeq_{H_a})$ by analogy with section 5. This is straightforward. Furthermore holds

**Theorem 6.1** ([Lor53, Satz 1, p. 18])**.** *$(\widetilde{G}, \preccurlyeq_{H_a})$ is a regular lattice group.*

The *proof* of theorem 6.1 takes 5 pages: [Lor53, pp. 18-22].

**Proposition 6.2.** *Let $(G, \preccurlyeq, \wedge, \vee)$ be a lattice group. T.f.a.e.*

1. *$G$ is regular.*
2. *$xa \wedge by \preccurlyeq xb \vee ay$.*
3. $\dfrac{a \wedge xax^{-1} \equiv 1}{a \equiv 1}$
4. $\dfrac{a \wedge b \preccurlyeq 1}{a \wedge xbx^{-1} \preccurlyeq 1}$

5. $a^{-1} \wedge xax^{-1} \preccurlyeq 1$.

It turns out [see Lor53, p. 23] that in any lattice group hold the following properties (without supposing regularity).

- $ab^{-1} \wedge ba^{-1} \preccurlyeq 1$.
- $c_1c_2^{-1} \wedge \cdots \wedge c_{n-1}c_n^{-1} \wedge c_nc_1^{-1} \preccurlyeq 1$.
- $a_1b_{\nu_1} \wedge \cdots \wedge a_mb_{\nu_m} \preccurlyeq a_{\mu_1}b_1 \vee \cdots \vee a_{\mu_n}b_n$ for any choice of $\nu_1, \ldots, \nu_m$ between 1 and $n$ and any choice of $\mu_1, \ldots, \mu_n$ between 1 and $m$.

# 7 "Die Erweiterung halbgeordneter Gruppen zu Verbandsgruppen" (1953), § 3.

In section 6, a regular lattice group $(\widetilde{G}, \preccurlyeq_{H_a})$ has been defined for every ideal semilattice preorder $\preccurlyeq_H$ for a preordered group $(G, \preccurlyeq_G)$. This lattice group is an ideal lattice domain for $G$ if the preorder $\preccurlyeq_{H_a}$ is an extension of the preorder of $G$: this is captured by

**Definition 7.1.** A group $G$ is $\preccurlyeq_H$*-closed* if

$$\frac{a \preccurlyeq_{G[\alpha_1^{\pm 1},\ldots,\alpha_l^{\pm 1}]_H} 1}{a \preccurlyeq_G 1}.$$

In the case of a field in which a relation of divisibility is defined by an integral domain $I$ and whose (commutative) multiplicative group $G$ is therefore associated to the ideal semilattice $(H_d, \preccurlyeq_d)$ of the Dedekind ideals of $I$, the $\preccurlyeq_d$-admissible preorders of $G$ are in bijection with the overrings for $I$. The preorder $\preccurlyeq_{G[\gamma]_d}$ corresponds for the pair $\gamma = (a, b)$ to the integral domain $I[a^{-1}b]$.

An element $a$ is $\preccurlyeq_d$-dependent from $I$ if and only if there are $c_1, \ldots, c_m$ such that $a \in I[c_1^{\pm 1}, \ldots, c_l^{\pm 1}]$ for every choice of signs; the condition of $\preccurlyeq_d$-closedness (the so-called "integral closedness") spells

$$\frac{a \in I[c_1^{\pm 1}, \ldots, c_l^{\pm 1}]}{a \in I}.$$

The definition of the regular lattice preorder $\preccurlyeq_{d_a}$ spells

$$\begin{aligned} a_1 \wedge \cdots \wedge a_m \preccurlyeq_{d_a} b_1 \vee \cdots \vee b_n \text{ iff } & \exists_{c_1,\ldots,c_e}\ 1 \in (a_1b_1^{-1}, \ldots, a_mb_n^{-1})I[c_1^{\pm 1}, \ldots, c_e^{\pm 1}] \\ \text{iff } & \exists_k\ 1 \in \sum_{\varkappa=1}^{k} (a_1b_1^{-1}, \ldots, a_mb_n^{-1})^{\varkappa}. \end{aligned} \tag{18}$$

*Remark* 7.2. The rôle of the hypothesis of $\preccurlyeq_d$-closedness here is not clear to me.

The last equivalence results from

**Theorem 7.3** ([Lor53, Satz 2, p. 24])**.**

$$\exists_{c_1,\ldots,c_e}\ 1 \in (a_1, \ldots, a_m)I[c_1^{\pm 1}, \ldots, c_e^{\pm 1}] \quad \textit{iff} \quad \exists_k\ 1 \in \sum_{\varkappa=1}^{k} (a_1, \ldots, a_m)^{\varkappa}.$$

This shows that in the case $n = 1$, the definition of $\preccurlyeq_{d_a}$ turns into the usual definition of integral dependence:

$$a_1 \wedge \cdots \wedge a_m \preccurlyeq_{d_a} b \text{ iff } b^k + c_1b^{k-1} + \cdots + c_k = 0 \text{ for some } k \text{ and } c_\varkappa \in (a_1, \ldots, a_m)^{\varkappa}.$$

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