Complete Log-Sobolev inequalities

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Quantum Markov semigroup

$(\mathcal{M}, \tau)$ finite von Neumann algebra with n.f. trace τ.
A quantum Markov semigroup (QMS) $(T_t)_{t\geq 0} : \mathcal{M} \to \mathcal{M}$ is a continuous family of maps satisfying

- T_t is normal unital completely positive
- $T_t \circ T_s = T_{s+t}, T_0 = \mathrm{id}$
- $t \mapsto T_t(x)$ is w^*-continuous for every $x \in \mathcal{M}$.
- (symmetric): $\tau(T_t(x)^* y) = \tau(x^* T_t(y))$.

Generator: $Ax = \lim_{t\to 0} -\frac{1}{t}(T_t(x) - x), T_t = e^{-At}$.

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Generator: $Ax = \lim_{t\to 0} -\frac{1}{t}(T_t(x) - x), T_t = e^{-At}$. Examples:

- Classical Markov semigroup: $\mathcal{M} = L_\infty(\Omega, \mu)$ probability space.
  E.g. $A = \Delta$ Laplacian on a compact Riemannian manifold (M, g).
- Quantum physics: $\mathcal{M} = (B(H), \mathrm{tr})$ dissipative open quantum systems.
  E.g. Lindbladian $A(x) = -\sum_j [a_j, [a_j, x]], a_j \in B(H)$
- Operator algebras: group von Neumann algebras, quantum group and q-deformed Gaussian.

Convergence and decoherence

Consider $\mathcal{M} = (M_d, \text{tr})$, $T_t = e^{-At} : M_d \to M_d$.
Fixed point space (subalgebra) $\mathcal{N} = \{x \in M_d \mid T_t(x) = x, \forall t\}$.
State space $S(M_d) = \{\rho \in M_d \mid \rho \geq 0\ , \text{tr}(\rho) = 1\}$. Then

$$T_t(\rho) \longrightarrow E_{\mathcal{N}}(\rho) \text{ as } t \to \infty$$

$E_{\mathcal{N}} : M_d \to \mathcal{N}$ conditional expectation, i.e. $\forall x \in \mathcal{N}, \text{tr}(x\rho) = \text{tr}(xE_{\mathcal{N}}(\rho))$.

How fast does the convergence $T_t(\rho) \longrightarrow E_{\mathcal{N}}(\rho)$ happen?

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Mixing time

$t_{\mathrm{mix}} = \inf\{t > 0 |\ \| T_t(\rho) - E_{\mathcal{N}}(\rho) \|_1 \leq 1/2\ , \forall \text{ state } \rho\}$

Spectral gap

$$\lambda_{\mathrm{gap}} := \inf_{E_{\mathcal{N}}(x)=0} \frac{\mathrm{tr}(x^* A x)}{\| x \|_2}$$ minimal positive eigenvalue A on $L_2(M_d)$.

- $\| T_t(x) - E_{\mathcal{N}}(x) \|_2 \leq e^{-\lambda_{\mathrm{gap}} t} \| x - E_{\mathcal{N}}(x) \|_2 \Rightarrow t_{\mathrm{mix}} \lesssim \frac{1}{\lambda_{\mathrm{gap}}} O(\ln d)$.

Modified Log-Sobolev Inequality

Relative entropy: for two quantum states ρ, σ,

$$D(\rho||\sigma) := \mathrm{tr}(\rho \log \rho - \rho \log \sigma)$$

Then $T_t(\rho) \to E_{\mathcal{N}}(\rho) \Longrightarrow D(T_t(\rho)||E_{\mathcal{N}}(\rho)) \to 0$.

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Definition (Bardet '17)

We say $T_t = e^{-At}$ satisfies λ-modified log-Sobolev inequality (λ-MLSI) for $\lambda > 0$ if

$$2\lambda \; D(\rho||E_{\mathcal{N}}(\rho)) \le \mathrm{tr}(A\rho \ln \rho) \; , \; \forall \rho \in S(M_d) \; .$$

Fisher information: $I_A(\rho) := \mathrm{tr}(A(\rho) \ln \rho) = -\frac{d}{dt}|_{t=0} D(T_t(\rho)||E_{\mathcal{N}}(\rho))$.
Then λ-MLSI means:

$$\frac{d}{dt} D(T_t(\rho)||E_{\mathcal{N}}(\rho)) \le -2\lambda D(T_t(\rho)||E_{\mathcal{N}}(\rho))$$
$$\iff D(T_t(\rho)||E_{\mathcal{N}}(\rho)) \le e^{-2\lambda t} D(\rho||E_{\mathcal{N}}(\rho)) \; .$$

λ-MLSI $\iff$ Exponential decay of relative entropy $\Rightarrow t_{\text{mix}} \le \frac{1}{\lambda} O(\ln(\ln d))$

Connection to other functional inequalities

L_2-Log-Sobolev inequality (LSI): for an ergodic semigroup $T_t = e^{-At}$ ($\mathcal{N} = \mathbb{C}1$),

$$\mathrm{tr}(x^2 \ln x^2) - \mathrm{tr}(x^2) \ln \mathrm{tr}(x^2) \le \frac{1}{\lambda} \mathrm{tr}(x^* A x)$$

$\Longleftrightarrow$ Hypercontractivity: (Olkiewicz-Zegarlinski '99)

$$\| T_t : L_2 \to L_p \| \le 1 \text{ for } p \le 1 + e^{2\lambda t}$$

(Temme-Kastoryano '13 & Bardet '17) For the optimal constant,

$$\lambda_{\mathrm{LSI}} \le \lambda_{\mathrm{MLSI}} \le \lambda_{\mathrm{gap}}$$

Moreover,

- MLSI $\Longrightarrow$ Transport cost inequality $\Longrightarrow$ concentration of measure. (Datta-Rouzé, Carlen-Maas '18)

Tensorization and complete MLSI

Tensorization

$T_t, S_t : L_\infty(\Omega) \to L_\infty(\Omega)$ has λ-MLSI $\Longrightarrow T_t \otimes S_t$ λ-MLSI (same for LSI)

- Unknown for MLSI in quantum cases. True for LSI on M_2 (King '14).

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Definition

$T_t = e^{-At}$ satisfies λ-complete log-Sobolev inequality (λ-CLSI) for $\lambda > 0$ if $\text{id}_{M_n} \otimes T_t$ satisfies λ-MLSI for all $n \geq 1$.

Tensorization of CLSI

$T_t, S_t : M_d \to M_d$ both has λ-CLSI $\Longrightarrow T_t \otimes S_t$ has λ-CLSI

- Applications in quantum lattice spin system.
- Estimating decay of entanglement: for a bipartite state ρ on $M_n \otimes M_d$,

$$D(\text{id}_n \otimes T_t(\rho) || \text{id}_n \otimes E_{\mathcal{N}}(\rho)) \to 0$$

Example: Depolarizing semigroup

Qubit depolarizing semigroup:

$$T_t : M_2 \to M_2, T_t(\rho) = e^{-t}\rho + (1 - e^{-t})\mathsf{tr}(\rho)\frac{1}{2}$$

- Optimal constant:

$$1/2 \le \lambda_{\mathsf{CLSI}}(T_t) \le \lambda_{\mathsf{MLSI}}(T_t \otimes \mathsf{id}_{M_2}) < \lambda_{\mathsf{MLSI}}(T_t) = \lambda_{\mathsf{LSI}}(T_t) = 1 \ .$$

  In particular, $\lambda_{\mathsf{CLSI}} \neq \lambda_{\mathsf{MLSI}}$.

- $\lambda_{\mathsf{LSI}}(T_t \otimes \mathsf{id}_{M_2}) = 0$. In general, LSI/Hypercontractivity fails for non ergodic T_t whenever $\mathcal{N}$ is noncommutative, hence for $\mathsf{id}_{M_n} \otimes T_t$.

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Question

Does every finite dimensional $T_t : M_d \to M_d$ has

$$\lambda_{\mathsf{CLSI}} := \inf_n \lambda_{\mathsf{MLSI}}(\mathrm{id}_{M_n} \otimes T_t) > 0?$$

- (Junge-Li-LaRacuente, '20) True for classical cases $T_t : l_\infty^d \to l_\infty^d$.

Finite dimensional results

Theorem. (G.-Rouzé, preprint '21)

Let $T_t = e^{-At} : M_d \to M_d$ be a symmetric Quantum Markov semigroup and $\mathcal{N}$ be the fixed point subalgebra. Then

$$\frac{\lambda_{gap}}{2C_{cb}(M_d : \mathcal{N})} \leq \lambda_{\text{CLSI}} \leq \lambda_{gap}$$

- $C(M_d : \mathcal{N})$ Pimsner-Popa index.
- Remain valid for GNS-symmetric semigroup: for a faithful state σ,

$$\text{tr}(T_t(x)^* y\sigma) = \text{tr}(x^* T_t(y)\sigma)$$

Pimsner-Popa index

$\mathcal{N} \subset \mathcal{M}$ finite von Neumann algebra. $E_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ cond. expectation.

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- (Pimsner-Popa, '86) $C(\mathcal{M} : \mathcal{N}) = [\mathcal{M} : \mathcal{N}]$ for II_1 subfactor & Explicit formula for finite dimensional $\mathcal{M}, \mathcal{N}$.
- CB-version: $C_{cb}(\mathcal{M} : \mathcal{N}) = \sup_n C(M_n(\mathcal{M}) : M_n(\mathcal{N}))$.
- E.g. $C(M_d : \mathbb{C}) = d,\; C_{cb}(M_d : \mathbb{C}) = d^2$ and

$$C_{cb}(M_d : \mathcal{N}) \leq C_{cb}(M_d : \mathbb{C}) = d^2.$$

Quantum χ_2-divergence

For $\rho \in S(M_d)$, define ρ weighted L_2-norm

$$\| X \|_{\rho^{-1}}^2 = \int_0^\infty \mathrm{tr}\big(X^* \frac{1}{\rho + s} X \frac{1}{\rho + s}\big) ds \ .$$

- commutative case, $\| X \|_{\rho^{-1}}^2 = \int \frac{|X|^2}{\rho} d\mu$
- for two quantum states, $\| \rho - \sigma \|_{\sigma^{-1}}^2 := \chi_2(\rho, \sigma)$ quantum χ_2-divergence.

Quantum χ_2-divergence

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Key Lemma

Recall the relative entropy $D(\rho||\sigma) = \text{tr}(\rho \log \rho - \rho \log \sigma)$.

i) $D(\rho||\sigma) \leq \| \rho - \sigma \|_{\sigma^{-1}}^2$

ii) If $\rho \leq C\sigma$, $C \| X \|_{\rho^{-1}}^2 \geq \| X \|_{\sigma^{-1}}^2$

Proof of Theorem.

Denote $\rho_{\mathcal{N}} = E_{\mathcal{N}}(\rho)$. Recall λ-MLSI is

$$2\lambda\, D(\rho||\rho_{\mathcal{N}}) \le I_A(\rho)$$

On M_d, $T_t = e^{-At}$ is given by $A(x) = -\sum_{j=1}^{k}[a_j,[a_j,x]]$. Then

$$A = \delta^*\delta\ ,\ \delta(x) = \oplus_{j=1}^{k}[a_j, x]$$

is a derivation. Then

$$\begin{aligned} I_A(\rho) =& \mathrm{tr}(A\rho\log\rho) = \mathrm{tr}(\delta(\rho)\delta(\log\rho)) \\ =& \int_0^\infty \mathrm{tr}(\delta(\rho)\frac{1}{\rho+s}\delta(\rho)\frac{1}{\rho+s})ds = \|\,\delta(\rho)\,\|^2_{\rho^{-1}} \\ D(\rho||\rho_{\mathcal{N}}) \le& \|\,\rho-\rho_{\mathcal{N}}\,\|^2_{\rho_{\mathcal{N}}^{-1}} \le \frac{1}{\lambda_{\mathrm{gap}}}\ \|\,\delta(\rho-\rho_{\mathcal{N}})\,\|^2_{\rho_{\mathcal{N}}^{-1}} \\ =& \frac{1}{\lambda_{\mathrm{gap}}}\ \|\,\delta(\rho)\,\|^2_{\rho_{\mathcal{N}}^{-1}} = \frac{C}{\lambda_{\mathrm{gap}}}\ \|\,\delta(\rho)\,\|^2_{\rho^{-1}} \quad (\rho \le C\rho_{\mathcal{N}}) \end{aligned}$$

Discussion on optimality

Since $C(M_d : \mathcal{N}) \le d, C_{cb}(M_d : \mathcal{N}) \le d^2$, we have shown

$$\frac{\lambda_{gap}}{2d} \le \frac{\lambda_{gap}}{2C(M_d : \mathcal{N})} \le \lambda_{\mathsf{MLSI}} \le \lambda_{gap}$$

$$\frac{\lambda_{gap}}{2d^2} \le \frac{\lambda_{gap}}{2C_{cb}(M_d : \mathcal{N})} \le \lambda_{\mathsf{CLSI}} \le \lambda_{gap}$$

- ergodic case: $\mathcal{N} = \mathbb{C}1$

$$\frac{\lambda_{gap}(T_t)}{\log d + 2} \le \lambda_{\mathsf{MLSI}}(T_t) \le \lambda_{gap}(T_t)$$ (Temme-Kastoryano '13)

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- ergodic case: $\mathcal{N} = \mathbb{C}1$

$$\frac{\lambda_{gap}(T_t)}{\log d + 2} \le \lambda_{\mathsf{MLSI}}(T_t) \le \lambda_{gap}(T_t) \qquad \text{(Temme-Kastoryano '13)}$$

Question

Can we have $\lambda_{\mathsf{MLSI}} \ge \lambda_{\mathsf{gap}} O(\frac{1}{\log d})$ or even $\lambda_{\mathsf{CLSI}} \ge \lambda_{\mathsf{gap}} O(\frac{1}{\log d})$?

- (Brannan-G.-Junge, '20) Ture for Schur multiplier and some Random unitary.

Heat Semigroup

- (Li-Junge-LaRacuente,'20) For a compact Riemannian Manifold (M, g), if Ricci$(M) \geq \lambda > 0$, the Heat semigroup $H_t = e^{-\Delta t}$ satisfy λ-CLSI.
- (Brannan-G.-Junge, '20) The Heat semigroup $H_t = e^{-\Delta t}$ on every compact Riemannian Manifold has $\lambda_{\text{CLSI}} > 0$.

Example: Unit Circle

$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and $H_t(z^m) = e^{-m^2 t} z^m$.

$$\frac{1}{4 \ln 3} \leq \lambda_{\text{CLSI}}(H_t) \leq \lambda_{\text{MLSI}}(H_t) = \lambda_{\text{LSI}}(H_t) = 1$$

Question

Does $\lambda_{\text{CLSI}} = \lambda_{\text{MLSI}}$ for classical semigroup?

Hypo-elliptic case

- (G.-Junge-LaRacuente, '18) For a sub-Laplacian $\Delta_X = -\sum_j X_j^2$ satisfying Hörmander condition, the subordinate semirgoup $e^{-\Delta_X^\theta t}$ satisfy has $\lambda_{\mathsf{CLSI}} > 0$ for $0 < \theta < 1$.

Example:Possion semigroup

$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and $P_t(z^m) = e^{-|m|t} z^m$.

$$\lambda_{\mathsf{CLSI}}(P_t) = \lambda_{\mathsf{MLSI}}(P_t) = \lambda_{\mathsf{LSI}}(P_t) = \lambda_{gap}(P_t) = 1$$

Question

Does the sub-Laplacian Δ_X itself satisfy CLSI?

- Implies dimension free estimate for all tranference semigroup of Δ_X
- No example known, even $SU(2)$.

Compact Quantum group

Central semigroup on compact quantum group has Ricci ≥ 0.

$$\|\delta(T_t(x))\|_{\rho^{-1}} \leq e^{-\lambda t} \|\delta(x)\|_{T_t(\rho)^{-1}} \qquad \text{(Ricci} \geq \lambda\text{)}$$

- Free orthogonal group O_N^+ & quantum permutation group S_N^+: Heat semigroup have $\lambda_{\text{CLSI}} \geq O(\frac{1}{N \log N})$.
- Group von Neumann algebra: Fourier multiplier $T_t(\lambda(g)) = e^{-\phi(g)t}\lambda(g)$ has $\lambda_{\text{CLSI}} > 0$ under some growth condition on ϕ.

Theorem (Brannan-G.-Junge & Wirth-Zhang, '20)

$\mathbb{F}_d$ free group and $P_t(\lambda(g)) = e^{-|g|t}\lambda(g)$.

$$\lambda_{\text{Ricci}}(P_t) = \lambda_{\text{CLSI}}(P_t) = \lambda_{gap}(P_t) = 1$$

- LSI constant $\lambda_{\text{LSI}}(P_t) \geq 1.17$ (Junge-Palazuelos-Parcet-Perrin-Ricard)

(1) Optimal Hypercontractivity for P_t?
(2) Positive curvature for O_N^+?

Thank you for listening!