# ASYMPTOTIC AND COARSE LIPSCHITZ STRUCTURES OF QUASI-REFLEXIVE BANACH SPACES

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Abstract. In this note, we extend to the setting of quasi-reflexive spaces a classical result of N. Kalton and L. Randrianarivony on the coarse Lipschitz structure of reflexive and asymptotically uniformly smooth Banach spaces. As an application, we show for instance, that for $1 \leq q < p$, a $q$-asymptotically uniformly convex Banach space does not coarse Lipschitz embed into a $p$-asymptotically uniformly smooth quasi-reflexive Banach space. This extends a recent result of B.M. Braga.

## 1. Introduction.

We start this note with some basic definitions on metric embeddings.
Let $(M,d)$ and $(N,\delta)$ be two metric spaces and $f$ be a map from $M$ into $N$. We define the *compression modulus* of $f$ by

$$\rho_f(t) = \inf\big\{\delta(f(x),f(y)),\ d(x,y) \geq t\big\},$$

and the *expansion modulus* of $f$ by

$$\omega_f(t) = \sup\big\{\delta(f(x),f(y)),\ d(x,y) \leq t\big\}.$$

We say that $f$ is a *Lipschitz embedding* if there exist $A, B$ in $(0,\infty)$ such that $\omega_f(t) \leq Bt$ and $\rho_f(t) \geq At$.
If the metric space $M$ is unbounded, we say that $f$ is a *coarse embedding* if $\lim_{t\to\infty} \rho_f(t) = \infty$ and $\omega_f(t) < \infty$ for all $t > 0$. Note that if $M$ is a Banach space, we have automatically in that case that $\omega_f$ is dominated by an affine function. This last observation can be deduced from the proof of Proposition 1.11 in [3], which in turn can be found in the earlier paper [5].
We say that $f$ is a *coarse Lipschitz embedding* if there exist $A, B, C, D$ in $(0,+\infty)$ such that $\omega_f(t) \leq Bt + D$ and $\rho_f(t) \geq At - C$.
In order to refine the scale of coarse embeddings, E. Guentner and J. Kaminker introduced in [8] the following notion. Let $X$ and $Y$ be two Banach spaces. We define $\alpha_Y(X)$ as the supremum of all $\alpha \in [0,1)$ for which there exists a

2010 *Mathematics Subject Classification.* 46B80, 46B20.
*Key words and phrases.* coarse Lipschitz embeddings, quasi-reflexive Banach spaces, asymptotic structure of Banach spaces.
The first named author was supported by the French "Investissements d'Avenir" program, project ISITE-BFC (contract ANR-15-IDEX-03) and as a participant of the "NSF Workshop in Analysis and Probability" at Texas A&M University.
The second named author was partially supported by the grants MINECO/FEDER MTM2014-57838-C2-1-P and Fundación Séneca CARM 19368/PI/14.

coarse embedding $f : X \to Y$ and $A, C$ in $(0,\infty)$ so that $\rho_f(t) \geq At^\alpha - C$ for all $t > 0$. Then, $\alpha_Y(X)$ is called the *compression exponent of $X$ in $Y$*.

We now turn to the definitions of the uniform asymptotic properties of norms that will be considered in this paper. For a Banach space $(X, \| \ \|)$ we denote by $B_X$ the closed unit ball of $X$ and by $S_X$ its unit sphere. The following definitions are due to V. Milman [14] and we follow the notation from [10]. For $t \in [0,\infty)$, $x \in S_X$ and $Y$ a closed linear subspace of $X$, we define

$$\overline{\rho}_X(t,x,Y) = \sup_{y\in S_Y} \big(\|x+ty\|-1\big) \quad \text{and} \quad \overline{\delta}_X(t,x,Y) = \inf_{y\in S_Y} \big(\|x+ty\|-1\big).$$

Then

$$\overline{\rho}_X(t,x) = \inf_{\dim(X/Y)<\infty} \overline{\rho}_X(t,x,Y) \quad \text{and} \quad \overline{\delta}_X(t,x) = \sup_{\dim(X/Y)<\infty} \overline{\delta}_X(t,x,Y)$$

and

$$\overline{\rho}_X(t) = \sup_{x\in S_X} \overline{\rho}_X(t,x) \quad \text{and} \quad \overline{\delta}_X(t) = \inf_{x\in S_X} \overline{\delta}_X(t,x).$$

The norm $\| \ \|$ is said to be *asymptotically uniformly smooth* (in short AUS) if

$$\lim_{t\to 0} \frac{\overline{\rho}_X(t)}{t} = 0.$$

It is said to be *asymptotically uniformly convex* (in short AUC) if

$$\forall t > 0 \quad \overline{\delta}_X(t) > 0.$$

Let $p \in (1,\infty)$ and $q \in [1,\infty)$.
We say that the norm of $X$ is *$p$-AUS* if there exists $c > 0$ such that for all $t \in [0,\infty)$, $\overline{\rho}_X(t) \leq ct^p$.
We say that the norm of $X$ is *$q$-AUC* if there exits $c > 0$ such that for all $t \in [0,1]$, $\overline{\delta}_X(t) \geq ct^q$.
Similarly, there is in $X^*$ a modulus of weak$^*$ asymptotic uniform convexity defined by

$$\overline{\delta}^*_X(t) = \inf_{x^*\in S_{X^*}} \sup_{E} \inf_{y^*\in S_E} \big(\|x^*+ty^*\|-1\big),$$

where $E$ runs through all weak$^*$-closed subspaces of $X^*$ of finite codimension. The norm of $X^*$ is said to be *weak$^*$ uniformly asymptotically convex* (in short weak$^*$-AUC) if $\overline{\delta}^*_X(t) > 0$ for all $t$ in $(0,\infty)$. If there exists $c > 0$ and $q \in [1,\infty)$ such that for all $t \in [0,1]$ $\overline{\delta}^*_X(t) \geq ct^q$, we say that the norm of $X^*$ is $q$-weak$^*$-AUC.

Let us recall the following classical duality result concerning these moduli (see for instance [6] Corollary 2.3 for a precise statement).

**Proposition 1.1.** *Let $X$ be a Banach space.*
*Then $\| \ \|_X$ is AUS if and and only if $\| \ \|_{X^*}$ is weak$^*$-AUC.*
*If $p, q \in (1,\infty)$ are conjugate exponents, then $\| \ \|_X$ is $p$-AUS if and and only if $\| \ \|_{X^*}$ is $q$-weak$^*$-AUC.*

The main purpose of this note is to extend to the quasi-reflexive case an important result obtained by N. Kalton and L. Randrianarivony in [12] about coarse Lipschitz embeddings into $p$-AUS reflexive spaces. In order to explain their result, we need to introduce special metric graphs that we shall call *Kalton-Randrianarivony's graphs.* For an infinite subset $\mathbb{M}$ of $\mathbb{N}$ and $k \in \mathbb{N}$, we denote

$$G_k(\mathbb{M}) = \{\overline{n} = (n_1, .., n_k),\ n_i \in \mathbb{M}\ \ n_1 < .. < n_k\}.$$

Then we equip $G_k(\mathbb{M})$ with the distance $d(\overline{n}, \overline{m}) = |\{j,\ n_j \neq m_j\}|$. The fundamental result of their paper (Theorem 4.2 in [12]) can be rephrased as follows.

**Theorem 1.2. (Kalton-Randrianarivony 2008)** *Let $p \in (1, \infty)$ and assume that $Y$ is a reflexive $p$-AUS Banach space. Then there exists a constant $C > 0$ such that for any $k \in \mathbb{N}$, any infinite subset $\mathbb{M}$ of $\mathbb{N}$, any $f : (G_k(\mathbb{M}), d) \to Y$ Lipschitz map and any $\varepsilon > 0$, there exists an infinite subset $\mathbb{M}'$ of $\mathbb{M}$, such that*

$$diam\ f(G_k(\mathbb{M}')) \leq CLip(f)k^{1/p} + \varepsilon.$$

We refer the reader to [12] and [11] for the various applications derived by the authors. Very recently, B.M. Braga used the above theorem in [4] to develop many other applications. We will only mention one of them in this introduction (Corollary 4.5 in [4]).

**Theorem 1.3. (Braga 2016)** *Let $1 \leq q < p$, $X$ be a $q$-AUC Banach space and $Y$ be a $p$-AUS reflexive Banach space. Then $\alpha_Y(X) \leq q/p$.*

We recall that a Banach space is said to be *quasi-reflexive* if the image of its canonical embedding into its bidual is of finite codimension in its bidual. The aim of this note is to obtain a version of the above Theorem 1.2 for quasi-reflexive Banach spaces. This is done in section 2 and our main results are Theorem 2.2 and Theorem 2.4. In section 3, we apply them in order to extend some results from [4] to the quasi-reflexive setting, including the above Theorem 1.3. Our applications are stated in Corollary 3.2, Theorem 3.5, Corollary 3.6 and Corollary 3.8.

## 2. The main result.

We first need the following simple property of the bidual of a Banach space $X$, in relation with the modulus of asymptotic uniform smoothness of $X$.

**Proposition 2.1.** *Let $X$ be a Banach space. Then the bidual norm on $X^{**}$ has the following property. For any $t \in (0, 1)$, any weak$^*$-null sequence $(x_n^{**})_{n=1}^\infty$ in $B_{X^{**}}$ and any $x \in S_X$ we have:*

$$\limsup_{n\to\infty} \|x + tx_n^{**}\| \leq 1 + \overline{\rho}_X(t, x).$$

*Proof.* Let $x$ in $S_X$ and $\varepsilon > 0$. By definition, there exists a finite codimensional subspace $Y$ of $X$ such that

$$x + tB_Y \subset (1 + \overline{\rho}_X(t,x) + \varepsilon)B_X. \tag{1}$$

There exist $x_1^*, .., x_k^* \in X^*$ such that

$$Y = \bigcap_{i=1}^{k} \{x \in X,\ x_i^*(x) = 0\}.$$

Note that the weak$^*$-closure of $Y$ in $X^{**}$ is

$$\overline{Y}^{w^*} = Y^{\perp\perp} = \bigcap_{i=1}^{k} \{x^{**} \in X^{**},\ x^{**}(x_i^*) = 0\}.$$

By Goldstine's Theorem we have that the weak$^*$-closures in $X^{**}$ of $B_Y$ and $B_X$ are respectively $B_{\overline{Y}^{w^*}}$ and $B_{X^{**}}$. So, taking the weak$^*$-closures in (1) we get that

$$x + tB_{\overline{Y}^{w^*}} \subset (1 + \overline{\rho}_X(t,x) + \varepsilon)B_{X^{**}}.$$

Since the sequence $(x_n^{**})_{n=1}^{\infty}$ is weak$^*$-null, we have that for all $i$ in $\{1, .., k\}$, $\lim_{n\to\infty} x_n^{**}(x_i^*) = 0$. It follows easily that $\lim_{n\to\infty} d(x_n^{**}, B_{\overline{Y}^{w^*}}) = 0$. We deduce, that for $n$ large enough

$$x + tx_n^{**} \in (1 + \overline{\rho}_X(t) + 2\varepsilon)B_{X^{**}},$$

which concludes our proof. □

We shall now give an analogue of Theorem 1.2 when $Y$ is only assumed to be quasi-reflexive and $p$-AUS for some $p \in (1,\infty)$. The idea is to adapt techniques from a work by F. Nétillard [15] on the coarse Lipschitz embeddings between James spaces. To this end, for $\mathbb{M}$ an infinite subset of $\mathbb{N}$, we denote $I_k(\mathbb{M})$ the set of strictly interlaced pairs in $G_k(\mathbb{M})$, namely :

$$I_k(\mathbb{M}) = \big\{(\overline{n}, \overline{m}) \in G_k(\mathbb{M}) \times G_k(\mathbb{M}),\ n_1 < m_1 < n_2 < m_2 < ... < n_k < m_k\big\}.$$

Note that for $(\overline{n}, \overline{m}) \in I_k(\mathbb{M})$, $d(\overline{n}, \overline{m}) = k$. Our statement is then the following.

**Theorem 2.2.** *Let $p \in (1,\infty)$ and $Y$ be a quasi-reflexive $p$-AUS Banach space. Then there exists a constant $C > 0$ such that for any $k \in \mathbb{N}$, any infinite subset $\mathbb{M}$ of $\mathbb{N}$, any $f : (G_k(\mathbb{M}), d) \to Y^{**}$ Lipschitz and any $\varepsilon > 0$ there exists an infinite subset $\mathbb{M}'$ of $\mathbb{M}$, such that*

$$\forall (\overline{n}, \overline{m}) \in I_k(\mathbb{M}') \quad \|f(\overline{n}) - f(\overline{m})\| \le C Lip(f) k^{\frac{1}{p}} + \varepsilon.$$

In fact, we will show a more general result, which follows the ideas of section 6 in [12]. Before to state it, we need some preparation. We briefly recall the setting of section 6 in [12].

So let $Y$ be a Banach space and denote by $\overline{\rho}_Y$ its modulus of asymptotic uniform smoothness. It is easily checked that $\overline{\rho}_Y$ is an Orlicz function. Then we define the Orlicz sequence space:

$$\ell_{\overline{\rho}_Y} = \Big\{x \in \mathbb{R}^{\mathbb{N}},\ \exists r > 0\ \ \sum_{n=1}^{\infty} \overline{\rho}_Y\big(\frac{|x_n|}{r}\big) < \infty\Big\},$$

equipped with the norm

$$\|x\|_{\overline{\rho}_Y} = \inf\Big\{r > 0,\ \sum_{n=1}^{\infty} \overline{\rho}_Y\big(\frac{|x_n|}{r}\big) \leq 1\Big\}.$$

Next we construct a sequence of norms $(N_k)_{k=1}^{\infty}$, where $N_k$ is a norm on $\mathbb{R}^k$, as follows.
For all $\xi \in \mathbb{R}$, $N_1(\xi) = |\xi|$.
$N_2(\xi, \eta) = |\eta|$ if $\xi = 0$ and

$$N_2(\xi, \eta) = |\xi|\Big(1 + \overline{\rho}_Y\big(\frac{|\eta|}{|\xi|}\big)\Big) \ \text{ if } \xi \neq 0.$$

Then, for $k \geq 3$, we define by induction the following norm on $\mathbb{R}^k$:

$$N_k(\xi_1, .., \xi_k) = N_2\big(N_{k-1}(\xi_1, .., \xi_{k-1}), \xi_k\big).$$

The following property is proved in [12].

**Proposition 2.3.** *For any $k \in \mathbb{N}$ and any $a \in \mathbb{R}^k$:*

$$N_k(a) \leq e\|a\|_{\overline{\rho}_Y}.$$

Fix now $a = (a_1, .., a_k)$ a sequence of non zero real numbers and define the following distance on $G_k(\mathbb{M})$, for $\mathbb{M}$ infinite subset of $\mathbb{N}$:

$$\forall \overline{n}, \overline{m} \in G_k(\mathbb{M}), \quad d_a(\overline{n}, \overline{m}) = \sum_{j,\ n_j \neq m_j} |a_j|.$$

We can now state our general result, from which Theorem 2.2 is easily deduced.

**Theorem 2.4.** *Let $Y$ be a quasi-reflexive Banach space, $a = (a_1, .., a_k)$ a sequence of non zero real numbers, $\mathbb{M}$ an infinite subset of $\mathbb{N}$ and let $f : (G_k(\mathbb{M}), d_a) \to Y^{**}$ be a Lipschitz map.*
*Then for any $\varepsilon > 0$ there exists an infinite subset $\mathbb{M}'$ of $\mathbb{M}$, such that*

$$\forall (\overline{n}, \overline{m}) \in I_k(\mathbb{M}') \quad \|f(\overline{n}) - f(\overline{m})\| \leq 2eLip(f)\|a\|_{\overline{\rho}_Y} + \varepsilon.$$

*Proof.* Under the assumptions of Theorem 2.4, we will show that there exists an infinite subset $\mathbb{M}'$ of $\mathbb{M}$, such that

$$\forall (\overline{n}, \overline{m}) \in I_k(\mathbb{M}') \quad \|f(\overline{n}) - f(\overline{m})\| \leq 2\mathrm{Lip}(f)N_k(a) + \varepsilon.$$

Then, the conclusion will follow from Proposition 2.3.

Since the graphs $G_k(\mathbb{N})$ are all countable, we may assume that $Y$ is separable. We can write $Y^{**} = Y \oplus E$, where $E$ is finite dimensional.

We will prove our statement by induction on $k \in \mathbb{N}$. It is clearly true for $k = 1$, so assume it is true for $k \in \mathbb{N}$ and let $a = (a_1, .., a_{k+1})$ be a sequence of

non zero real numbers and $f:(G_{k+1}(\mathbb{M}),d_a)\to Y^{**}$ be a Lipschitz map. Let $\varepsilon>0$ and fix $\eta\in(0,\frac{\varepsilon}{2})$ (our initial choice of a small $\eta$ will be made precise later).
Since $Y$ is separable and quasi-reflexive, $Y^*$ is also separable. So, using weak*-compactness in $Y^{**}$, we can find an infinite subset $\mathbb{M}_0$ of $\mathbb{M}$ such that

$$\forall \overline{n}\in G_k(\mathbb{M}_0)\qquad w^*-\lim_{n_{k+1}\in\mathbb{M}_0} f(\overline{n},n_{k+1})=g(\overline{n})\in Y^{**}.$$

Using the weak* lower semi continuity of $\|\ \|_{Y^{**}}$ we get that the map $g:G_k(\mathbb{M}_0,d_{(a_1,..a_k)})\to Y^{**}$ satisfies $\mathrm{Lip}(g)\le\mathrm{Lip}(f)$. For $\overline{n}\in G_k(\mathbb{M}_0)$, we can write $g(\overline{n})=h(\overline{n})+e(\overline{n})$, with $h(\overline{n})\in Y$ and $e(\overline{n})\in E$. It then follows from Ramsey's theorem and the norm compactness of bounded sets in $E$ that there exists an infinite subset $\mathbb{M}_1$ of $\mathbb{M}_0$ such that

$$\forall \overline{n},\overline{m}\in G_k(\mathbb{M}_1)\quad \|e(\overline{n})-e(\overline{m})\|\le\eta.$$

For $\overline{n},\overline{m}\in G_k(\mathbb{M}_1)$ and $t,l\in\mathbb{M}_1$, set

$$u_{\overline{n},\overline{m},t,l}=\big(f(\overline{n},t)-g(\overline{n})\big)-\big(f(\overline{m},l)-g(\overline{m})\big).$$

Since $\|\ \|_{Y^{**}}$ is weak* lower semi continuous, we have that

$$\|u_{\overline{n},\overline{m},t,l}\|\le 2\mathrm{Lip}(f)|a_{k+1}|.$$

On the other hand, it follows from our induction hypothesis that there exists an infinite subset $\mathbb{M}_2$ of $\mathbb{M}_1$ such that

$$\forall(\overline{n},\overline{m})\in I_k(\mathbb{M}_2)\quad \|g(\overline{n})-g(\overline{m})\|\le 2\mathrm{Lip}(f)N_k(a_1,..,a_k)+\eta.$$

Therefore

$$\forall(\overline{n},\overline{m})\in I_k(\mathbb{M}_2)\quad \|h(\overline{n})-h(\overline{m})\|\le 2\mathrm{Lip}(f)N_k(a_1,..,a_k)+2\eta.$$

Assume first that $h(\overline{n})\neq h(\overline{m})$. Then it follows from Proposition 2.1 and the fact that $u_{\overline{n},\overline{m},t,l}$ is tending to 0 in the weak* topology, as $t,l$ tend to $\infty$, that there exists $N_{(\overline{n},\overline{m})}\in\mathbb{M}_2$ such that for all $t,l\in\mathbb{M}_2$ satisfying $t,l\ge N_{(\overline{n},\overline{m})}$:

$$\begin{aligned}\|h(\overline{n})-h(\overline{m})+u_{\overline{n},\overline{m},t,l}\|&\le\|h(\overline{n})-h(\overline{m})\|\Big(1+\overline{\rho}_Y\big(\frac{2\mathrm{Lip}(f)|a_{k+1}|}{\|h(\overline{n})-h(\overline{m})\|}\big)\Big)+\eta\\&\le N_2\big(\|h(\overline{n})-h(\overline{m})\|,2\mathrm{Lip}(f)|a_{k+1}|\big)+\eta.\end{aligned}$$

Note that if $h(\overline{n})=h(\overline{m})$, the above inequality is clearly true for all $t,l\in\mathbb{M}_2$. Therefore, we have that for all $(\overline{n},\overline{m})\in I_k(\mathbb{M}_2)$ there exists $N_{(\overline{n},\overline{m})}\in\mathbb{M}_2$ such that for all $t,l\ge N_{(\overline{n},\overline{m})}$:

$$\begin{aligned}\|h(\overline{n})-h(\overline{m})+u_{\overline{n},\overline{m},t,l}\|&\le N_2\big(2\mathrm{Lip}(f)N_k(a_1,..,a_k)+2\eta,2\mathrm{Lip}(f)|a_{k+1}|\big)+\eta\\&\le 2\mathrm{Lip}(f)N_{k+1}(a_1,..,a_{k+1})+\frac{\varepsilon}{2},\end{aligned}$$

if $\eta$ was initially chosen small enough.
So we have proved that for all $(\overline{n},\overline{m})\in I_k(\mathbb{M}_2)$ there exist $N_{(\overline{n},\overline{m})}\in\mathbb{M}_2$ such that for all $t,l\ge N_{(\overline{n},\overline{m})}$:

$$\|f(\overline{n},t)-f(\overline{m},l)\|\le 2\mathrm{Lip}(f)N_{k+1}(a_1,..,a_{k+1})+\varepsilon. \tag{2}$$

We now wish to construct $\mathbb{M}'$ infinite subset of $\mathbb{M}_2$ satisfying our conclusion. For that purpose, for a finite subset $F$ of $\mathbb{N}$ of cardinality at least $2k$, we denote $I_k(F)$ the set of all $(\overline{n},\overline{m}) \in I_k(\mathbb{N})$ such that $n_1 < m_1 < .. < n_k < m_k \in F$. Assume now $\mathbb{M}_2 = \{n_1 < .. < n_j < ..\}$. We shall define inductively the elements of our set $\mathbb{M}' = \{m_1 < ... < m_j < ..\}$.
First, we set $m_1 = n_1, ..., m_{2k} = n_{2k}$. Then, for $j > 2k$, we define $m_j = n_{\phi(j)} > m_{j-1}$ in such a way that for all $(\overline{n},\overline{m}) \in I_k(\{m_1,..,m_{j-1}\})$ we have that $m_j \geq N_{(\overline{n},\overline{m})}$. Then, it should be clear from equation (2) and the construction of $\mathbb{M}'$ that

$$\forall (\overline{n},\overline{m}) \in I_{k+1}(\mathbb{M}') \quad \|f(\overline{n}) - f(\overline{m})\| \leq 2\mathrm{Lip}(f) N_{k+1}(a_1,..,a_{k+1}) + \varepsilon.$$

This finishes our induction. □

**Remark.** Suppose now that $Y$ is a non reflexive Banach space and fix $\theta \in (0,1)$. Then, James' Theorem (see [9]) insures the existence of a sequence $(x_n)_{n=1}^\infty$ in $S_X$ and a sequence $(x_n^*)_{n=1}^\infty$ in $S_{X^*}$ such that

$$x_n^*(x_i) = \theta \text{ if } n \leq i \quad \text{and} \quad x_n^*(x_i) = 0 \text{ if } n > i.$$

In particular, for all $n_1 < .. < n_k < m_1 < .. < m_k$:

$$\|x_{n_1} + .. + x_{n_k} - (x_{m_1} + .. + x_{m_k})\| \geq \theta k. \tag{3}$$

Define now, for $k \in \mathbb{N}$ and $\overline{n} \in G_k(\mathbb{N})$, $f(\overline{n}) = x_{n_1} + .. + x_{n_k}$. Then $f$ is clearly 2-Lipschitz. On the other hand, it follows from (3) that for any infinite subset $\mathbb{M}$ of $\mathbb{N}$, $\mathrm{diam}\,\big(f(G_k(\mathbb{M}))\big) \geq \theta k$. This shows that the conclusion of Theorem 1.2 cannot hold as soon as $Y$ is non reflexive. This obstacle is overcome in our Theorems 2.2 and 2.4 by considering particular pairs of elements in $G_k(\mathbb{N})$ that are $k$-separated, namely the strictly interlaced pairs from $I_k(\mathbb{N})$. Note also that the pairs considered in our above application of James' Theorem are $k$-separated but at the "opposite" of being interlaced, since $n_1 < .. < n_k < m_1 < .. < m_k$.

## 3. Applications.

Let us start with the following definitions.

**Definition 3.1.**
(i) A Banach space $X$ has the *Banach-Saks property* if every bounded sequence in $X$ admits a subsequence whose Cesàro means converge in norm.
(ii) A Banach space $X$ has the *alternating Banach-Saks property* if for every bounded sequence $(x_n)_{n=1}^\infty$ in $X$, there exists a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ and a sequence $(\varepsilon_k)_{k=1}^\infty \in \{-1,1\}^{\mathbb{N}}$ such that the Cesàro means of the sequence $(\varepsilon_k x_{n_k})_{k=1}^\infty$ converge in norm.

Our last remark of section 2 was used in [1] (Theorem 4.1), to show that if a Banach space coarse Lipschitz embeds into a reflexive AUS Banach space, then $X$ is reflexive. This result, was recently improved by B.M Braga [4] (Theorem 1.3) who showed that actually $X$ must have the Banach-Saks property (which clearly implies reflexivity). As a first application of our result we obtain.

**Proposition 3.2.** *Assume that $X$ is a Banach space which coarse Lipschitz embeds into a quasi reflexive AUS Banach space $Y$. Then $X$ has the alternating Banach-Saks property.*

*Proof.* Since $Y$ is AUS, there exists $p \in (1,\infty)$ such that $Y$ is $p$-AUS renormable. This a consequence of a result of Knaust, Odell and Schlumprecht [13] in the separable case and of the second named author for the general case [16]. So we may as well assume that $Y$ is $p$-AUS for some $p > 1$.
Assume also that $X$ does not have the alternating Banach-Saks property. Then it follows from the work of B. Beauzamy (Theorems II.2 and III.1 in [2]) that there exists a sequence $(x_n)_{n=1}^\infty$ in $X$ such that for all $k \in \mathbb{N}$, all $\varepsilon_1, .., \varepsilon_k \in \{-1, 1\}$ and all $n_1 < .. < n_k$:

$$\frac{1}{2} \leq \Big\| \frac{1}{k} \sum_{i=1}^{k} \varepsilon_i x_{n_i} \Big\| \leq \frac{3}{2}. \tag{4}$$

Assume now that $X$ coarse Lipschitz embeds into $Y$. Then, after a linear change of variable if necessary, there exists $f : X \to Y$ and $A, B > 0$ such that

$$\forall x, x' \in X \quad \|x - x'\| \geq \frac{1}{2} \Rightarrow A\|x - x'\| \leq \|f(x) - f(x')\| \leq B\|x - x'\|.$$

We then define $\varphi_k : (G_k(\mathbb{N}), d) \to X$ as follows:

$$\varphi_k(\overline{n}) = x_{n_1} + .. + x_{n_k}, \quad \text{for } \overline{n} = (n_1, .., n_k) \in G_k(\mathbb{N}).$$

We clearly have that $\mathrm{Lip}(\varphi_k) \leq 3$. Moreover, for $\overline{n} \neq \overline{m} \in G_k(\mathbb{N})$ we have $\|\varphi_k(\overline{n}) - \varphi_k(\overline{m})\| \geq 1$. We deduce that for all $k \in \mathbb{N}$, $\mathrm{Lip}(f \circ \varphi_k) \leq 3B$.
It now follows from Theorem 2.2 that there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$, there is an infinite subset $\mathbb{M}_k$ of $\mathbb{N}$ so that

$$\forall (\overline{n}, \overline{m}) \in I_k(\mathbb{M}_k) \ \ \|(f \circ \varphi_k)(\overline{n}) - (f \circ \varphi_k)(\overline{m})\| \leq Ck^{1/p}.$$

On the other hand, it follows from (4) that for all $(\overline{n}, \overline{m}) \in I_k(\mathbb{M}_k)$, we have: $\|\varphi_k(\overline{n}) - \varphi_k(\overline{m})\| \geq k$. Therefore

$$\forall (\overline{n}, \overline{m}) \in I_k(\mathbb{M}_k) \ \ \|(f \circ \varphi_k)(\overline{n}) - (f \circ \varphi_k)(\overline{m})\| \geq Ak.$$

This yields a contradiction for $k$ large enough.

□

In order to state some more quantitative results, we will need the following definition.

**Definition 3.3.** Let $q \in (1,\infty)$ and $X$ be a Banach space. We say that $X$ has the $q$-co-Banach-Saks property if for every semi-normalized weakly null sequence $(x_n)_{n=1}^\infty$ in $X$ there exists a subsequence $(x'_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ and $c > 0$ such that for all $k \in \mathbb{N}$ and all $k \leq n_1 < .. < n_k$:

$$\|x'_{n_1} + ... + x'_{n_k}\| \geq ck^{1/q}.$$

For the proof of the following result, we refer the reader to Proposition 4.6 in [11], or Proposition 2.3 in [4] and references therein.

**Proposition 3.4.** *Let $q \in (1,\infty)$ and $X$ be a Banach space. If $X$ is $q$-AUC, then $X$ has the $q$-co-Banach-Saks property.*

Using our Theorem 2.2 and adapting the arguments of Braga in [4] we obtain.

**Theorem 3.5.** *Let $1 < q < p$ in $(1,\infty)$. Assume that $X$ is an infinite dimensional Banach space with the $q$-co-Banach-Saks property and $Y$ is a $p$-AUS and quasi reflexive Banach space. Then $X$ does not coarse Lipschitz embed into $Y$. More precisely, the compression exponent $\alpha_Y(X)$ of $X$ into $Y$ satisfies the following.*
*(i) If $X$ contains an isomorphic copy of $\ell_1$, then $\alpha_Y(X) \le \frac{1}{p}$.*
*(ii) Otherwise, $\alpha_Y(X) \le \frac{q}{p}$.*

*Proof.* We follow the proof of Theorem 4.1 in [4].

Assume first that $X$ contains an isomorphic copy of $\ell_1$ and that $\alpha_Y(X) > \frac{1}{p}$. Then there exists $f : X \to Y$ and $\theta, A, B > 0$ and $\alpha > \frac{1}{p}$ such that

$$\forall x, x' \in X \quad \|x - x'\| \ge \theta \Rightarrow A\|x - x'\|^\alpha \le \|f(x) - f(x')\| \le B\|x - x'\|.$$

Let $(x_n)_{n=1}^\infty$ be a sequence in $X$ which is equivalent to the canonical basis of $\ell_1$. We may assume, after a dilation, that there exists $K \ge 1$ such that

$$\forall a_1, .., a_k \in \mathbb{R}, \quad \theta \sum_{i=1}^k |a_i| \le \Big\| \sum_{i=1}^k a_i x_i \Big\| \le K\theta \sum_{i=1}^k |a_i|.$$

Then define $\varphi_k : (G_k(\mathbb{N}), d) \to X$ by

$$\varphi_k(\overline{n}) = x_{n_1} + .. + x_{n_k}, \quad \text{for } \ \overline{n} = (n_1, .., n_k) \in G_k(\mathbb{N}).$$

We clearly have that $\mathrm{Lip}(\varphi_k) \le 2K\theta$. Moreover, for $\overline{n} \neq \overline{m} \in G_k(\mathbb{N})$ we have $\|\varphi_k(\overline{n}) - \varphi_k(\overline{m})\| \ge 2\theta$. We deduce that for all $k \in \mathbb{N}$, $\mathrm{Lip}(f \circ \varphi_k) \le 2\theta KB$. It now follows from Theorem 2.2 that there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$, there is an infinite subset $\mathbb{M}_k$ of $\mathbb{N}$ so that

$$\forall (\overline{n}, \overline{m}) \in I_k(\mathbb{M}_k) \ \ \|(f \circ \varphi_k)(\overline{n}) - (f \circ \varphi_k)(\overline{m})\| \le Ck^{1/p}.$$

On the other hand, for all $(\overline{n}, \overline{m}) \in I_k(\mathbb{M}_k)$, $\|\varphi_k(\overline{n}) - \varphi_k(\overline{m})\| \ge 2k\theta$. Therefore

$$\forall (\overline{n}, \overline{m}) \in I_k(\mathbb{M}_k) \ \ \|(f \circ \varphi_k)(\overline{n}) - (f \circ \varphi_k)(\overline{m})\| \ge A2^\alpha \theta^\alpha k^\alpha.$$

This yields a contradiction if $k$ was chosen large enough.

Assume now that $X$ does not contain an isomorphic copy of $\ell_1$ and that $\alpha_Y(X) > \frac{q}{p}$. Then there exists $f : X \to Y$, $\alpha > \frac{q}{p}$ and $\theta, A, B > 0$ such that

$$\forall x, x' \in X \quad \|x - x'\| \ge \theta \Rightarrow A\|x - x'\|^\alpha \le \|f(x) - f(x')\| \le B\|x - x'\|.$$

Now, by Rosenthal's theorem, we can pick a normalized weakly null sequence $(x_n)_{n=1}^\infty$ in $X$. By extracting a subsequence, we may also assume that $(x_n)_{n=1}^\infty$ is a 2-basic sequence in $X$. We then define $\varphi_k : (G_k(\mathbb{N}), d) \to X$ as follows:

$$\varphi_k(\overline{n}) = 2\theta(x_{n_1} + .. + x_{n_k}), \quad \text{for } \ \overline{n} = (n_1, .., n_k) \in G_k(\mathbb{N}).$$

Note that $\mathrm{Lip}(\varphi_k) \le 4\theta$. Since $(x_n)_{n=1}^\infty$ is 2-basic, we have that for all $\overline{n} \neq \overline{m} \in G_k(\mathbb{N})$, $\|\varphi_k(\overline{n}) - \varphi_k(\overline{m})\| \ge \theta$. Therefore $\mathrm{Lip}(f \circ \varphi_k) \le 4\theta B$, for all $k \in \mathbb{N}$. It now follows from Theorem 2.2 that, for any $k \in \mathbb{N}$, there exists an infinite subset $\mathbb{M}_k$ of $\mathbb{N}$ such that

$$\forall(\overline{n},\overline{m}) \in I_k(\mathbb{M}_k) \quad \|(f\circ\varphi_k)(\overline{n}) - (f\circ\varphi_k)(\overline{m})\| \le Ck^{1/p}, \tag{5}$$

where $C > 0$ is a constant independent of $k$.
We can make sure in our construction that for all $k \in \mathbb{N}$, $\mathbb{M}_{k+1} \subset \mathbb{M}_k$. Let now $\mathbb{M}$ be the diagonalization of the sequence $(\mathbb{M}_k)_{k=1}^\infty$ and enumerate $\mathbb{M} = \{r_1 < .. < r_i < ..\}$. Denote $z_n = x_{r_{2n}} - x_{r_{2n+1}}$. By applying the $q$-co-Banach-Saks property to the semi normalized weakly null sequence $(z_n)_{n=1}^\infty$, we can find $\lambda : \mathbb{N} \to \mathbb{N}$ increasing and a constant $d > 0$ such that for all $k \in \mathbb{N}$ and all $k \le n_1 < .. < n_k$:

$$\begin{aligned}\Big\|\sum_{j=1}^k z_{\lambda(n_j)}\Big\| &= \Big\|\sum_{j=1}^k (x_{r_{2\lambda(n_j)}} - x_{r_{2\lambda(n_j)+1}})\Big\| \\ &= \|\varphi_k(r_{2\lambda(n_1)},..,r_{2\lambda(n_k)}) - \varphi_k(r_{2\lambda(n_1)+1},..,r_{2\lambda(n_k)+1})\| \ge dk^{1/q}.\end{aligned}$$

If $k$ is chosen large enough so that $dk^{1/q} \ge \theta$, we get that for all $k \le n_1 < .. < n_k$:

$$\|(f\circ\varphi_k)(r_{2\lambda(n_1)},..,r_{2\lambda(n_k)}) - (f\circ\varphi_k)(r_{2\lambda(n_1)+1},..,r_{2\lambda(n_k)+1})\| \ge Ad^\alpha k^{\alpha/q}.$$

It is now important to note that, due to the diagonal construction of $\mathbb{M}$, we have that for all $k \le n_1 < .. < n_k$

$$\big((r_{2\lambda(n_1)},..,r_{2\lambda(n_k)}),(r_{2\lambda(n_1)+1},..,r_{2\lambda(n_k)+1})\big) \in I_k(\mathbb{M}_k).$$

Therefore, this yields a contradiction with (5) if $k$ is chosen large enough. □

The following result is now a direct consequence of Theorem 3.5 and Proposition 3.4.

**Corollary 3.6.** *Let $1 < q < p < \infty$. Assume that $X$ is $q$-AUC and $Y$ is a $p$-AUS and quasi reflexive Banach space. Then*

$$\alpha_Y(X) \le \frac{q}{p}.$$

For our next application, we need to recall the definition of the Szlenk index. This ordinal index was first introduced by W. Szlenk [18], in a slightly different form, in order to prove that there is no separable reflexive Banach space universal for the class of all separable reflexive Banach spaces.
So, let $X$ be a Banach space, $K$ a weak$^*$-compact subset of its dual $X^*$ and $\varepsilon > 0$. Then we define

$s'_\varepsilon(K) = \{x^* \in K$ s.t. for any weak$^*$neighborhood $U$ of $x^*$, $\mathrm{diam}(K\cap U) \ge \varepsilon\}$

and inductively the sets $s_\varepsilon^\alpha(K)$ for $\alpha$ ordinal as follows: $s_\varepsilon^{\alpha+1}(K) = s'_\varepsilon(s_\varepsilon^\alpha(K))$ and $s_\varepsilon^\alpha(K) = \bigcap_{\beta<\alpha} s_\varepsilon^\beta(K)$ if $\alpha$ is a limit ordinal.
Then $Sz(K,\varepsilon) = \inf\{\alpha,\ s_\varepsilon^\alpha(K) = \emptyset\}$ if it exists and we denote $Sz(K,\varepsilon) = \infty$ otherwise. Next we define $Sz(K) = \sup_{\varepsilon>0} Sz(K,\varepsilon)$. The Szlenk index of $X$

is $Sz(X) = Sz(B_{X^*})$. We also denote $Sz(X,\varepsilon) = Sz(B_{X^*},\varepsilon)$.
We shall apply the following renorming theorem, which is proved in [7].

**Theorem 3.7.** *Let $q \in (1,\infty)$ and $X$ be a Banach space. Assume that there exists $C > 0$ such that*

$$\forall \varepsilon > 0 \quad Sz(X,\varepsilon) \le C\varepsilon^{-q}.$$

*Then, for all $r \in (q,\infty)$, $X$ admits an equivalent norm whose dual norm is $r$-weak*-AUC.*

Since a dual norm which is $r$-weak*-AUC is also $r$-AUC, we obtain the following statement as an immediate consequence of our Theorem 3.7 and Corollary 3.6.

**Corollary 3.8.** *Let $1 < q < p < \infty$. Assume that $Y$ is a $p$-AUS and quasi reflexive Banach space. Assume also that there exists $C > 0$ such that*

$$\forall \varepsilon > 0 \quad Sz(X,\varepsilon) \le C\varepsilon^{-q}.$$

*Then*

$$\alpha_Y(X^*) \le \frac{q}{p}.$$

**Aknowledgements.**
The authors wish to thank F. Baudier and Th. Schlumprecht for pointing out the application to the alternating Banach-Saks property. The first named author also wants to thank for their hospitality the Universidad de Murcia and Texas A&M University, where part of this work was completed.

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