Kato inequalities for scalar convection-diffusion problems

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based upon joint works with
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1. **Kato inequalities and their consequences**
2. **How to get Kato inequalities ? (work hard to get it !?)**
   - Parabolic case
   - Hyperbolic, degenerate parabolic, fractional cases
   - Kato inequality: inherit it ?
3. **How to exploit Kato inequalities in $\mathbb{R}^N$?**
   - Hyperbolic, degenerate parabolic, fractional cases
   - The quasilinear elliptic case
4. **How to exploit the Kato inequality in a bounded domain ?**
   - General setting of dissipative boundary conditions.
   - Dissipative BC in the hyperbolic setting.
   - Dirichlet BC in the degenerate parabolic case. Trace-regularity.
   - Neumann (zero-flux) BC. Up-to-the-boundary Kato inequalities.

Kato inequalities and their consequences

Classical Kato inequality

Inequality due to [Kato'72] . Let Ω be a bounded domain of $\mathbb{R}^N$.
Let $W \in L^1_{loc}(\Omega)$ such that $-\Delta W \in L^1_{loc}(\Omega)$. Then

$$-\Delta|W| \leq \mathrm{sign}\,(W)(-\Delta W) \;\text{ in } \mathcal{D}'(\Omega).$$

Proof: in the general case, by approximation of W.
In the case of W in H^1_{loc} at least, by approximation of $\mathrm{sign}\,(\cdot)$.
Heuristics: "$\leq$" is due to $-\Delta|W| = \mathrm{sign}\,(W)(-\Delta W) + \delta_0(W)|\nabla W|^2$.

Generalization ([Brézis '84]): Under the same assumptions on W, let $S : \mathbb{R} \to \mathbb{R}$ non-decreasing, Lipschitz , piecewise C^1. Then one has

$$-\Delta S(W) \leq S'(W)(-\Delta W) \;\text{ in } \mathcal{D}'(\Omega).$$

"Quasilinear diffusion" variant (following [Blanchard,Porretta'05]) :
Let $\varphi : \mathbb{R} \to \mathbb{R}$ continuous, non-decreasing .
Let $u, w, g \in L^1_{loc}(\Omega)$ verifying $w = \varphi(u)$ and $u - \Delta w = g$;
let $\hat{u}, \hat{w}, \hat{g} \in L^1_{loc}(\Omega)$ verifying $\hat{w} = \varphi(\hat{u})$ and $\hat{u} - \Delta \hat{w} = \hat{g}$. Then

$$(u - \hat{u})^+ - \Delta(w - \hat{w})^+ \leq \mathrm{sign}^+(u - \hat{u})(g - \hat{g}) \;\text{ in } \mathcal{D}'(\Omega).$$

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“Kato inequalities” for convection-diffusion equations

Typical form of equations we consider :

$$u_t + \operatorname{div} F[u] = 0 \ \text{ in } (0,T)\times\Omega,\ \Omega\subseteq\mathbb{R}^N$$

where the flux F is defined in terms of $u(\cdot)$ by

$$F[u] = f(u) - \nabla\varphi(u),$$

f continuous (at least), φ continuous, non-decreasing.

Kato inequality:

$$|u-\hat{u}|_t + \operatorname{div} \operatorname{sign}(u-\hat{u})(F[u]-F[\hat{u}]) \leq 0 \ \text{ in } \mathcal{D}'((0,T)\times\Omega).$$

Stationary version: $u + \operatorname{div} F[u] = g$ in $\Omega\subseteq\mathbb{R}^N$.

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“Fractional diffusion” version: $u_t + \operatorname{div} f(u) + (-\Delta)^{\alpha/2}[\varphi(u)] = 0$.

Kato inequality ([Alibaud’07]):

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Consequences of the Kato inequality

Well-posedness of such equations is derived (at least for a part) from the Kato inequalities. E.g. for the stationary case:

Kato inequality "integrated over Ω" (NB: not always possible !)
$\Rightarrow$ **Ok for uniqueness, L^1 continuous dependence, comparaison.**

Uniqueness Take the test function $\xi = 1\!\!1_\Omega$ in the place of $\xi \in \mathcal{D}'(\Omega)$?
Sometimes impossible (L^1_{loc} setting of [Brézis'84]), often delicate.
Therefore one tries to work with a suitable sequence $(\xi_n)_n$

Order $S(r) = r^+$ in the place of $S(r) = |r| \Rightarrow$ comparison principle

Existence Comparison principle
$+ \exists$ for a dense subset of data[1] + uniform bounds
$\Rightarrow$ existence (via monotone convergence arguments)

Continuous dependence Similar, using comparison principle combined with lim inf-lim sup arguments

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How to obtain Kato inequalities

Obtain KI in the case of non-degenerate diffusion

Prototype eqn.: $u_t + \operatorname{div} f(u) - \Delta w = 0$, $w = \varphi(u)$, φ increasing
Setting : weak (variational) solutions $(H^1_{loc}(\Omega))$
Principle for getting KI: multiply by $\operatorname{sign}_\alpha(w - \hat{w})$, $\alpha \to 0$.

- Works for stationary problem or for solutions such that $u_t \in L^1_{loc}$, provided f is Lipschitz or at least Hölder of order 1/2.
- If u_t is a distribution, use doubling of time variable ([Otto'96]): consider $\hat{w} = \hat{w}(s,x)$ and test functions in $\xi_n(t,s,x)$ with support that concentrates on $t = s$ as $n \to \infty$
  If f is not regular enough, also doubling in space is needed
- Variant (exotic): if the diffusion Δw is replaced by "$p(u)$-laplacian" $\operatorname{div} |\nabla w|^{p(u)-2} \nabla w$, one can obtain KI if one of the two solutions is "regular" (here, regularity of $\hat{u}$ means that $\nabla \hat{u} \in L^\infty$)
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How to obtain KI in the hyperbolic or degenerate parabolic context

Prototype eqn.: $u_t + \operatorname{div} f(u) - \Delta w = 0$, $w = \varphi(u)$, φ non-decreasing
Setting : entropy solutions

Principle for getting KI :
double the variables in the (suitable) entropy formulation

- Hyperbolic case $\varphi \equiv 0$ [Kruzhkov'69] :

$$\forall k \in \mathbb{R} \quad |u-k|_t + \operatorname{div} \operatorname{sign}(u-k)(f(u)-f(k)) \leq 0 \quad \text{in } D',$$

i.e., KI is imposed with $\hat{u} \equiv k$! Doubling : "put" $\hat{u}$ in the place of k.
$\Rightarrow$ general KI can be deduced from a restricted family of KI's !

- Parabolic-hyperbolic case [Carrillo'99] :

$$|u-k|_t + \ldots - \Delta|\varphi(u)-\varphi(k)| \leq -\limsup_{\alpha\to 0} \frac{1}{\alpha}\int_{[|u-k|<\alpha]} |\nabla\varphi(u)|^2,$$

one keeps the "parabolic dissipation" "$\delta_0(u-k)|\nabla\varphi(u)|^2$".

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Kato inequality passes to the limit under strong enough convergence

It occurs frequently that the KI is inherited after passage to the limit.

- Vanishing viscosity method:

$$u = \lim_{\varepsilon \to 0^+} u^\varepsilon, \qquad u_t^\varepsilon + \operatorname{div} f(u^\varepsilon) = \varepsilon \Delta u^\varepsilon,$$

  i.e., passage to the limit from non-degenerate parabolic (easy to get IK) to a hyperbolic equation (very delicate KI !)

- Numerical approximation of $u_t + f(u)_x + ... = 0$ by monotone finite volume schemes.
  Use [Crandall,Tartar'80] $\Rightarrow$ easy discrete KI (since every test function is admissible)

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Inherit the contraction property

NB: For this slide, the setting is L^1 contraction (a bit different from KI).

- Approximation of data
  Start from partial contraction between general solution u and a restricted class of "regular" solutions $\hat{u}$.
  Extend the partial contraction to general contraction.
  Condition: L^1 density of data $\hat{g}$ for which $\exists$ "regular" $\hat{u}$.

  Two examples:

- Non variational solutions of $u + (-\Delta)^{\frac{\alpha}{2}} w = f \in L^1$, $\alpha \leq 2$:
  renormalized [Murat-Lions] or entropy solutions [Bénilan et al.'95] ,
  contraction proof is much easier if $\hat{u}$ is a bounded solution
- Setting of m-accretive operators
  If contraction between general $u(t,.)$ and stationary $\hat{u}(.)$
  $\Rightarrow$ contraction for $u, \hat{u}$ general solutions
  (notion of integral solution to abstract evolution pbs, [Bénilan'72])

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Exploit

Kato inequalities:

the case of the whole $\mathbb{R}^N$

Usual hyperbolic techniques for exploiting KI

Known facts for $u_t + \operatorname{div} f(u) = 0$ set in $L^\infty((0,T)\times\mathbb{R}^N)$:

- f Lipschitz $\Rightarrow$ one can "integrate KI locally" [Kruzhkov'70]
  $\Rightarrow$ **Ok for uniqueness, comparison, L^1 contraction**
- f merely continuous $\Rightarrow$ non-uniqueness examples [Panov'91]
- $f \in C^{1-\frac{1}{N}} \Rightarrow$ Ok (KI can be integrated globally) [Bénilan'72]
- (subtle) conditions of anisotropic Hölder kind
  on moduli of continuity of components of the flux
  $\Rightarrow$ Ok [Kruzhkov,Panov'94],[Bénilan,Kruzhkov'96]

KI is exploited by choosing sequences of test functions $(\xi_n)_n$.

- $(N-1)$ components of the flux vector f are monotone
  $+\ u$ decreases at infinity $\Rightarrow$ Ok [A.,Bénilan,Kruzhkov'00]

Very different techniques in use

Open problem :
uniqueness or counterexamples for $u \in L^1$, f merely continuous

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Same techniques in the parabolic-hyperbolic setting

Extensions to parabolic-hyperbolic (and fractional) cases :

- f, φ Lipschitz $\Rightarrow$ Ok :
  "finite-infinite speed of propagation", a splitting argument
  [Alibaud'07] (designed for $\alpha < 2$ but also works for $\alpha = 2$).
  [Endal,Jakobsen'14] : localized estimates for $\alpha \leq 2$.
- $f \in C^{1-\frac{1}{N}}$ and $\varphi \in C^{1-\frac{2}{N}} \Rightarrow$ Ok [Maliki,Touré'03] :
  Bénilan-Kruzhkov technique works for $\alpha = 2$,
  and one expects that it work for $\alpha < 2$ if $\varphi \in C^{1-\frac{\alpha}{N}}$

Optimality quest:
in these techniques, the monotonicity of φ is not exploited.

Question : the assumption of $C^{1-\frac{2}{N}}$ regularity of φ is it optimal ?

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Classical and less classical techniques for Fast Diffusion

Model case: $u - \Delta\varphi(u) = g$ (with φ as general as possible)

Goal: convert KI into uniqueness-comparison-... for L^∞ solutions

[A.,Maliki'??] : three well-posedness classes containing L^∞...
two of them were, roughly speaking, known.

Common assumption: uniform continuity of φ (OK in L^∞ solutions).
Notation: ω_φ modulus of continuity, $\Omega_\varphi = \omega_\varphi^{-1}$.

- Class L^1_{loc}, under the Keller-Osserman condition $\int^{+\infty} \frac{dz}{z\Omega_\varphi(z)} < +\infty$
typically verified for Fast Diffusion $\varphi(u) = u^m$, $0 < m < 1$
[Brézis'84],[Gallouët,Morel'87] . Technique: $\exists$ super-solutions W_R of $\Omega_\varphi(W) - \Delta W = 0$ blowing-up on $|x| = R$ and s.t. $W_R \to 0$, $R \to \infty$.
- Weighted class $L^1(\mathbb{R}^N; \exp(-c|\cdot|)$,
assuming that φ grows at most linearly at infinity, $\varphi'(+\infty) > c$.
Technique of linearization of the generalized Kato inequality
(choice of S such that $S'\Omega_\varphi$"="cS – not exactly possible...)
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NB: Use of superharmonic weights is compatible with $+\mathrm{div}\, f(u)$!

Classical and less classical techniques for Fast Diffusion

Model case: $u - \Delta\varphi(u) = g$ (with φ as general as possible)

Goal: convert KI into uniqueness-comparison-... for L^∞ solutions

[A.,Maliki'??] : three well-posedness classes containing L^∞... two of them were, roughly speaking, known.

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Exploit

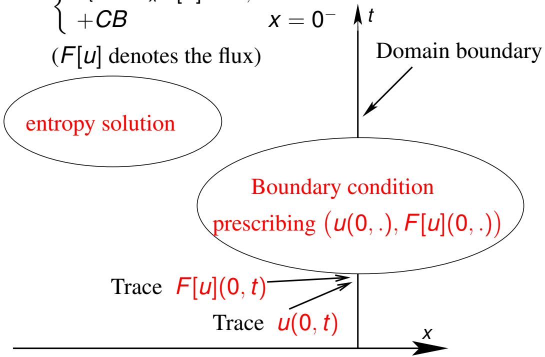
Kato inequalities:

the case of a bounded domain

Boundary-value problems: approach by local Kato inequalities

$$\begin{cases} \partial_t u + \partial_x F[u] = 0, & x < 0 \\ +CB & x = 0^- \end{cases}$$

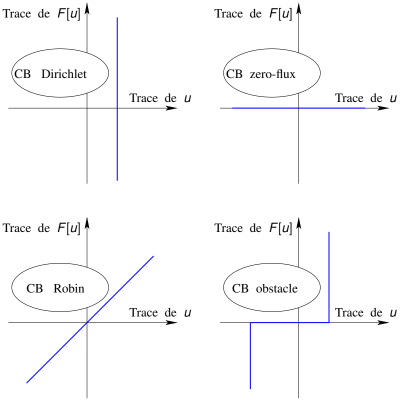
($F[u]$ denotes the flux)



Exploit KI near the boundary: test fct. $\xi_n \to \mathbf{1}_\Omega$ with $\nabla \xi_n \rightharpoonup -\delta|_{\partial\Omega}\mathbf{n} \;\Rightarrow$

$$\int_\Omega |u-\hat{u}|(T,x) - \int_\Omega |u_0 - \hat{u}_0| \leq - \int_0^T \gamma_{ad\ hoc}\Big\{ \text{sign}\,(u-\hat{u})(F[u]-F[\hat{u}])\cdot\mathbf{n} \Big\}(t)\, dt$$

Classical boundary conditions



In these cases, $(u, F[u]) \in \beta$ for some maximal monotone graph β .
General framework: BC set up in terms of a maximal monotone dependence between the solution u and flux $F[u]$ at the boundary
Boundary dissipation:
$\mathrm{sign}\,(u-\hat{u})\big(F[u]-F[\hat{u}]\big) = \mathrm{sign}\,(u-\hat{u})\big(\beta(u)-\beta(\hat{u})\big) \geq 0$!

Dissipative BC for hyperbolic conservation law. Projection.

Hyperbolic equation $u_t + f(u)_x = 0$ + formal BC $(u, F[u]) \in \beta$:

- Uniqueness is obvious for the formal problem
- Formal problem ill-posed (in general, existence fails)
- Problem with ... $= \varepsilon\partial^2_{xx}u$ is well posed, KI and L^1 contraction are inherited as $\varepsilon \to 0$. The limit is a local entropy solution verifying effective BC $(u, F[u]) \in \tilde{\beta}$ where $\tilde{\beta}$ is a projection of β
  Problem with effective BC (i.e., $\tilde{\beta}$ in BC) is well posed
- One can easily grasp the projection procedure by picturing $\tilde{\beta}$.
  One observes : $\tilde{\beta}$ is the maximal monotone subgraph of f which is the closest to β !
  One can describe $\tilde{\beta}$ in terms of the "Godunov numerical flux":

$$\tilde{\beta} = \left\{ (u, \mathcal{F}) \,\middle|\, \mathcal{F} = f(u) = \mathrm{God}(u, \tilde{u}) \in \beta(\tilde{u}) \right\}$$

Example: BLN condition [Bardos,LeRoux,Nédélec'79] can be reformulated this way [Dubois,LeFloch'88]

Détails : [Thesis Sbihi'06],[A.,Sbihi'15]

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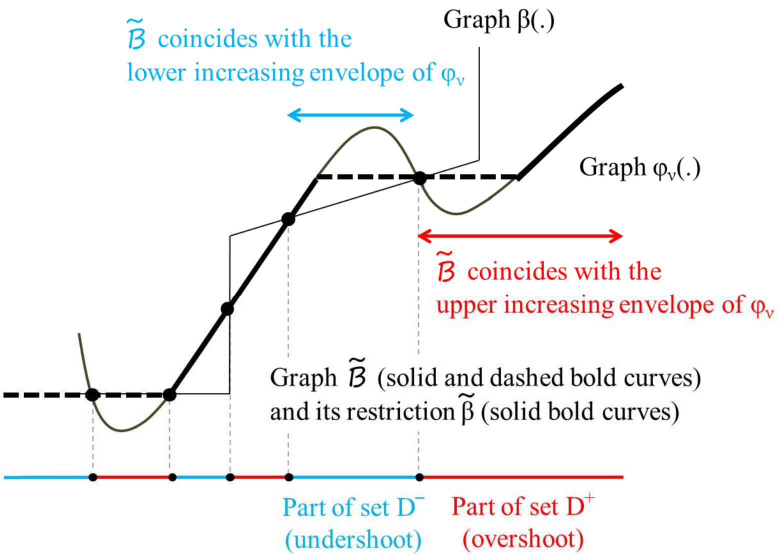
Example for a general BC: the projection procedure



BC in the degenerate parabolic case ?

In principle, dissipative BC would work for all kind of flux $F[u]$, e.g. (sedimentation models) $F[u] = f(u) - \nabla\varphi(u)$; think of $\varphi(u) = (u - u_c)^+$

Difficulty: existence of traces – for the solution ? – for the flux ???
Which sense can be given to $\gamma_{ad\ hoc}\Big\{ \mathrm{sign}\,(u - \hat{u})(F[u] - F[\hat{u}]) \cdot \mathbf{n} \Big\}$?
NB: in the hyperbolic case, strong traces exist [Vasseur'01,Panov'07]
What can be said in the degenerate parabolic case ?

$\Rightarrow$ [Thesis Gazibo'13],[A.,Gazibo'15?] : make explicit the effective BC

Results :

- for the Dirichlet pb., we describe the effective graph $\tilde{\beta}$
- we put forward the sufficient "trace-regularity" notion for solutions needed to extend the KI up to the boundary starting from a local KI. Yet, trace-regularity is not guaranteed, except for stationary 1D pb.

Outcome : An heuristic formulation ? Enough to do numerics !

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Up-to-the-boundary entropy and Kato inequalities.

Finally, the idea "local KI $\Rightarrow$ up to the boundary KI" may not work.

Natural idea: obtain directly up-to-the-boundary KI
NB: after all, for $u - \Delta\varphi(u) = g$ with Dirichlet BC, it's straightforward !

Yet for the case with hyperbolic degeneracy, this means

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- learn to do up-to-the-boundary doubling of variables...?

Ex.: "zero-flux" condition : [Bürger,Frid,Karlsen'09] , hyperb. case

Open problem: Uniqueness for zero-flux parab.-hyperb. multi-D ?

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GRAZIE — THANK YOU

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