Local conditions insuring bifurcation from the continuous spectrum *

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1 Introduction

We consider a family of equations

$$-\Delta u(x) + \lambda u(x) = f(x, u(x)),\ \lambda > 0,\ x \in \mathbb{R}^N, \qquad (1)_\lambda$$

where the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfies $f(x, 0) = 0$, a.e. $x \in \mathbb{R}^N$. We say that $\lambda = 0$ is a bifurcation point for $(1)_\lambda$ if there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R}^+ \times H^1(\mathbb{R}^N)$ of nontrivial solutions of $(1)_{\lambda_n}$ with $\lambda_n \to 0$ and $||u_n||_{H^1(\mathbb{R}^N)} \to 0$. In this case $\{(\lambda_n, u_n)\}$ is called a bifurcating sequence. The aim of the paper is to show that weak conditions on $f(x, .)$ around zero suffice to guarantee that $\lambda = 0$ is a bifurcation point for $(1)_\lambda$. More precisely suppose there is $\delta > 0$ such that

(H1) $f : \mathbb{R}^N \times [-\delta, \delta] \to \mathbb{R}$ is Caratheodory.

(H2) $\lim_{|x|\to\infty} f(x, s) = 0$ uniformly for $s \in [-\delta, \delta]$.

(H3) There exists $K > 0$ such that $\limsup_{s\to 0} |\frac{f(x, s)}{s}| \leq K$ uniformly in $x \in \mathbb{R}^N$.

(H4) $\lim_{s\to 0} \frac{F(x, s)}{s^2} = 0$ uniformly in $x \in \mathbb{R}^N$ with $F(x, s) := \int_0^s f(x, t)\, dt$.

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(H5) One of the following conditions holds

(i) $N \geq 1$ and there exist $A > 0$, $d \in]0, 2[$ and $\alpha \in]0, \frac{2(2-d)}{N}[$ such that

$$F(x, s) \geq A(1 + |x|)^{-d}|s|^{2+\alpha} \text{ for all } s \in [-\delta, \delta] \text{ and a.e. } x \in \mathbb{R}^N.$$

(ii) $N = 1$ and there exists $\alpha \in]0, 2[$ such that

$$F(x, s) \geq r(x)|s|^{2+\alpha} \text{ for all } s \in [-\delta, \delta] \text{ and a.e. } x \in \mathbb{R},$$

where $r \in L^\infty(\mathbb{R})$, $r \geq 0$ and $\int_{\mathbb{R}} r(x)\, dx > 0$ (the value $+\infty$ is possible).

Our main result is the following:

Theorem 1.1 *Assume that (H1)-(H5) hold. Then $\lambda = 0$ is a bifurcation point for $(1)_\lambda$.*

Note that, when $\partial_s f(x, s) = 0$ for $s = 0$, $\lambda = 0$ corresponds to the infimum of the (essential) spectrum of the linearisation of $(1)_\lambda$. The study of bifurcation, from the infimum of the spectrum, for equations of the type of $(1)_\lambda$ started in the late's 70. The equations being set on $\mathbb{R}^N$ these problems are caracterised by a lack of compactness. Also $\lambda = 0$ is not an eigenvalue. Thus Rabinowitz's alternative (see [8]) does not apply and in general one expects weaker results. This is why different definitions of bifurcation points have been introduced. They range from requiring as in our case that $(0, 0) \in \mathbb{R} \times H^1(\mathbb{R}^N)$ is an accumulation point of nontrivial solutions of $(1)_\lambda$ to $(0, 0)$ being an ending point of a smooth curve of solutions parametrized by λ. A whole array of methods, analytic, topological or variational have been used or developed to handle these questions. In our terminology for a bifurcation point, the sharpest results have been obtained by variational methods either free [14, 16] or with constraint [9, 12, 15] (see also [3, 7] for related results).

A first originality of our result is that only conditions on $f(x, .)$ around zero are required. To this end we modify $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ outside of $\mathbb{R}^N \times [-\delta, \delta]$ and we show that the new family of equations has a bifurcating sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R}^+ \times H^1(\mathbb{R}^N)$. Then we prove that $||u_n||_{L^\infty(\mathbb{R}^N)} \to 0$ and thus $\lambda = 0$ is also a bifurcation point for $(1)_\lambda$. This idea of modifying the nonlinearity and passing through L^∞- bifurcation to avoid requiring global conditions of f had been forseen by some specialists [11] but so far was not written down (with the exception of [6] where $N = 1$ and thus the argument is trivial). In that direction we mention a recent paper [2] where this idea is used on a problem which, even if not dealing with bifurcation, is related.

Our main achievement however is that our local conditions on f are weak local conditions. Indeed, in the papers studying bifurcation through variational methods, the following condition is always required

(SQC) $\exists \mu > 2$ such that $0 \leq \mu F(x, s) \leq f(x, s)s$ for all $s \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^N$.

(In case of constrained variational approachs $\mu \geq 2$). Thus even if, using the L^∞-bifurcation trick, we can forget about $f(x,.)$ outside of an interval $[-\delta,\delta]$, (SQC) with $s \in [-\delta,\delta]$ replacing $s \in \mathbb{R}$ must still be required. This assumption is much stronger than ours. To see this let

$$F(x,s) = (1+|x|)^{-1}|s|^{2+\alpha}(2+\sin\frac{1}{|s|^\alpha})$$

where $\alpha \in]0, \frac{2}{N}[$. Clearly $f(x,s) := \partial_s F(x,s)$ satisfies (H1)-(H5). But $\lim_{s\to 0} f(x,s)s^{-1}$ does not exist and there are sequences $s_n \searrow 0$ with $f(x,s_n) < 0 < f(x,s_{n+1})$ for all $n \in \mathbb{N}$ and a.e. $x \in \mathbb{R}^N$. Namely $f(x,.)$ is indefinite in sign. In contrast when (SQC) holds necessarily $f(x,s)s^{-1} \to 0$ if $s \to 0$ and $f(x,s) \geq 0$ for all $s \geq 0$ and a.e. $x \in \mathbb{R}^N$.

The condition (SQC) is a key ingredient in the two main steps of the classical (unconstrained) variational approaches. The first step is to find a solution u_λ of $(1)_\lambda$ for any $\lambda > 0$. The condition (SQC) insures that the functional associated to $(1)_\lambda$, $J_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + \lambda u^2)\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx,$$

has bounded Palais-Smale sequences. Thanks to the compactness condition (H2), this permits to use the Mountain-Pass theorem (see [1]) to get a critical point u_λ at the Mountain-Pass level. In the second step one proves that $||u_\lambda||_{H^1(\mathbb{R}^N)} \to 0$ as $\lambda \to 0$. Here (SQC) gives a crucial relation between $J_\lambda(u_\lambda)$ and $||u_\lambda||_{H^1(\mathbb{R}^N)}$ and the bifurcation is proved controlling the decrease of $J_\lambda(u_\lambda) \to 0$ as $\lambda \to 0$ with test functions.

In our paper we still study bifurcation with a variational method. However to overcome the lack of (SQC) we need to introduce new ideas. We exploit some monotonicity properties of $(1)_\lambda$ and this is reminicient of Struwe's work on the so called "monotonicity trick" (see [10], Chapter II, Section 9). The sketch of the proof of Theorem 1.1 is as follows. We denote by $(I_\lambda)_{\lambda>0}$ the family of functionals obtained by modifying $f(x,.)$ outside $[-\delta,\delta]$. We show there exists $\lambda_0 > 0$, such that for all $\lambda \in]0,\lambda_0]$, the following sets are non empty

$$\Gamma_\lambda = \{\gamma \in C([0,1], H^1(\mathbb{R}^N)), \gamma(0) = 0 \text{ and } I_\lambda(\gamma(1)) < 0\}$$

and

$$c(\lambda) := \inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]} I_\lambda(\gamma(t)) > I_\lambda(0) = 0.$$

Namely I_λ has, for $\lambda \in]0,\lambda_0]$, a Mountain-Pass geometry. The function $\lambda \to c(\lambda)$ is non decreasing and thus differentiable almost everywhere. We prove that for each $\lambda \in]0,\lambda_0]$ where the derivative $c'(\lambda)$ of $c(\lambda)$ exists, I_λ has a bounded Palais-Smale sequence contained in a ball of $H^1(\mathbb{R}^N)$ centred at the origin whose radius goes to zero when both $c(\lambda) \to 0$ and $c'(\lambda) \to 0$. Using the compactness properties of $(1)_\lambda$ insured by (H2) we deduce that $(1)_\lambda$ has, for such λ, a nontrivial solution lying in the ball. Now test functions show that $c(\lambda)\lambda^{-1} \to 0$ as $\lambda \to 0$. It implies the existence of a strictly decreasing sequence $\lambda_n \to 0$ such that $c(\lambda_n) \to 0$ and $c'(\lambda_n) \to 0$. For the corresponding solutions we have $||u_n||_{H^1(\mathbb{R}^N)} \to 0$ and this proves the bifurcation for the modified problem. Since the bifurcation also occurs in the L^∞- norm this ends the proof of Theorem 1.1.

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Notation Throughout the article the letter C will denote various positive constants whose exact value may change from line to line but are not essential to the analysis of the problem. Also we make the convention that when we take a subsequence of a sequence $\{u_n\}$ we denote it again $\{u_n\}$.

2 The Variational Setting

In this section we modify $f(x,.)$ outside $[-\delta,\delta]$. We show that to the new family of equations $(\tilde{1})_\lambda$ corresponds, for $\lambda > 0$ sufficiently small, a family of C^1- functionals having a Mountain-Pass geometry. Moreover the Mountain-Pass value $c(\lambda)$ is a non decreasing function of λ.

We denote by $\|\cdot\|_p$ for each $p \in [1,\infty]$ the standard norm of the Lebesgue space $L^p(\mathbb{R}^N)$. Our working space is the Sobolev space $H^1(\mathbb{R}^N) := H$ equipped with the norm $||u||^2 := ||\nabla u||_2^2 + ||u||_2^2$. Let $\tilde{f} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\tilde{f}(x,s) = \begin{cases} f(x,-\delta) \text{ if } s \le -\delta, \text{ a.e. } x \in \mathbb{R}^N \\ f(x,s) \text{ if } s \in [-\delta,\delta], \text{ a.e. } x \in \mathbb{R}^N \\ f(x,\delta) \text{ if } s \ge \delta, \text{ a.e. } x \in \mathbb{R}^N. \end{cases}$$

¿From (H3) we see that for all $s \in \mathbb{R}$ and a.e. $x \in \mathbb{R}^N$

$$|\tilde{f}(x,s)| \le K|s| \quad \text{and} \quad |\tilde{F}(x,s)| \le \frac{K}{2}|s|^2 \quad \text{with} \quad \tilde{F}(x,s) := \int_0^s \tilde{f}(x,t)\,dt. \tag{2.1}$$

It follows in particular that for all $u \in H$

$$\int_{\mathbb{R}^N} \tilde{F}(x,u)\,dx \le \frac{K}{2}||u||_2^2. \tag{2.2}$$

We denote by $(\tilde{1})_\lambda$ the new family of equations obtained replacing f by $\tilde{f}$ and we introduce the functionals $I_\lambda : H \to \mathbb{R}$ with

$$I_\lambda(u) = ||\nabla u||_2^2 + \lambda\,||u||_2^2 - 2\int_{\mathbb{R}^N} \tilde{F}(x,u)\,dx.$$

It is standard that I_λ is a C^1-functional for each $\lambda \ge 0$. Moreover to a critical point u_λ of I_λ corresponds a solution $(\lambda, u_\lambda) \in \mathbb{R} \times H$ of $(\tilde{1})_\lambda$ (see [14] for such results). Next we show that, for $\lambda > 0$ small, I_λ has a Mountain-Pass geometry.

Lemma 2.1 *Assume that (H1)-(H5) hold. There exists $\lambda_0 > 0$ such that for all $\lambda \in]0,\lambda_0]$ the sets*

$$\Gamma_\lambda = \{\gamma \in C([0,1],H), \gamma(0) = 0 \text{ and } I_\lambda(\gamma(1)) < 0\}$$

are non empty. Moreover for $\lambda \in]0, \lambda_0]$

$$c(\lambda) := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > I_\lambda(0) = 0$$

and the function $\lambda \to c(\lambda)$ *is non decreasing.*

Note that $I_{\lambda_1}(u) \leq I_{\lambda_2}(u)$ for all $u \in H$ and $0 \leq \lambda_1 \leq \lambda_2$. Thus $\Gamma_{\lambda_2} \subset \Gamma_{\lambda_1}$ whenever $0 \leq \lambda_1 \leq \lambda_2$ and $\lambda \to c(\lambda)$ is non decreasing. Since $I_\lambda(0) = 0$ for all $\lambda \geq 0$ the lemma is now a direct consequence of Lemmas 2.2 and 2.3.

Lemma 2.2 *Assume that (H1)-(H4) hold. Then* $I_\lambda(u) \geq \lambda ||u||^2 + o(\|u\|^2)$ *as* $u \to 0$ *for all* $\lambda \in]0, 1]$.

Proof. From (H4) and (2.1) we see that for any $\epsilon > 0$ and $q \in]2, \frac{2N}{N-2}]$ ($q > 2$ if $N = 1, 2$) there exists $C(\epsilon, q) > 0$ such that

$$\tilde{F}(x, s) \leq \epsilon |s|^2 + C(\epsilon, q)|s|^q.$$

Thus, by the Sobolev's embeddings, there exists $\tilde{C}(\epsilon, q) > 0$ such that for all $u \in H$

$$\int_{\mathbb{R}^N} \tilde{F}(x, u)\, dx \leq \epsilon ||u||^2 + \tilde{C}(\epsilon, q)||u||^q$$

and we deduce that

$$\int_{\mathbb{R}^N} \tilde{F}(x, u)\, dx = o(\|u\|^2) \text{ as } u \to 0.$$

Since for $u \in H$ and $\lambda \in]0, 1]$, $I_\lambda(u) \geq \lambda ||u||^2 - 2\int_{\mathbb{R}^N} \tilde{F}(x, u)\, dx$, the lemma follows. ♠

Lemma 2.3 *Assume that (H1)-(H5) hold. There are* $\lambda_0 > 0$ *and* $v_0 \in H$ *with* $I_{\lambda_0}(v_0) < 0$.

Proof. The proof is based on the use of test functions that were first introduced in [13]. For $\lambda > 0$ we define $v_\lambda(x) = \delta e^{-\sqrt{\lambda}|x|}$. Straightforward calculations give

$$||v_\lambda||_2^2 = B(N)\lambda^{-\frac{N}{2}} \text{ and } ||\nabla v_\lambda||_2^2 = B(N)\lambda^{1-\frac{N}{2}}$$

with

$$B(N) = \delta^2 \int_{\mathbb{R}^N} e^{-2|y|} dy. \tag{2.3}$$

Thus

$$||\nabla v_\lambda||_2^2 + \lambda\, ||v_\lambda||_2^2 = 2B(N)\lambda^{1-\frac{N}{2}}. \tag{2.4}$$

Now when (H5)(i) holds

$$\int_{\mathbb{R}^N} \tilde{F}(x, v_\lambda)\, dx \geq A \int_{\mathbb{R}^N} (1+|x|)^{-d} |v_\lambda|^{\alpha+2}\, dx$$

with

$$\begin{aligned} A\int_{\mathbb{R}^N}(1+|x|)^{-d}|v_\lambda|^{\alpha+2}\,dx &\geq \delta^{\alpha+2}A\ \lambda^{-\frac{N}{2}}\int_{\mathbb{R}^N}(1+\frac{|y|}{\sqrt{\lambda}})^{-d}e^{-(\alpha+2)|y|}\,dy \\ &\geq \delta^{\alpha+2}A\ \lambda^{-\frac{N}{2}}\lambda^{\frac{d}{2}}\ 2^{-d}\int_{|y|\geq 1}|y|^{-d}e^{-(\alpha+2)|y|}\,dy \\ &:= D(N,d,\alpha)\lambda^{\frac{d}{2}-\frac{N}{2}}. \end{aligned} \tag{2.5}$$

It follows that

$$I_\lambda(v_\lambda) \leq 2B(N)\lambda^{1-\frac{N}{2}} - 2D(N,d,\alpha)\lambda^{\frac{d}{2}-\frac{N}{2}}.$$

Since $d < 2$ we clearly have $I_\lambda(v_\lambda) < 0$ for all $\lambda > 0$ sufficiently small. On the other hand when (H5)(ii) holds we have

$$\int_{\mathbb{R}} \tilde{F}(x, v_\lambda)\, dx \geq \int_{\mathbb{R}} r(x)|v_\lambda|^{\alpha+2}\, dx$$

with, for all $\lambda > 0$ sufficiently small,

$$\begin{aligned} \int_{\mathbb{R}} r(x)|v_\lambda|^{\alpha+2}\,dx &= \delta^{\alpha+2}\int_{\mathbb{R}} r(x)e^{-(\alpha+2)\sqrt{\lambda}|x|}\,dx \\ &\geq \delta^{\alpha+2}e^{-(\alpha+2)\sqrt{\lambda}\mu}\int_{|x|\leq\mu} r(x)\,dx \text{ (for all } \mu>0) \\ &\geq \frac{1}{2}\delta^{\alpha+2}e^{-(\alpha+2)\sqrt{\lambda}\mu}\min\{\int_{\mathbb{R}} r(x)\,dx,\ 1\} \text{ (for } \mu>\mu_0) \\ &\geq \frac{1}{4}\delta^{\alpha+2}\min\{\int_{\mathbb{R}} r(x)\,dx,\ 1\} \\ &:= E(\alpha). \end{aligned} \tag{2.6}$$

Thus using (2.4) we get

$$I_\lambda(v_\lambda) \leq 2B(N)\sqrt{\lambda} - 2E(\alpha)$$

and as precedingly $I_\lambda(v_\lambda) < 0$ for $\lambda > 0$ sufficiently small. It is important to note that our proof only uses the behaviour of $\tilde{f}(x,.)$ on $[-\delta,\delta]$, namely when it corresponds to $f(x,.)$. Thus the lemma is independent of the particular choice of $\tilde{f}$ that we have made. ♠

3 Some Nice Paths

Since the map $\lambda \to c(\lambda)$ is non decreasing, $c'(\lambda)$, the derivative of $c(\lambda)$, exists almost everywhere. We claim that for any $\lambda \in]0,\lambda_0]$, where $c'(\lambda)$ exists, there is a sequence of paths $\{\gamma_m\} \subset \Gamma_\lambda$ with

$$\max_{t\in[0,1]} I_\lambda(\gamma_m(t)) \to c(\lambda)$$

having "nice" localisation properties. Namely, starting from a level strictly below $c(\lambda)$, the "top" of each path is contained in same ball centred at the origin whose radius $\beta(\lambda) > 0$ has the property $\beta(\lambda) \to 0$ when both $c(\lambda) \to 0$ and $c'(\lambda) \to 0$. To see this let $\lambda \in]0, \lambda_0[$ be an arbitrary but fixed value where $c'(\lambda)$ exists. Let $\{\lambda_m\} \subset]0, \lambda_0[$ be a strictly decreasing sequence with $\lambda_m \to \lambda$. Our claim is a direct consequence of the following result.

Proposition 3.1 *For any $\epsilon > 0$ there exists a sequence of paths $\{\gamma_m\} \subset \Gamma_\lambda$ such that for $m \in \mathbb{N}$ sufficiently large*

(i) $||\gamma_m(t)||_2^2 \leq c'(\lambda) + 3\epsilon$ *when*

$$I_\lambda(\gamma_m(t)) \geq c(\lambda) - \epsilon(\lambda_m - \lambda). \tag{3.1}$$

(ii) $\max_{t\in[0,1]} I_\lambda(\gamma_m(t)) \leq c(\lambda) + (c'(\lambda) + 2\epsilon)(\lambda_m - \lambda).$

Making the choice $\epsilon = c(\lambda) > 0$ we have when (3.1) hold

$$||\gamma_m(t)||^2 \leq (5 + 3K)\ c(\lambda) + (1 + K)\ c'(\lambda) := \beta^2(\lambda)$$

for $K > 0$ defined in (H3) and $m \in \mathbb{N}$ sufficiently large.

Proof. Let $\{\gamma_m\} \subset \Gamma_\lambda$ be an arbitrary sequence such that

$$\max_{t\in[0,1]} I_{\lambda_m}(\gamma_m(t)) \leq c(\lambda_m) + \epsilon(\lambda_m - \lambda). \tag{3.2}$$

Note that such sequence exists since $\Gamma_{\lambda_m} \subset \Gamma_\lambda$ for all $m \in \mathbb{N}$. When $\gamma_m(t)$ satisfies (3.1) we have

$$\begin{aligned} \frac{I_{\lambda_m}(\gamma_m(t)) - I_\lambda(\gamma_m(t))}{\lambda_m - \lambda} &\leq \frac{c(\lambda_m) + \epsilon(\lambda_m - \lambda) - c(\lambda) + \epsilon(\lambda_m - \lambda)}{\lambda_m - \lambda} \\ &= \frac{c(\lambda_m) - c(\lambda)}{\lambda_m - \lambda} + 2\epsilon. \end{aligned}$$

Also since $c'(\lambda)$ exists, there is $m_0 = m_0(\lambda, \epsilon)$ such that for all $m \geq m_0$

$$c'(\lambda) - \epsilon \leq \frac{c(\lambda_m) - c(\lambda)}{\lambda_m - \lambda} \leq c'(\lambda) + \epsilon. \tag{3.3}$$

Thus for all $m \geq m_0$

$$||\gamma_m(t)||_2^2 = \frac{I_{\lambda_m}(\gamma_m(t)) - I_\lambda(\gamma_m(t))}{\lambda_m - \lambda} \leq c'(\lambda) + 3\epsilon$$

and this proves (i). Now to get (ii) notice that from (3.3) we have for all $m \geq m_0$

$$c(\lambda_m) \leq c(\lambda) + (c^{'}(\lambda) + \epsilon)(\lambda_m - \lambda). \tag{3.4}$$

Using (3.2), (3.4) and since $I_{\lambda_m}(v) \geq I_\lambda(v)$ for all $v \in H$, it follows that for all $t \in [0,1]$

$$\begin{aligned} I_\lambda(\gamma_m(t)) &\leq I_{\lambda_m}(\gamma_m(t)) \\ &\leq c(\lambda_m) + \epsilon(\lambda_m - \lambda) \\ &\leq c(\lambda) + (c^{'}(\lambda) + \epsilon)(\lambda_m - \lambda) + \epsilon(\lambda_m - \lambda). \end{aligned}$$

This ends the proof of (ii). Finally if we choose $\epsilon = c(\lambda) > 0$ we both have when (3.1) holds and $m \in \mathbb{N}$ is sufficiently large

$$||\gamma_m(t)||_2^2 \leq c^{'}(\lambda) + 3c(\lambda) \ \text{ and } \ I_\lambda(\gamma_m(t)) \leq 2c(\lambda).$$

Thus using (2.2) we get

$$\begin{aligned} ||\nabla\gamma_m(t)||_2^2 &= I_\lambda(\gamma_m(t)) - \lambda||\gamma_m(t)||_2^2 + 2\int_{\mathbb{R}^N} \tilde{F}(x, \gamma_m(t))\, dx \\ &\leq I_\lambda(\gamma_m(t)) + 2\int_{\mathbb{R}^N} \tilde{F}(x, \gamma_m(t))\, dx \\ &\leq I_\lambda(\gamma_m(t)) + K||\gamma_m(t)||_2^2 \\ &\leq 2c(\lambda) + K(c^{'}(\lambda) + 3c(\lambda)). \end{aligned}$$

We deduce that, when (3.1) holds,

$$\begin{aligned} ||\gamma_m(t)||^2 &:= ||\nabla\gamma_m(t)||_2^2 + ||\gamma_m(t)||_2^2 \\ &\leq (5 + 3K)\ c(\lambda) + (1 + K)\ c^{'}(\lambda). \end{aligned}$$

This ends the proof of the proposition. ♠

4 A Critical Point For Almost Every $\lambda \in]0, \lambda_0]$

In this section we prove that when $c^{'}(\lambda)$ exists the functional I_λ has a nontrivial critical point which is contained in the ball of radius $2\beta(\lambda)$ centred at the origin. For $a > 0$ we define

$$F_a = \{u \in H : ||u|| \leq 2\beta(\lambda) \text{ and } |I_\lambda(u) - c(\lambda)| \leq a\}$$

Proposition 4.1 *For all $a > 0$*

$$\inf\{||I_\lambda^{'}(u)|| : u \in F_a\} = 0. \tag{4.1}$$

Proof. Seeking a contradiction we assume that (4.1) does not hold. Then, because of the Mountain-Pass geometry (see Lemma 2.1), $a > 0$ can be chosen such that for any $u \in F_a$

$$||I_\lambda^{'}(u)|| \geq a \text{ and } 0 < a < \frac{1}{2}c(\lambda). \tag{4.2}$$

A classical deformation argument says that there exist $\mu \in]0, a[$ and a homeomorphism $\eta : H \to H$ such that

$$\eta(u) = u \text{ if } |I_\lambda(u) - c(\lambda)| \geq a \tag{4.3}$$

$$I_\lambda(\eta(u)) \leq I_\lambda(u) \text{ for all } u \in H \tag{4.4}$$

and

$$I_\lambda(\eta(u)) \leq c(\lambda) - \mu \text{ for all } u \in H \text{ satisfying } ||u|| \leq \beta(\lambda) \text{ and } I_\lambda(u) \leq c(\lambda) + \mu. \tag{4.5}$$

Let $\{\gamma_m\} \subset \Gamma_\lambda$ be the sequence obtained in Proposition 3.1 where the choice $\epsilon = c(\lambda) > 0$ is made. By Proposition 3.1 (ii) we can select a $k \in \mathbb{N}$ sufficiently large so that

$$\max_{t \in [0,1]} I_\lambda(\gamma_k(t)) \leq c(\lambda) + \mu. \tag{4.6}$$

Clearly by (4.2) and (4.3), $\eta \circ \gamma_k \in \Gamma_\lambda$. Now if $u = \gamma_k(t)$ with $I_\lambda(u) \leq c(\lambda) - c(\lambda)(\lambda_k - \lambda)$ then (4.4) implies that

$$I_\lambda(\eta(u)) \leq c(\lambda) - c(\lambda)(\lambda_k - \lambda). \tag{4.7}$$

On the other hand if $u = \gamma_k(t)$ with $I_\lambda(u) > c(\lambda) - c(\lambda)(\lambda_k - \lambda)$ then Proposition 3.1 and (4.6) imply that u is such that $||u|| \leq \beta(\lambda)$ with $I_\lambda(u) \leq c(\lambda) + \mu$. Now (4.5) gives that

$$I_\lambda(\eta(u)) \leq c(\lambda) - \mu \tag{4.8}$$

which, combined with (4.7), yields

$$\max_{t \in [0,1]} I_\lambda(\eta \circ \gamma_k(t)) < c(\lambda).$$

This contradicts the variational characterisation of $c(\lambda)$ and proves the proposition. ♠

Lemma 4.1 *Assume that (H1)-(H5) hold. For all $\lambda \in]0, \lambda_0]$ where $c^{'}(\lambda)$ exists there is $u_\lambda \in H$ with $0 < ||u_\lambda|| \leq 2\beta(\lambda)$ such that $(\lambda, u_\lambda) \in \mathbb{R}^+ \times H$ is a solution of $(\tilde{1})_\lambda$.*

Proof. ¿From Proposition 4.1, when $c^{'}(\lambda)$ exists, I_λ has a Palais-Smale sequence $\{u_m\} \subset H$ at the level $c(\lambda)$ which is contained in the ball of radius $2\beta(\lambda)$ centred at the origin. Since $\{u_m\}$ is bounded, without loss of generality we can assume that $u_m \rightharpoonup u_\lambda$ weakly in H. To end the proof we just need to show that $u_m \to u_\lambda$ strongly in H. The condition $I_\lambda^{'}(u_m) \to 0$ in H^{-1} is just

$$-\Delta u_m + \lambda u_m - \tilde{f}(x, u_m) \to 0 \;\text{ in } H^{-1}.$$

Because of (H2), $\tilde{f}(x, u_m) \to \tilde{f}(x, u_\lambda)$ in H^{-1}. This is a classical result (see for example [13], Lemma 5.2) that we give without proof. Actually, thanks to the vanishing of $\tilde{f}(x, s)$ at infinity we are somehow back to the case of a bounded domain. We deduce that

$$-\Delta u_m + \lambda u_m \to \tilde{f}(x, u_\lambda) \;\text{ in }\; H^{-1}. \tag{4.9}$$

Now let $L : H \to H^{-1}$ be defined by

$$(Lu)v = \int_{\mathbb{R}^N} (\nabla u \nabla v + \lambda uv)\, dx.$$

Since $\lambda > 0$, L is invertible and we deduce from (4.9) that $u_m \to L^{-1}\tilde{f}(x, u_\lambda)$ in H. Consequently by uniqueness of the limit, $u_m \to u_\lambda$ in H. ♠

5 A Special Sequence $\lambda_n \to 0$

In this section we show that (H5), which insures a lower bound on $F(x, .)$ around zero, implies that $c(\lambda)\lambda^{-1} \to 0$ as $\lambda \to 0$. From this result we deduce the existence of a strictly decreasing sequence $\lambda_n \to 0$ with $c(\lambda_n) \to 0$ and $c^{'}(\lambda_n) \to 0$. In particular, then $\beta(\lambda_n) \to 0$ and from Lemma 4.1 we get a bifurcating sequence for $(\tilde{1})_\lambda$.

Lemma 5.1 *Assume that (H1)-(H5) hold. Then $c(\lambda)\lambda^{-1} \to 0$ as $\lambda \to 0$.*

Proof. We use again the test functions $v_\lambda(x) = \delta e^{-\sqrt{\lambda}|x|}$ introduced in Lemma 2.3. By the proof of Lemma 2.3 we know that $I_\lambda(v_\lambda) < 0$ for all $\lambda \in]0, \lambda_0]$. Thus, because of the variational characterisation of $c(\lambda)$, necessarily $c(\lambda) \leq \max_{t\in[0,1]} I_\lambda(tv_\lambda)$. ¿From (2.4) we have for all $\lambda \in]0, \lambda_0]$ and all $t \in [0, 1]$

$$\begin{aligned} I_\lambda(tv_\lambda) &= t^2 \left[||\nabla v_\lambda||_2^2 + \lambda\; ||v_\lambda||_2^2 \right] - 2\int_{\mathbb{R}^N} \tilde{F}(x, tv_\lambda)\, dx \\ &= 2B(N)\lambda^{1-\frac{N}{2}} t^2 - 2\int_{\mathbb{R}^N} \tilde{F}(x, tv_\lambda)\, dx \end{aligned}$$

where $B(N)$ is defined in (2.3). We distinguish the cases (i) and (ii) in (H5). When (H5)(i) holds we have using (2.5) that for all $\lambda \in]0, \lambda_0]$ and all $t \in [0, 1]$

$$\begin{aligned} \int_{\mathbb{R}^N} \tilde{F}(x, tv_\lambda)\, dx &\geq A \int_{\mathbb{R}^N} (1 + |x|)^{-d} |tv_\lambda|^{\alpha+2}\, dx \\ &= t^{\alpha+2}\; A \int_{\mathbb{R}^N} (1 + |x|)^{-d} |v_\lambda|^{\alpha+2}\, dx \\ &\geq D(N, d, \alpha)\lambda^{\frac{d}{2}-\frac{N}{2}} t^{\alpha+2}. \end{aligned}$$

Thus for all $\lambda \in]0, \lambda_0]$ and all $t \in [0,1]$

$$I_\lambda(tv_\lambda) \leq 2B(N)\lambda^{1-\frac{N}{2}}t^2 - 2D(N,d,\alpha)\lambda^{\frac{d}{2}-\frac{N}{2}}t^{\alpha+2}. \tag{5.1}$$

Elementary calculations show that the maximum of the right hand side of (5.1), viewed as a function of $t \in [0,1]$, is of the form $C(N,d,\alpha)\lambda^\theta$ with $\theta = (1-\frac{N}{2})(\frac{\alpha+2}{\alpha}) - \frac{2}{\alpha}(\frac{d}{2}-\frac{N}{2})$. Thus recording that, by (H5)(i), $\alpha < 2(2-d)N^{-1}$ we deduce that $c(\lambda)\lambda^{-1} \to 0$ as $\lambda \to 0$. Now when (H5)(ii) holds we have, using (2.6), that for all $\lambda \in]0, \lambda_0]$ and all $t \in [0,1]$

$$\begin{aligned} \int_{\mathbb{R}} \tilde{F}(x, tv_\lambda)\,dx &\geq \int_{\mathbb{R}} r(x)|tv_\lambda|^{\alpha+2}\,dx \\ &= t^{\alpha+2}\int_{\mathbb{R}} r(x)|v_\lambda|^{\alpha+2}\,dx \\ &\geq E(\alpha)t^{\alpha+2}. \end{aligned}$$

Thus for all $\lambda \in]0, \lambda_0]$ and all $t \in [0,1]$

$$I_\lambda(tv_\lambda) \leq 2B(1)\sqrt{\lambda}t^2 - 2E(\alpha)t^{\alpha+2}. \tag{5.2}$$

The maximum of the right hand side of (5.2) is of the form $C(\alpha)\lambda^\theta$ with $\theta = (\frac{\alpha+2}{2\alpha})$ and here also we deduce, since $\alpha < 2$ by (H5)(ii), that $c(\lambda)\lambda^{-1} \to 0$ as $\lambda \to 0$. ♠

Lemma 5.2 *Assume that (H1)-(H5) hold. There exists a strictly decreasing sequence $\lambda_n \to 0$ with $c(\lambda_n) \to 0$ and $c'(\lambda_n) \to 0$.*

Proof. ¿From Lemma 5.1 we know that $c(\lambda)\lambda^{-1} \to 0$ as $\lambda \to 0$ and thus trivially $c(\lambda) \to 0$ as $\lambda \to 0$. We claim there exists $\lambda_n \searrow 0$ with $c'(\lambda_n) \to 0$. Seeking a contradiction we assume that

$$a := \liminf_{\lambda \to 0} c'(\lambda) > 0.$$

Since the function $\lambda \to c(\lambda)$ is non decreasing and positive, we have for $\lambda > 0$ sufficiently small,

$$\begin{aligned} c(\lambda) &\geq c(\lambda) - \lim_{h\to 0} c(h) \\ &\geq \lim_{h\to 0}\int_h^\lambda c'(t)\,dt \\ &\geq \lim_{h\to 0}\int_h^\lambda \frac{a}{2}\,dt = \frac{a}{2}\lambda. \end{aligned}$$

This is in contradiction with $c(\lambda)\lambda^{-1} \to 0$ as $\lambda \to 0$ and the lemma is proved. ♠

6 Conclusion

In this last section we end the proof of Theorem 1.1 showing that the bifurcating sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R}^+ \times H$ obtained in Section 5 for $(\tilde{1})_\lambda$ also satisfies $||u_n||_\infty \to 0$. We have for all $n \in \mathbb{N}$

$$\Delta u_n = \lambda_n u_n - \tilde{f}(x, u_n)$$

and thus because of (2.1)

$$|\Delta u_n| \leq \lambda_n |u_n| + |\tilde{f}(x, u_n)| \leq \lambda_0 |u_n| + K|u_n|. \tag{6.1}$$

Using the Calderon-Zymund estimate ([4], Chap. 2, 3, Prop. 8):

$$||\frac{\partial^2 u}{\partial x_i \partial x_j}||_p \leq C(p)||\Delta u||_p \text{ for all } u \in W^{2,p}(\mathbb{R}^N),\ 1 < p < \infty,$$

we see from (6.1) that if $\{u_n\} \subset L^p(\mathbb{R}^N)$ for a $p \in]1, \infty[$ then also $\{u_n\} \subset W^{2,p}(\mathbb{R}^N)$ and

$$||u_n||_{W^{2,p}(\mathbb{R}^N)} \leq C(p)||u_n||_p \text{ for all } n \in \mathbb{N}. \tag{6.2}$$

We recall the continuous embedding $W^{2,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ for any $p > \frac{N}{2}$. Thus, when $N \leq 3$, using the estimate (6.2) with $p = 2$, we directly check that $||u_n||_\infty \leq C||u_n||_2$ for a $C > 0$ and all $n \in \mathbb{N}$. When $N \geq 4$ we use the embedding $W^{2,p}(\mathbb{R}^N) \hookrightarrow L^{\frac{Np}{N-2p}}(\mathbb{R}^N)$ and iterate (6.2) starting from $p = 2$. After a finite number of steps we also obtain that $||u_n||_\infty \leq C||u_n||_2$ for a $C > 0$ and all $n \in \mathbb{N}$. Since $||u_n||_2 \to 0$ it completes the proof of Theorem 1.1.

Remark 6.1 The boot-strap argument is particularly simple here because of the "linear" estimate (6.1). This estimate follows from our choice of $\tilde{f}$. In the classical variational approach, where (SQC) must hold, proving that the bifurcation in L^∞ also occurs is more difficult (see also [2]). □

Remark 6.2 The condition (H5) used to show that I_λ has a Mountain-Pass geometry (see Lemma 2.1) and that $c(\lambda)\lambda^{-1} \to 0$ as $\lambda \to 0$ (see Lemma 5.1) is sharp. Indeed for $N \geq 3$ the nonlinearity defined on $\mathbb{R}^N \times \mathbb{R}$ by $f(x, s) = (1 + |x|)^{-d}|s|^{2+\alpha}$ with $d \in]0, 2[$ and $\alpha \in]\frac{2(2-d)}{N}, \frac{4}{N}]$ satisfies (H1)-(H4) but not (H5). It is proved in Theorem 4.8 of [14] that with this f, $(1)_\lambda$ has no bifurcating sequence. For this f, Lemma 2.1 holds and thus, necessarily, $c(\lambda)\lambda^{-1} \to 0$ as $\lambda \to 0$ is no longer true. □

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