MULTIPLE SOLUTIONS FOR AN INDEFINITE ELLIPTIC PROBLEM WITH CRITICAL GROWTH IN THE GRADIENT

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Abstract. We consider the problem

$$(P) \qquad -\Delta u = c(x)u + \mu|\nabla u|^2 + f(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 3$, $\mu > 0$ and $c, f \in L^q(\Omega)$ for some $q > \frac{N}{2}$ with $f \gneqq 0$. Here c is allowed to change sign and we assume that $c^+ \not\equiv 0$. We show that when c^+ and μf are suitably small this problem has at least two positive solutions. This result contrasts with the case $c \leq 0$, where uniqueness holds. To show this multiplicity result we first transform (P) into a semilinear problem having a variational structure. Then we are led to the search of two critical points for a functional whose superquadratic part is indefinite in sign and has a so called *slow growth* at infinity. The key point is to show that the Palais-Smale condition holds.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ with $N \geq 3$. In this paper we are concerned with the boundary value problem

$$(P) \qquad -\Delta u = c(x)u + \mu|\nabla u|^2 + f(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

where

$$(\mathcal{H}) \qquad \mu > 0, \quad f \gneqq 0 \quad \text{and} \quad c, f \in L^q(\Omega) \text{ for some } q > \frac{N}{2}.$$

Quasilinear elliptic equations with a gradient dependence up to the critical growth $|\nabla u|^2$ were first studied by Boccardo, Murat and Puel in the 80's [12, 13, 14] and have been an active field of research until now, see for example [2, 18, 19]. To situate our problem we underline that we are interested in bounded solutions. The main goal of this paper is to carry on the study of non-uniqueness of solutions for such problems, which (P) is a prototype of.

The sign of c plays in (P) a central role regarding uniqueness, as well as existence, of bounded solutions. We refer to [20] for a heuristic discussion on the influence of the sign of c on the nature of the problem. The case $c \leq -\alpha_0$ a.e. in Ω for some $\alpha_0 > 0$ is referred to as the coercive case. In this case, the existence of solutions holds under very general assumptions and it was shown in [9, 10] (see also [8, 11]) that there is a unique bounded solution. When one just requires $c \leq 0$ (in particular when $c \equiv 0$) the situation is already more complex. The fact that restrictions on the data are necessary for (P) to have a solution was first observed in [16, 17]. Concerning uniqueness, some partial results are given in [9, 10], but it was only in [6] that uniqueness of bounded solutions was established under the mere condition $c \leq 0$. See also [7] for an extension to a larger class of problems.

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The case $c \gneqq 0$ started to be studied only recently. Surely in part because it was not accessible by the methods traditionally used in the coercive case. In [20] it was shown that when $c \gneqq 0$ and c, μ and f are sufficiently small in an appropriate sense, (P) has two solutions. See also [1, 3, 24] for related results. Note that the case where μ is allowed to be non constant was treated in [6] leading also, when $c \gneqq 0$ and under appropriate conditions, to the existence of two bounded solutions.

In view of these results it remained to analyse the case where c is allowed to change sign, which is the aim of the present paper. Roughly speaking we shall show that the uniqueness is lost as soon as $c^+ \not\equiv 0$, where $c^+ = \max\{0, c\}$, see Theorem 1.1.

We first observe that (P) is equivalent to

$$(P') \qquad -\Delta w = c(x)w + |\nabla w|^2 + \mu f(x), \quad w \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Indeed, it is easy to check that u is a solution of (P) if and only if $w = \mu u$ is a solution of (P'). Now, we use the change of variable

$$v = e^w - 1, \tag{1.1}$$

which goes back to [21] and rids the gradient term of (P'), reducing it to a semilinear problem with a variational structure, namely,

$$(Q) \qquad -\Delta v - (c(x) + \mu f(x))v = c(x)g(v) + \mu f(x), \quad v \in H_0^1(\Omega) \cap L^\infty(\Omega),$$

where

$$g(s) = \begin{cases} (1+s)\ln(1+s) - s & \text{if } s \geq 0 \\ 0 & \text{if } s \leq 0. \end{cases} \tag{1.2}$$

We shall prove in Lemma 2.1 that if v is a non negative solution of (Q) then w defined by (1.1) is a non negative (and therefore positive, by Harnack inequality) solution of (P'). Solutions of (Q) will be obtained as critical points of the functional

$$I(v) = \frac{1}{2}\int_\Omega \left[|\nabla v|^2 - [c(x) + \mu f(x)](v^+)^2\right] - \int_\Omega c(x)G(v^+) - \mu \int_\Omega f(x)v$$

defined on $H_0^1(\Omega)$ and where $G(s) = \int_0^s g(t)\,dt$. Note that since $f \geq 0$, critical points of I are necessarily non-negative, see Lemma 2.1. Since g behaves essentially as $s \ln s$ for s large, the superquadratic part of I has at infinity a growth which is usually referred to as a *slow superlinear growth*.

To obtain two critical points we start following the strategy used in [20]. Note that if the positive part of $c + \mu f$ is not 'too large' in a suitable sense (cf. Lemma 2.2) then

$$\int_\Omega \left[|\nabla v|^2 - [c(x) + \mu f(x)](v^+)^2\right]$$

is coercive. Moreover, as g is superlinear, we shall prove that I takes positive values on a sphere $\|v\| = \rho$ if either c or μf is sufficiently small. Moreover it is easily seen that since $f \not\equiv 0$, I takes negative values in the ball $B(0, \rho)$. Finally, since $c^+ \not\equiv 0$, it is possible to show that I takes a negative value at some point v_0 outside of the ball $B(0, \rho)$. Thus I has a mountain-pass geometry and it is reasonable to search for a first critical point as a minimizer of I in $B(0, \rho)$ and a second one at the mountain pass level. The existence of a minimizer will follow from a standard lower semi continuity argument, whereas in the proof of the existence of a mountain-pass critical point we will face the difficulty of showing that Palais-Smale sequences are bounded.

We recall that the Palais-Smale condition holds for I if any sequence $(u_n) \subset H_0^1(\Omega)$ such that $(I(u_n)) \subset \mathbb{R}$ is bounded and $||I'(u_n)||_* \to 0$ admits a convergent subsequence. The boundedness of such sequences proves to be a delicate issue due to the fact that c is sign-changing and g has a slow growth at infinity. In particular g does not satisfy an

Ambrosetti-Rabinowitz type condition. Let us recall that a nonlinearity f is said to satisfy the Ambrosetti-Rabinowitz condition if

$$(\mathcal{AR}) \qquad \text{There exist } \theta > 2 \text{ and } s_1 > 0 \text{ such that } 0 < \theta F(s) \le s f(s) \quad \forall s \ge s_1,$$

where $F(s) = \int_0^s f(t)\,dt$. This condition is known to be central when proving that Palais-Smale sequences are bounded. When the domain $\Omega \subset \mathbb{R}^N$ is bounded and the nonlinearity is subcritical, the boundedness leads directly to the strong convergence of a subsequence.

In the case where the superquadratic term is positive, many efforts have been done to weaken the condition $(\mathcal{AR})$. However, to the best of our knowledge, this issue has not been considered for functionals of the type

$$J(u) = \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - c(x) F(u) \right), \quad u \in H_0^1(\Omega)$$

when c changes sign and f is a superlinear function not satisfying $(\mathcal{AR})$. A typical example of such a nonlinearity is $f(s) = s \ln(s+1)$.

When $f(s) = s^{p-1}$ with $p \in [2, 2^*)$, using the homogeneity of f it is straightforward that J satisfies the Palais-Smale condition. When f is not powerlike, this issue becomes delicate, as shown in [5] (see also [4]), where the authors assume that f is superlinear and asymptotically powerlike at infinity, i.e.

$$(\mathcal{G}) \qquad \text{There exist } p > 2 \text{ such that } \lim_{s \to \infty} \frac{f(s)}{s^{p-1}} = 1.$$

Note that this condition implies $(\mathcal{AR})$. Furthermore, in [5] one needs to assume the so called *thick zero set* condition on $c \in \mathcal{C}(\overline{\Omega})$:

$$(\mathcal{AT}) \qquad \overline{(\Omega_+)} \cap \overline{(\Omega_-)} = \emptyset,$$

where

$$\Omega_+ := \{x \in \Omega;\ c(x) > 0\} \quad \text{and} \quad \Omega_- := \{x \in \Omega;\ c(x) < 0\}.$$

In [23], still under $(\mathcal{G})$, the authors were able to remove $(\mathcal{AT})$, but at the expense of some alternative strong conditions on c.

In our problem we prove that the Palais-Smale condition is satisfied without assuming $(\mathcal{AT})$ nor any special condition on c. Given $V \in L^q(\Omega)$, with $q > \frac{N}{2}$, we denote by $\lambda_1(V) = \lambda_1(V, \Omega)$ the first eigenvalue of the problem

$$\begin{cases} -\Delta u + V(x) u = \mu u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Let us recall that $\lambda_1(V)$ is given by

$$\lambda_1(V) = \inf \left\{ \int_\Omega \left(|\nabla v|^2 + V(x) v^2 \right);\ u \in H_0^1(\Omega),\ \|u\|_2 = 1 \right\}.$$

It is well-known that $\lambda_1(V)$ is simple, so that it is achieved by an unique $\varphi_1 > 0$ such that $\|\varphi_1\|_2 = 1$, cf [22].

Our main result is the following:

Theorem 1.1. *Assume $(\mathcal{H})$ and $c^+ \not\equiv 0$. Then (P) has two positive solutions if either one of the following conditions hold:*

(1) $\lambda_1(-\mu f) > 0$ and $\|c^+\|_q < K$, where K is a constant depending on f and μ.
(2) $\lambda_1(-c) > 0$ and $\|\mu f\|_q < K$, where K is a constant depending on c.

Remark 1.2. *In* [20, Theorem 2], *assuming $c \gneqq 0$, it is proved that if*

$$\|\mu f\|_{\frac{N}{2}} < C_N, \tag{1.3}$$

where C_N denotes the best Sobolev constant for the embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$, then there exists $\overline{c} > 0$ such that (P) has at least two bounded solutions if $\|c\|_q < \overline{c}$. We observe that under (1.3) we have $\lambda_1(-\mu f) > 0$. Thus Theorem 1.1 (2) is consistent with [20, Theorem 2]. *We point out however that f is allowed to be sign changing in* [20].

In [6], see Corollary 3.2 and Remark 3.2, it is shown that when $c \equiv 0$, (P) has a solution if and only if $\lambda_1(-\mu f) > 0$. We now complement this result.

Lemma 1.3. *Assume $(\mathcal{H})$.*

(1) If $c \geq 0$ then $\lambda_1(-c-\mu f) > 0$ is necessary for (P) to have a non-negative solution.
(2) $\lambda_1(-c) > 0$ is necessary for (P) to have a non negative solution and under this condition every solution of (P) is non-negative.

Remark 1.4. *As far as non negative solutions are concerned, Lemma 1.3 (1) shows that when $c \geq 0$ the condition $\lambda_1(-c-\mu f) > 0$ is necessary in Theorem 1.1. However, other kinds of solutions of (P), namely negative or sign-changing solutions, may exist if $\lambda_1(-c-\mu f) \leq 0$. See* [15] *in this direction.*

This paper is organized as follows. In Section 2 we prove some preliminary results and show that the functional I has the geometry described above. Section 3 is devoted to the Palais-Smale condition for I. Finally in Section 4 we prove Theorem 1.1 and Lemma 1.3. Also in Remark 4.1 we discuss the necessity of some assumptions in Theorem 1.1.

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1.1. **Notation.**

- The Lebesgue norm in $L^r(\Omega)$ will be denoted by $\|\cdot\|_r$ and the usual norm of $H_0^1(\Omega)$ by $\|\cdot\|$, i.e. $\|u\| = \|\nabla u\|_2$. The Holder conjugate of r is denoted by r'.
- The weak convergence is denoted by $\rightharpoonup$.
- The positive and negative parts of a function u are defined by $u^{\pm} := \max\{\pm u, 0\}$.
- We denote by $B(0,R)$ the ball of radius R centered at 0 in $H_0^1(\Omega)$.

2. Preliminaries

Lemma 2.1. *Assume $(\mathcal{H})$.*

(1) If v is a non-negative solution of (Q) then $w = \ln(1+v)$ is a non-negative solution of (P'). Similarly if w is a non negative solution of (P') then v given by (1.1) is a non negative solution of (Q).
(2) If v is a critical point of I then v is a non-negative solution of (Q).
(3) If u is a non-negative solution of (P) then u is positive.

Proof. Let $v \geq 0$ be a solution of (Q). From the expression of g it is seen that v solves

$$-\Delta v = c(x)(1+v)\ln(1+v) + \mu f(x)(1+v). \tag{2.1}$$

Let $w = \ln(1+v)$, i.e. $e^w = 1+v$. Since $v \geq 0$ and $\nabla w = \frac{\nabla v}{1+v}$, one may easily see that $w \in H_0^1(\Omega)$. If $\phi \in H_0^1(\Omega)$ then $\psi = \frac{\phi}{1+v} \in H_0^1(\Omega)$, so that (2.1) provides

$$\begin{aligned} \int_\Omega \nabla v \nabla \psi &= \int_\Omega c(x)\psi(1+v)\ln(1+v) + \mu \int_\Omega f(x)\psi(1+v) \\ &= \int_\Omega c(x)\phi \ln(1+v) + \mu \int_\Omega f(x)\phi. \end{aligned} \tag{2.2}$$

Now, from $\nabla v = e^w \nabla w$ and $\nabla \psi = \frac{\nabla \phi}{1+v} - \frac{\phi \nabla v}{(1+v)^2}$, we get

$$\begin{aligned}\int_\Omega \nabla v \nabla \psi &= \int_\Omega e^w \nabla w \left(\frac{\nabla \phi}{1+v} - \frac{\phi \nabla v}{(1+v)^2} \right) = \int_\Omega \nabla w \left(\nabla \phi - \frac{\phi \nabla v}{1+v} \right) \\ &= \int_\Omega \nabla w \left(\nabla \phi - \frac{\phi e^w \nabla w}{1+v} \right) = \int_\Omega \left(\nabla w \nabla \phi - |\nabla w|^2 \phi \right).\end{aligned}$$

Furthermore, we have

$$\int_\Omega c(x) \phi \ln(1+v) = \int_\Omega c(x) w \phi,$$

so we deduce from (2.2) that u is a solution of (P'). By similar arguments we prove the reverse statement. This proves (1).

To prove (2), let v be a critical point of I. Then

$$\int_\Omega \left[\nabla v \nabla \varphi - (c(x) + \mu f(x)) v^+ \varphi \right] - \int_\Omega c(x) g(v^+) \varphi - \mu \int_\Omega f(x) \varphi = 0 \tag{2.3}$$

for all $\varphi \in H_0^1(\Omega)$. Taking $\varphi = -v^-$ we get

$$\int_\Omega |\nabla v^-|^2 + \mu \int_\Omega f(x) v^- = 0.$$

Since $f \geq 0$, we get

$$\int_\Omega |\nabla v^-|^2 \leq 0$$

and it follows that $v^- \equiv 0$, i.e. $v \geq 0$. The proof that $v \in L^\infty(\Omega)$ can be found in [20, Lemma 13], so we omit it.

Finally, if $u \geq 0$ is a solution of (P) then, since $\mu > 0$ and $f \geq 0$, u is a bounded weak supersolution of

$$-\Delta u = c(x) u, \quad u \in H_0^1(\Omega).$$

By a standard argument relying on the Harnack inequality, see [25, Theorem 1.2], we have either $u \equiv 0$ or $u > 0$. Since $f \gneqq 0$, we get $u > 0$. □

We shall now prove that when $\lambda_1(-c - \mu f) > 0$ the functional I takes positive values on a sphere centered at the orign if either $\|c^+\|_q$ or $\|\mu f\|_q$ is small enough.

Lemma 2.2. *Let $V \in L^q(\Omega)$, with $q > \frac{N}{2}$. If $\lambda_1(V) > 0$ then there exists $K_1 > 0$ such that*

$$\int_\Omega \left(|\nabla v|^2 + V(x)(v^+)^2 \right) \geq K_1 \|v\|^2 \quad \forall v \in H_0^1(\Omega). \tag{2.4}$$

Proof. Let us first prove that there exists a constant $K_1 > 0$ such that

$$\int_\Omega \left(|\nabla v|^2 + V(x) v^2 \right) \geq K_1 \|v\|^2 \quad \forall v \in H_0^1(\Omega). \tag{2.5}$$

Indeed, assume by contradiction that there is a sequence $(v_n) \subset H_0^1(\Omega)$ such that

$$\int_\Omega \left(|\nabla v_n|^2 + V(x)(v_n)^2 \right) \leq \frac{\|v_n\|^2}{n}.$$

Setting $w_n = \frac{v_n}{\|v_n\|}$ we may assume that, up to a subsequence,

$$w_n \rightharpoonup w_0 \text{ in } H_0^1(\Omega) \quad \text{and} \quad w_n \to w_0 \text{ in } L^r(\Omega) \text{ for } r \in [1, 2^*).$$

In particular since $q > \frac{N}{2}$ we have that $w_n \to w_0$ in $L^{2q'}(\Omega)$. Thus from

$$\int_\Omega \left(|\nabla w_n|^2 + V(x)(w_n)^2\right) \leq \frac{1}{n} \tag{2.6}$$

it follows that

$$\int_\Omega \left(|\nabla w_0|^2 + V(x)(w_0)^2\right) \leq 0. \tag{2.7}$$

We claim that $w_0 \not\equiv 0$. Indeed, if $w_0 \equiv 0$ then $w_n \to 0$ in $L^{2q'}(\Omega)$ and (2.6) yields $w_n \to 0$ in $H_0^1(\Omega)$, which is impossible since $\|w_n\| = 1$. Hence $w_0 \not\equiv 0$ and consequently (2.7) provides $\lambda_1(V) \leq 0$, which contradicts our assumption. Thus (2.5) is proved. Finally, we may assume that $K_1 \leq 1$, so that

$$\begin{aligned} \int_\Omega \left(|\nabla v|^2 + V(x)(v^+)^2\right) &= \int_\Omega |\nabla v^-|^2 + \int_\Omega \left(|\nabla v^+|^2 + V(x)(v^+)^2\right) \\ &\geq \|v^-\|^2 + K_1\|v^+\|^2 \geq K_1\|v\|^2. \end{aligned}$$

□

We are now ready to prove that I has the appropriate geometry. Note that g given by (1.2) satisfies

$$\lim_{s\to 0} \frac{g(s)}{s^p} = \lim_{s\to\infty} \frac{g(s)}{s^p} = 0$$

if $p \in (1, 2)$. As a consequence, there exists a constant $C > 0$ such that

$$0 \leq G(s) \leq Cs^{p+1}, \quad \forall s \in \mathbb{R}. \tag{2.8}$$

Proposition 2.3. *Assume that $\lambda_1(-c-\mu f) > 0$. Given $R > 0$ sufficiently large, there exist $K, M > 0$ depending on R and such that:*

(1) If $\|c^+\|_q < K$ then $I(v) \geq M$ for every $v \in H_0^1(\Omega)$ with $\|v\| = R$.
(2) If $\|\mu f\|_q < K$ then $I(v) \geq M$ for every $v \in H_0^1(\Omega)$ with $\|v\| = R^{-1}$.

Proof. Since $\lambda_1(-c - \mu f) > 0$, by Lemma 2.2 there exists $K_1 > 0$ such that

$$\int_\Omega \left(|\nabla v|^2 - [c(x) + \mu f(x)](v^+)^2\right) \geq K_1\|v\|^2 \quad \forall v \in H_0^1(\Omega).$$

Let $p \in (1, 2)$. By (2.8) we have

$$I(v) \geq K_1\|v\|^2 - C_1\|c^+\|_q\|v|^{p+1} - C_2\|\mu f\|_q\|v\|$$

for some $C_1, C_2 > 0$. If $\|v\| = R$ and $\|c^+\|_q \leq R^{-\beta}$, with $\beta > p - 1$, then

$$I(v) \geq K_1R^2 - C_1R^{p+1-\beta} - C_2\mu\|f\|_qR \geq R$$

for R sufficiently large. Thus (1) holds with $K = R^{-\beta}$ and $M = R$.

In a similar way, if now $\|v\| = R^{-1}$ and $\|\mu f\|_q \leq R^{-\beta}$, with $\beta > 1$ then

$$I(v) \geq K_1R^{-2} - C_1\|c^+\|_qR^{-(p+1)} - C_2R^{-\beta-1} \geq R^{-3}$$

for R sufficiently large. Hence we may take $K = R^{-\beta}$ and $M = R^{-3}$ to get (2). □

3. The Palais-Smale condition

We set

$$\alpha_c = \inf\left\{\int_\Omega \left(|\nabla u|^2 - \mu f(x)(u^+)^2\right); u \in H_0^1(\Omega),\ \|u\|_2 = 1,\ cu^+ \equiv 0\right\}.$$

In the next proposition, we shall use an explicit expression of G, namely,

$$G(s) = \frac{s^2}{2}\ln(s+1) - \frac{3}{4}s^2 + s\ln(s+1) - \frac{s}{2} + \frac{1}{2}\ln(s+1) \tag{3.1}$$

for $s > 0$.

Proposition 3.1. *If $\alpha_c > 0$ then I satisfies the Palais-Smale condition.*

Proof. Let (u_n) be a Palais-Smale sequence for I at the level $d \in \mathbb{R}$, i.e.

$$I(u_n) \to d \quad \text{and} \quad \|I'(u_n)\|_* \to 0. \tag{3.2}$$

From (3.2) we have

$$\frac{1}{2}\int_\Omega \left[|\nabla u_n|^2 - (c(x) + \mu f(x))(u_n^+)^2\right] - \int_\Omega c(x)G(u_n^+) - \mu\int_\Omega f(x)u_n = d + o(1) \tag{3.3}$$

and

$$\left|\int_\Omega \left[\nabla u_n \nabla\varphi - (c(x) + \mu f(x))u_n^+\varphi\right] - \int_\Omega c(x)g(u_n^+)\varphi - \mu\int_\Omega f(x)\varphi\right| \le \varepsilon_n\|\varphi\| \tag{3.4}$$

for some sequence $\varepsilon_n \to 0$ and for every $\varphi \in H_0^1(\Omega)$. In particular, we have

$$|\langle I'(u_n), u_n\rangle| \le \varepsilon_n\|u_n\|. \tag{3.5}$$

Let us assume that $\|u_n\| \to \infty$ and set $v_n = \frac{u_n}{\|u_n\|}$. Up to a subsequence, we have

$$v_n \rightharpoonup v_0 \text{ in } H_0^1(\Omega), \quad v_n \to v_0 \text{ in } L^r(\Omega),\ \forall r \in [1, 2^*), \quad \text{and} \quad v_n \to v_0 \ a.e. \text{ in } \Omega.$$

We claim that $cv_0^+ \equiv 0$. Indeed, from (3.4) we have, using the convergences above,

$$\int_\Omega c(x)\frac{g(u_n^+)}{\|u_n\|}\varphi = \int_\Omega \left[\nabla v_0\nabla\varphi - (c(x) + \mu f(x))v_0^+\varphi\right] + o(1) < \infty, \tag{3.6}$$

for every $\varphi \in H_0^1(\Omega)$. If $cv_0^+ \not\equiv 0$ then we may choose $\varphi \in H_0^1(\Omega)$ and a measurable subset $\Omega_\varphi \subset \Omega$ such that

$$|\Omega_\varphi| > 0, \quad cv_0^+\varphi > 0 \text{ on } \Omega_\varphi \subset \Omega, \quad \text{and} \quad cv_0^+\varphi = 0 \text{ on } \Omega \setminus \Omega_\varphi.$$

Now, using that $\lim_{s\to\infty}\frac{g(s)}{s} = \infty$, we have

$$\liminf c(x)\frac{g(u_n^+)}{\|u_n\|}\varphi = \liminf c(x)v_n^+\frac{g(\|u_n\|v_n^+)}{\|u_n\|v_n^+}\varphi = +\infty \quad \text{on} \quad \Omega_\varphi.$$

Fatou's lemma then yields a contradiction with (3.6). Therefore $cv_0^+ \equiv 0$. On the other hand, taking $\varphi = v_0$ in (3.4) and dividing it by $\|u_n\|$ we get

$$\int_\Omega \left[\nabla v_n\nabla v_0 - (c(x) + \mu f(x))v_n^+ v_0\right] \to 0,$$

so that, using $v_n \rightharpoonup v_0$ in $H_0^1(\Omega)$ and $cv_0^+ \equiv 0$, we get

$$\int_\Omega \left[|\nabla v_0|^2 - \mu f(x)(v_0^+)^2\right] = 0.$$

Thus $v_0 \equiv 0$ (otherwise $\alpha_c \leq 0$). Now from (3.4) we have, taking $\varphi = u_n$ and using the definition (1.2) of g,

$$\left| \int_\Omega (|\nabla u_n|^2 - \mu f(x))(u_n^+)^2 - \int_\Omega c(x)(1+u_n^+)\ln(1+u_n^+)u_n^+ - \mu \int_\Omega f(x) u_n^+ \right| \leq \varepsilon_n \|u_n\|. \tag{3.7}$$

Dividing by $||u_n||^2$ and using that $v_n \to 0$ in $L^r(\Omega), \forall r \in [1, 2^*)$ we get

$$1 - \int_\Omega c(x)(v_n^+)^2 \ln(1 + ||u_n|| v_n^+) \to 0.$$

Now, using the property $\ln(st) = \ln s + \ln t$, it follows that

$$1 - \ln(||u_n||) \int_\Omega c(x)(v_n^+)^2 - \int_\Omega c(x)(v_n^+)^2 \ln\left(v_n^+ + \frac{1}{||u_n||}\right) \to 0.$$

We claim that

$$\ln(||u_n||) \int_\Omega c(x)(v_n^+)^2 \to 0. \tag{3.8}$$

In that case we would get

$$\int_\Omega c(x)(v_n^+)^2 \ln\left(v_n^+ + \frac{1}{||u_n||}\right) \to 1,$$

which clearly contradicts the fact that $v_0 = 0$. To prove (3.8) we define for every $s > 0$

$$H(s) = \frac{1}{2} g(s)s - G(s).$$

From (1.2) and (2.8) it follows that

$$H(s) = \frac{s^2}{4} - s\ln(s+1) + \frac{s}{2} - \frac{1}{2}\ln(1+s). \tag{3.9}$$

From (3.5) we get

$$I(u_n) - \frac{1}{2}\langle I'(u_n), u_n \rangle = c + \varepsilon_n \|u_n\| + o(1),$$

which leads, using the definition of H, to

$$\int_\Omega c(x) H(u_n^+) - \frac{\mu}{2}\int_\Omega f(x) u_n = c + \varepsilon_n \|u_n\| + o(1). \tag{3.10}$$

Now, combining (3.9) and (3.10), we obtain

$$\begin{aligned} \frac{1}{4}\int_\Omega c(x)(u_n^+)^2 \quad &= \quad c + \varepsilon_n \|u_n\| + \frac{1}{2}\int_\Omega c(x) u_n^+ - \int_\Omega c(x) u_n^+ \ln(1+u_n^+) \\ &\quad + \frac{1}{2}\int_\Omega c(x)\ln(1+u_n^+) + \frac{\mu}{2}\int_\Omega f(x) u_n + o(1). \end{aligned}$$

Hence

$$\begin{aligned} \ln(||u_n||)\int_\Omega c(x)(v_n^+)^2 \quad &= \quad \frac{4\ln||u_n||}{||u_n||^2}\Big(c + \varepsilon_n\|u_n\| + \frac{1}{2}\int_\Omega c(x)u_n^+ - \int_\Omega c(x)u_n^+\ln(1+u_n^+) \\ &\quad + \frac{1}{2}\int_\Omega c(x)\ln(1+u_n^+) + \frac{\mu}{2}\int_\Omega f(x)u_n + o(1)\Big) \to 0. \end{aligned}$$

Thus (3.8) is proved and we reach a contradiction. Therefore (u_n) must be bounded and, up to subsequence, we have $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$ and $u_n \to u_0$ in $L^p(\Omega)$ for $p \in [1, 2^*)$. At this point the strong convergence follows in a standard way. We refer to [20, Lemma 11] for a proof. □

Corollary 3.2. *If $\lambda_1(-c - \mu f) > 0$ then I satisfies the Palais-Smale condition.*

Proof. Let $\|u\|_2 = 1$ with $cu^+ \equiv 0$. Since $\lambda_1(-c-\mu f) > 0$, by Lemma 2.2 there is a constant $K_1 > 0$ such that

$$\begin{aligned}\int_\Omega \left(|\nabla u|^2 - \mu f(x)(u^+)^2\right) &= \int_\Omega \left(|\nabla u|^2 - (c(x) + \mu f(x))(u^+)^2\right) \\ &\geq K_1\|u\|^2 \geq SK_1\|u\|_2^2 = SK_1 > 0,\end{aligned}$$

where S is the best Sobolev constant for the embedding $H_0^1(\Omega) \subset L^2(\Omega)$. Thus $\alpha_c > 0$ and by Proposition 3.1 we get the conclusion. □

4. Proof of Theorem 1.1 and Lemma 1.3

We are now ready to prove our main results.

Proof of Theorem 1.1: First of all, we fix $K > 0$ such that $\lambda_1(-c - \mu f) > 0$ if either $\lambda_1(-\mu f) > 0$ and $\|c^+\|_q < K$ or $\lambda_1(-c) > 0$ and $\|\mu f\|_q < K$. This is possible in view of the continuity of $\lambda_1(V)$ with respect to $V \in L^q(\Omega)$. Decreasing K if necessary, we fix R sufficiently large so that, by Proposition 2.3, if $\|c^+\|_q < K$ (respect. $\|\mu f\|_q < K$) then $I(v) \geq M > 0$ for $\|v\| = R$ (respect. $\|v\| = R^{-1}$). We set $\rho = R$ if $\|c^+\|_q < K$ and $\rho = R^{-1}$ if $\|\mu f\|_q < K$. It easily seen that if $f \not\equiv 0$ then I takes negative values in the ball $B(0,\rho)$. Therefore, by weak lower semi-continuity, we infer that if either $\|c^+\|_q < K$ or $\|\mu f\|_q < K$ then the infimum of I in $B(0,\rho)$ is achieved by some $w_0 \not\equiv 0$, which is a critical point of I. Furthermore, since $G(s)/s^2 \to \infty$ as $s \to \infty$, if $v \in H_0^1(\Omega)$ is such that $\int_\Omega c(x)G(v^+) > 0$ then $I(tv) \to -\infty$ as $t \to \infty$. We fix $t > 0$ and v such that $v_0 = tv$ satisfies $\|v_0\| > \rho$ and $I(v_0) < 0$. Now let

$$\Gamma := \{\gamma \in \mathcal{C}([0,1], H_0^1(\Omega));\ \gamma(0) = 0, \gamma(1) = v_0\}$$

and

$$d := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)).$$

Since I satisfies the Palais-Smale condition, by the mountain-pass theorem it is straightforward that I has a critical point w_1, which, by Proposition 2.3, satisfies $I(w_1) = d > 0$. In particular, we have $w_0 \neq w_1$. Finally, from Lemma 2.1, we know that these two critical points provide two positive solutions of (P'), and consequently, two positive solutions of (P). □

Proof of Lemma 1.3. By Lemma 2.1, we know that if $u \geq 0$ is a solution of (P) then u is positive so that $w = \mu u$ is a positive solution of (P'). Thus v given by (1.1) is a positive solution of (Q). Taking $\phi > 0$, the first positive eigenfunction associated to $\lambda_1(-c-\mu f)$, as test function and using that $g \geq 0$ on $\mathbb{R}$ we obtain

$$\int_\Omega (\nabla v \nabla \phi - c(x)v\phi - \mu f(x)v\phi) = \int_\Omega (c(x)g(v)\phi + \mu f(x)\phi) > 0,$$

so that

$$\lambda_1(-c-\mu f)\int_\Omega v\phi > 0.$$

Thus $\lambda_1(-c-\mu f) > 0$.

Similarly, let $\varphi > 0$ be an eigenfunction associated to $\lambda_1(-c)$ and assume that $u \geq 0$ is a solution of (P). Taking $\varphi > 0$ as test function we get

$$\int_\Omega (\nabla u \nabla \varphi - c(x)u\varphi) = \int_\Omega \left(\mu|\nabla u|^2\varphi + f(x)\varphi\right) > 0.$$

Thus

$$\lambda_1(-c)\int_\Omega u\varphi > 0,$$

so that $\lambda_1(-c) > 0$. Finally, let u be a solution of (P). Using u^- as test function in (P), we obtain

$$-\int_\Omega (|\nabla u^-|^2 - c(x)|u^-|^2) = \int_\Omega (\mu|\nabla u|^2 u^- + f(x)u^-) \geq 0.$$

Hence

$$\int_\Omega (|\nabla u^-|^2 - c(x)|u^-|^2) \leq 0,$$

so that under the condition $\lambda_1(-c) > 0$ we get $u^- \equiv 0$, i.e. $u \geq 0$. □

Our last result show that when $\lambda_1(-c) > 0$ a restriction on the size of μf is necessary in Theorem 1.1.

Remark 4.1. *Assume $(\mathcal{H})$, $\lambda_1(-c) > 0$, and $c \geq 0$ in some open set $\Omega_0 \subset \Omega$. Then there exist a $R > 0$ and a $f \in L^q(\Omega)$ with $\|\mu f\|_q = R$ such that (P) has no non negative solutions.*

Proof. Equivalently we shall prove that (P') has no non negative solutions. We choose $\phi \in C_0^\infty(\Omega_0)$ and $f \in L^q(\Omega)$ such that $f > 0$ on $supp\, \phi$. In particular we have

$$\int_\Omega f(x)\phi^2 > 0. \tag{4.1}$$

By Cauchy-Schwartz inequality we have

$$\int_\Omega \nabla u \nabla(\phi^2) = \int_\Omega 2\phi\nabla u\nabla\phi \leq \int_\Omega |\nabla\phi|^2 + |\nabla u|^2\phi^2. \tag{4.2}$$

Now assume that (P') has a non negative solution. Using ϕ^2 as test function in (P') and (4.2) we get

$$\int_\Omega |\nabla\phi|^2 \geq \int_\Omega c(x)u\phi^2 + \mu\int_\Omega f(x)\phi^2 \geq \mu\int_\Omega f(x)\phi^2.$$

Because of (4.1) we get a contradiction for $\mu > 0$ large enough. □

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