On global minimizers for a mass constrained problem

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Abstract. In any dimension $N \geq 1$, for given mass $m > 0$ and for the C^1 energy functional

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(u) dx,$$

we revisit the classical problem of finding conditions on $F \in C^1(\mathbb{R}, \mathbb{R})$ insuring that I admits global minimizers on the mass constraint

$$S_m := \left\{ u \in H^1(\mathbb{R}^N) \mid \|u\|^2_{L^2(\mathbb{R}^N)} = m \right\}.$$

Under assumptions that we believe to be nearly optimal, in particular without assuming that F is even, any such global minimizer, called energy ground state, proves to have constant sign and to be radially symmetric monotone with respect to some point in $\mathbb{R}^N$. Moreover, we manage to show that any energy ground state is a least action solution of the associated free functional. This last result settles, under general assumptions, a long standing open problem.

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1 Introduction

Let $N \geq 1$ and $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ be a C^1 functional defined by

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(u)dx$$

where $F(t) := \int_0^t f(\tau)d\tau$ for some function $f \in C(\mathbb{R},\mathbb{R})$.

In this paper we focus on the minimization problem

$$E_m := \inf_{u\in S_m} I(u), \tag{Inf_m}$$

where $m > 0$ is prescribed and

$$S_m := \left\{ u \in H^1(\mathbb{R}^N) \mid \|u\|^2_{L^2(\mathbb{R}^N)} = m \right\}.$$

By a direct application of Lagrange multiplier's rule, if $u \in S_m$ solves (Inf_m) then there exists $\mu = \mu(u) \in \mathbb{R}$ such that

$$-\Delta u = f(u) - \mu u \qquad \text{in } H^1(\mathbb{R}^N). \tag{1.1}$$

The study of problem (Inf_m) naturally arises in the search of standing waves for nonlinear scalar field equations the form

$$i\psi_t + \Delta\psi + f(\psi) = 0, \qquad \psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}. \tag{1.2}$$

By standing waves, we mean solutions of (1.2) of the special form $\psi(t,x) = e^{i\mu t}u(x)$ with $\mu \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^N)$. Clearly $\psi(t,x)$ satisfies (1.2) if $u(x)$ satisfies (1.1) for the corresponding $\mu \in \mathbb{R}$.

The study of such type of equations, which already saw major contributions forty years ago, [1, 6, 7, 22, 23, 27], now lies at the root of several models linked with current physical applications (such as nonlinear optics, the theory of water waves, ...). For these equations, finding solutions with a prescribed L^2-norm is particularly relevant since this quantity is preserved along the time evolution. In addition, if the solutions correspond to ground states, then, in most situations, it is possible to prove that the associated standing waves are orbitally stable. This likely explains why the study of problem (Inf_m) is still the object of an intense activity. Among many others possible choices, we refer to [4, 5, 10, 12, 13, 15, 21, 25, 26] and to the references therein.

Our first main result concerns the solvability of (Inf_m). It can be viewed as an extension of the one of [25] already obtained in a very general setting. The following assumptions on $f \in C(\mathbb{R},\mathbb{R})$ will be required.

($f1$) $\lim_{t\to 0} f(t)/t = 0$.

($f2$) When $N \geq 3$,

$$\limsup_{t\to\infty} \frac{|f(t)|}{|t|^{\frac{N+2}{N-2}}} < \infty;$$

when $N = 2$,

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t^2}} = 0 \quad \forall \alpha > 0;$$

and also for any $N \geq 1$,

$$\limsup_{t\to\infty} \frac{f(t)t}{|t|^{2+\frac{4}{N}}} \leq 0.$$

$(f3)$ There exists $\zeta \neq 0$ such that $F(\zeta) > 0$.

Theorem 1.1 *Assume that $N \geq 1$, $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $(f1) - (f3)$. Then*

$$E_m := \inf_{u \in S_m} I(u) > -\infty$$

and the map $m \mapsto E_m$ is nonincreasing and continuous. Moreover,

(i) *there exists a number $m^* \in [0, \infty)$ such that*

$$E_m = 0 \quad \text{if } 0 < m \leq m^*, \qquad E_m < 0 \quad \text{when } m > m^*;$$

(ii) *when $m > m^*$, the global infimum E_m is reached and thus (Inf_m) has a ground state solution $v \in S_m$ with $I(v) = E_m < 0$;*

(iii) *when $0 < m < m^*$, $E_m = 0$ is not reached;*

(iv) *$m^* = 0$ if in addition*

$$\lim_{t\to 0} \frac{F(t)}{|t|^{2+\frac{4}{N}}} = +\infty, \tag{A.1}$$

and $m^* > 0$ if in addition

$$\limsup_{t\to 0} \frac{F(t)}{|t|^{2+\frac{4}{N}}} < +\infty. \tag{A.2}$$

Remark 1.2 (i) *As it will be clear from the proof of Theorem 1.1 (ii), see also Remark 2.3, when $m > m^*$ we also show that any minimizing sequence for (Inf_m) is, up to a subsequence and up to translations in $\mathbb{R}^N$, strongly convergent.*

(ii) *When $0 < m < m^*$, it is proved in Theorem 1.1 (iii) that the global infimum $E_m = 0$ is not reached, but this does not mean that the constrained functional $I_{|S_m}$ may not admit critical points with positive energies, see the companion work [17].*

(iii) *In the case $m^* > 0$, studying existence and nonexistence of global minimizers with respect to $E_{m^*} = 0$ seems to be a delicate issue. Since it exceeds our scope of the present paper, we shall not explore general further conditions on f that ensure the existence or nonexistence but refer the interested reader to [17] and [25, Theorem 1.4] for some existence results.*

(iv) *For convenience of statement, we introduce the notation*

$$m \succeq_f m^*$$

with the understanding that $m \geq m^*$ *if* $m^* > 0$ *and* $E_{m^*} = 0$ *is reached, and* $m > m^*$ *if otherwise. As one may observe, when* $m \succeq_f m^*$ *and for any minimizer* $v \in S_m$ *of* (Inf_m)*, the associated Lagrange multiplier* $\mu = \mu(v)$ *is positive. Indeed, from the Pohozaev identity corresponding to* (1.1)*, see [1, Proposition 1],*

$$P(v) := \frac{N-2}{2N}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2}\mu \int_{\mathbb{R}^N} |v|^2 dx - \int_{\mathbb{R}^N} F(v) dx = 0$$

and the fact that $I(v) = E_m \leq 0$*, we have*

$$0 \geq I(v) = I(v) - P(v) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{1}{2}\mu m$$

and hence $\mu > 0$.

Remark 1.3 *Let us give some examples of nonlinearities satisfying* $(f1) - (f3)$.

(i) $f(t) = |t|^{p-2}t + A|t|^{q-2}t$ *with*

$$A \in \mathbb{R} \qquad \text{and} \qquad 2 < q < p < 2 + \frac{4}{N}.$$

In particular, (A.1) *and* (A.2) *hold when* $A \geq 0$ *and when* $A < 0$ *respectively.*

(ii) $f(t) = |t|^{p-2}t - |t|^{q-2}t$ *with*

$$\begin{cases} 2 < p < q < \infty, & \text{if } N = 1, 2, \\ 2 < p < q \leq \dfrac{2N}{N-2}, & \text{if } N \geq 3. \end{cases}$$

In particular, (A.1) *and* (A.2) *hold if* $p < 2 + \frac{4}{N}$ *and if* $p \geq 2 + \frac{4}{N}$ *respectively, and when* $N = 2, 3$ *we cover the so-called cubic-quintic nonlinearity*

$$f(t) = |t|^2 t - |t|^4 t$$

which attracts much attention due to its physical relevance, see for example [4, 5, 20, 21].

These example are only some special cases of course, our Theorem 1.1 and the subsequent Theorems 1.4 and 1.6 apply to more general nonlinearities, in particular to those which are not a sum of powers.

The next result shows further that any energy ground state has constant sign and enjoys symmetry and monotonicity properties.

Theorem 1.4 *Assume that* $N \geq 1$, $f \in C(\mathbb{R}, \mathbb{R})$ *satisfies* $(f1) - (f3)$*, and in addition* f *is locally Lipschitz continuous when* $N = 1$*. Let* $m \succeq_f m^*$*, where* $m^* \geq 0$ *is the number given by Theorem 1.1. Then any minimizer* $v \in S_m$ *of* (Inf_m) *satisfies the following properties:*

(i) v has constant sign,

(ii) v is radially symmetric up to a translation in $\mathbb{R}^N$,

(iii) v is monotone with respect to the radial variable.

Our final main theorem is the heart of the present paper, it answers for nonlinear scalar field equations a long standing open problem. To explain what is at stake we need the following definition.

Definition 1.5 *For given $\mu > 0$, a nontrivial solution $w \in H^1(\mathbb{R}^N)$ of the free problem*

$$\begin{cases} -\Delta u = f(u) - \mu u \quad \text{in } \mathbb{R}^N, \\ \quad u \in H^1(\mathbb{R}^N), \end{cases} \tag{Q_μ}$$

is said to be a least action solution if it reaches the infimum of the C^1 action functional

$$J_\mu(u) := I(u) + \frac{1}{2}\mu \int_{\mathbb{R}^N} |u|^2 dx$$

among all the nontrivial solutions, namely

$$J_\mu(w) = A_\mu := \inf\{J_\mu(u) \mid u \in H^1(\mathbb{R}^N) \setminus \{0\}, J'_\mu(u) = 0\}.$$

For future reference, the value A_μ is called the least action of (Q_μ).

The open problem can be now formulated as follows.

Open Problem : We know that a ground state energy minimizer $v \in S_m$ is a nontrivial solution to (Q_μ), where $\mu = \mu(v) > 0$ is the associated Lagrange multiplier. Is the minimizer $v \in S_m$ a least action solution to (Q_μ)? In other words, does an energy ground state is necessarily a least action solution?

For some odd $f \in C(\mathbb{R}, \mathbb{R})$ satisfying $(f1) - (f3)$ it is known that there exists a unique positive solution to (Q_μ) and that it is a least action solution, see for example [5, 20, 21]. Thus, in such situations, Theorem 1.4 implies that any energy ground state is a least action solution. However, apart in some particular cases of this type where some uniqueness property was used, the Open Problem remained unsolved until recently. In 2020 a positive answer was given in [11] for a related problem of biharmonic type with a power nonlinearity, see [11, Proposition 3.9 and Theorem 1.3]. Also, very recently in [10], the authors answered positively the Open Problem assuming that the nonlinearity f induces a Nehari manifold for which the minimizers of J_μ on this manifold coincide with the least action solutions. Essentially, this property holds when the function $t \mapsto f(t)/t$ is nondecreasing on $(0, \infty)$, see [10, Theorem 1.3] for more details. Note that the results of [10] also hold when the analog of problem (Q_μ) is set on an arbitrary domain $\Omega \subset \mathbb{R}^N$. Finally, we mention [14] in which the Open Problem was claimed to be solved for a nonlinearity which is a sum of powers.

Our result in that direction covers all the previous particular cases, at least when the associated equations are set on all the space $\mathbb{R}^N$.

Theorem 1.6 *Assume that $N \geq 1$, $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $(f1)-(f3)$, and in addition f is locally Lipschitz continuous when $N = 1$. Let $m \succeq_f m^*$ and denote by $\mu(v)$ the Lagrange multiplier corresponding to an arbitrary minimizer $v \in S_m$ of (Inf_m), where $m^* \geq 0$ is the number given by Theorem 1.1. Then the following statements hold.*

(i) *Any minimizer $v \in S_m$ of (Inf_m) is a least action solution of (Q_μ) with $\mu = \mu(v) > 0$. In particular,*

$$A_\mu = E_m + \frac{1}{2}\mu m.$$

(ii) *For given $\mu \in \{\mu(v) \mid v \in S_m \text{ is a minimizer of } (\mathrm{Inf}_m)\}$, any least action solution $w \in H^1(\mathbb{R}^N)$ of (Q_μ) is a minimizer of (Inf_m), namely*

$$\|w\|^2_{L^2(\mathbb{R}^N)} = m \qquad \text{and} \qquad I(w) = E_m.$$

Remark 1.7 (i) *The conclusions of Theorem 1.6 (ii) were also observed in [10, 11, 14] in the corresponding frames.*

(ii) *For alternative variational characterizations of the energy ground states, in related problems, we refer to [8, 9, 13]. Note that in [8, 9], a variational characterization of the associated Lagrange multiplier is proposed, see also [10, Theorem 1.2] in that direction.*

(iii) *It is known, see for example [4, 17, 21], that under the assumptions $(f1)-(f3)$ there may exist least action solutions which are not energy ground states.*

The paper is organized as follows. In Section 2 we prove Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.4. Finally, in Section 4 we prove Theorem 1.6.

Notations. Throughout this paper, for any $p \in [1, \infty)$, $L^p(\mathbb{R}^N)$ is the usual Lebesgue space endowed with the norm

$$\|u\|_{L^p(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |u|^p dx\right)^{1/p},$$

and $H^1(\mathbb{R}^N)$ the usual Sobolev space endowed with the norm

$$\|u\|_{H^1(\mathbb{R}^N)} := \left(\|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \|u\|^2_{L^2(\mathbb{R}^N)}\right)^{1/2}.$$

Moreover, for given $u \in H^1(\mathbb{R}^N)$ and any $s \in \mathbb{R}$, we define the scaling function

$$s \star u := e^{Ns/2} u(e^s \cdot),$$

which remains in $H^1(\mathbb{R}^N)$ and preserves the L^2 norm when $s \in \mathbb{R}$ varies.

2 Existence and nonexistence

This section aims to prove Theorem 1.1 and in particular we shall show the existence and nonexistence of minimizers of (Inf_m) for suitable range of the mass $m > 0$. As a necessary preparation, we have the following lemma the proof of which is standard.

Lemma 2.1 *Assume that $N \geq 1$ and $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $(f1) - (f2)$. Then the following statements hold.*

(i) For any bounded sequence $\{u_n\}$ in $H^1(\mathbb{R}^N)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx = 0$$

if $\lim_{n\to\infty} \|u_n\|_{L^\infty(\mathbb{R}^N)} = 0$, and

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} F(u_n)dx \leq 0$$

if $\lim_{n\to\infty} \|u_n\|_{L^{2+4/N}(\mathbb{R}^N)} = 0$.

(ii) There exists $C = C(f, N, m) > 0$ such that

$$I(u) \geq \frac{1}{4}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} - C(f, N, m)$$

for any $u \in H^1(\mathbb{R}^N)$ satisfying $\|u\|^2_{L^2(\mathbb{R}^N)} \leq m$. In particular, I is coercive on S_m.

To proceed further, we recall the global infimum

$$E_m := \inf_{u\in S_m} I(u)$$

and make below a detailed study of its basic properties.

Lemma 2.2 *Assume that $N \geq 1$ and $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $(f1) - (f3)$. Then the following statements hold.*

(i) $-\infty < E_m \leq 0$ for all $m > 0$.

(ii) There exists $m_0 > 0$ such that $E_m < 0$ for any $m > m_0$.

(iii) $E_m < 0$ for all $m > 0$ if (A.1) holds, and $E_m = 0$ for small $m > 0$ if (A.2) holds.

(iv) For any $m > m' > 0$ one has

$$E_m \leq \frac{m}{m'} E_{m'}. \tag{2.1}$$

If $E_{m'}$ is reached then the inequality is strict.

(v) The function $m \mapsto E_m$ is nonincreasing and continuous.

Proof. (i) By Lemma 2.1 (ii), I is bounded from below on S_m and thus $E_m > -\infty$. For a fixed $u \in S_m \cap L^\infty(\mathbb{R}^N)$, we have $\|\nabla(s \star u)\|_{L^2(\mathbb{R}^N)} \to 0$ and $\|s \star u\|_{L^\infty(\mathbb{R}^N)} \to 0$ as $s \to -\infty$. In view of Lemma 2.1 (i), $E_m \leq \lim_{s\to-\infty} I(s \star u) = 0$.

(ii) From $(f3)$ and Step 1 of [1, Proof of Theorem 2], there exists $u \in H^1(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} F(u)dx > 0$. For any $m > 0$, set $u_m := u(m^{-1/N} \cdot \|u\|^{2/N}_{L^2(\mathbb{R}^N)} \cdot x) \in S_m$. Since

$$\begin{aligned} I(u_m) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u_m|^2 dx - \int_{\mathbb{R}^N} F(u_m)dx \\ &= \frac{m^{\frac{N-2}{N}}}{2\|u\|^{2(N-2)/N}_{L^2(\mathbb{R}^N)}} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{m}{\|u\|^2_{L^2(\mathbb{R}^N)}} \int_{\mathbb{R}^N} F(u)dx \\ &=: Am^{\frac{N-2}{N}} - Bm =: g(m), \end{aligned}$$

it follows that $E_m \leq I(u_m) = g(m) < 0$ for any sufficiently large $m > 0$.

(iii) When (A.1) holds, we choose $u \in S_m \cap L^\infty(\mathbb{R}^N)$. For

$$D := \int_{\mathbb{R}} |\nabla u|^2 dx \Big/ \int_{\mathbb{R}^N} |u|^{2+4/N} dx > 0,$$

by (A.1), there exists $\delta > 0$ such that $F(t) \geq D|t|^{2+4/N}$ for all $|t| \leq \delta$. Since $\|s \star u\|_{L^\infty(\mathbb{R}^N)} \leq \delta$ for some $s < 0$, it is clear that

$$\begin{aligned} E_m \leq I(s \star u) &\leq \frac{1}{2}\int_{\mathbb{R}} |\nabla(s \star u)|^2 dx - D\int_{\mathbb{R}^N} |s \star u|^{2+4/N} dx \\ &= \frac{1}{2}e^{2s}\int_{\mathbb{R}} |\nabla u|^2 dx - De^{2s}\int_{\mathbb{R}^N} |u|^{2+4/N} dx \\ &= -\frac{1}{2}e^{2s}\int_{\mathbb{R}} |\nabla u|^2 dx < 0. \end{aligned}$$

When (A.2) is satisfied, there exists $C_f > 0$ such that $F(t) \leq C_f|t|^{2+4/N}$ for any $t \in \mathbb{R}$. By the Gagliardo-Nirenberg inequality,

$$\int_{\mathbb{R}^N} F(u)dx \leq C_f C_N m^{2/N} \|\nabla u\|^2_{L^2(\mathbb{R}^N)} \qquad \text{for all } u \in S_m.$$

For any $m > 0$ small enough such that $C_f C_N m^{2/N} \leq 1/4$, we have

$$I(u) \geq \frac{1}{4}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} > 0,$$

and thus $E_m \geq 0$. From Item (i), it follows that $E_m = 0$ for $m > 0$ small.

(iv) Let $t := m/m' > 1$. For any $\varepsilon > 0$ there exists $u \in S_{m'}$ such that $I(u) \leq E_{m'} + \varepsilon$. Clearly, $v := u(t^{-1/N}\cdot) \in S_m$ and then

$$\begin{aligned} E_m \leq I(v) &= tI(u) + \frac{1}{2}t^{\frac{N-2}{N}}\left(1 - t^{\frac{2}{N}}\right)\int_{\mathbb{R}^N} |\nabla u|^2 dx \\ &< tI(u) \\ &\leq \frac{m}{m'}(E_{m'} + \varepsilon). \end{aligned} \tag{2.2}$$

Since $\varepsilon > 0$ is arbitrary, we see that the inequality (2.1) holds. If $E_{m'}$ is reached, for example, at some $u \in S_{m'}$, then we can let $\varepsilon = 0$ in (2.2) and thus the strict inequality follows.

(v) By Item (i) and (2.1), it is clear that E_m is nonincreasing. Our remaining task is to prove the continuity and this is equivalent to show that for given $m > 0$ and any positive sequence $\{m_k\}$ such that $m_k \to m$ as $k \to \infty$, one has $\lim_{k\to\infty} E_{m_k} = E_m$. We first claim that

$$\limsup_{k\to\infty} E_{m_k} \le E_m. \tag{2.3}$$

Indeed, for any $u \in S_m$ and each $k \in \mathbb{N}^+$, set $u_k := \sqrt{m_k/m} \cdot u \in S_{m_k}$. Since $u_k \to u$ strongly in $H^1(\mathbb{R}^N)$, it is clear that $\lim_{k\to\infty} I(u_k) = I(u)$ and thus

$$\limsup_{k\to\infty} E_{m_k} \le \limsup_{k\to\infty} I(u_k) = I(u).$$

By the arbitrariness of $u \in S_m$, we conclude that (2.3) holds. To complete the proof of the continuity, we only need to show

$$\liminf_{k\to\infty} E_{m_k} \ge E_m. \tag{2.4}$$

For each $k \in \mathbb{N}^+$, there exists $v_k \in S_{m_k}$ such that

$$I(v_k) \le E_{m_k} + \frac{1}{k}. \tag{2.5}$$

Setting $t_k := (m/m_k)^{1/N}$ and $\tilde{v}_k := v_k(\cdot/t_k) \in S_m$, we have

$$\begin{aligned} E_m \le I(\tilde{v}_k) &\le I(v_k) + \big|I(\tilde{v}_k) - I(v_k)\big| \\ &\le E_{m_k} + \frac{1}{k} + \big|I(\tilde{v}_k) - I(v_k)\big| =: E_{m_k} + \frac{1}{k} + C(k), \end{aligned}$$

where

$$\begin{aligned} C(k) &\le \frac{1}{2}|t_k^{N-2} - 1| \cdot \int_{\mathbb{R}^N} |\nabla v_k|^2 dx + |t_k^N - 1| \cdot \int_{\mathbb{R}^N} |F(v_k)| dx \\ &=: \frac{1}{2}|t_k^{N-2} - 1| \cdot A(k) + |t_k^N - 1| \cdot B(k). \end{aligned}$$

Since $t_k \to 1$, the proof of (2.4) can be reduced to showing that $A(k)$ and $B(k)$ are bounded. To justify the boundedness, by (2.5) and (2.3), we have $\limsup_{k\to\infty} I(v_k) \le E_m$. Noting that $v_k \in S_{m_k}$ and $m_k \to m$, it follows from Lemma 2.1 (ii) that $\{v_k\}$ is bounded in $H^1(\mathbb{R}^N)$. Since $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $(f1)$ and $(f2)$, it is clear that $A(k)$ and $B(k)$ are both bounded and thus the continuity is proved. □

Proof of Theorem 1.1. We define

$$m^* := \inf\{m > 0 \mid E_m < 0\}.$$

It is easily seen from Lemma 2.2 that $m^* \in [0, \infty)$,

$$E_m = 0 \quad \text{if } 0 < m \le m^*, \qquad\qquad E_m < 0 \quad \text{when } m > m^*; \tag{2.6}$$

in particular, $m^* = 0$ if (A.1) holds, and $m^* > 0$ if (A.2) holds. Let us first show that if $0 < m < m^*$ then $E_m = 0$ is not reached. Indeed, assuming by contradiction that $E_m = 0$ is reached for some $m \in (0, m^*)$, we infer from Lemma 2.2 (iv) that

$$E_{m^*} < \frac{m^*}{m} E_m = 0$$

and this leads a contradiction since $E_{m^*} = 0$ by (2.6). The rest is to prove that the global infimum E_m is reached when $m > m^*$.

Fix $m > m^*$ and let $\{u_n\} \subset S_m$ be any minimizing sequence with respect to E_m. It is clear that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ by Lemma 2.1 (ii) and then one may assume that up to a subsequence $\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx$ and $\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n) dx$ exist. Since $E_m < 0$ by (2.6), we deduce that $\{u_n\}$ is non-vanishing, namely

$$\lim_{n\to\infty} \left(\sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n|^2 dx \right) > 0. \tag{2.7}$$

Indeed, if (2.7) were not true, then $u_n \to 0$ in $L^{2+4/N}(\mathbb{R}^N)$ by Lions Lemma [23, Lemma I.1] and thus

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n) dx \leq 0$$

via Lemma 2.1 (i); noting that $I(u_n) \geq -\int_{\mathbb{R}^N} F(u_n) dx$, we obtain a contradiction:

$$0 > E_m = \lim_{n\to\infty} I(u_n) \geq - \lim_{n\to\infty} \int_{\mathbb{R}^N} F(u_n) dx \geq 0.$$

Since $\{u_n\}$ is non-vanishing, there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ and a nontrivial element $v \in H^1(\mathbb{R}^N)$ such that up to a subsequence $u_n(\cdot + y_n) \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and $u_n(\cdot + y_n) \to v$ almost everywhere on $\mathbb{R}^N$. Set $m' := \|v\|^2_{L^2(\mathbb{R}^N)} \in (0, m]$ and $w_n := u_n(\cdot + y_n) - v$. It is clear that

$$\lim_{n\to\infty} \|w_n\|^2_{L^2(\mathbb{R}^N)} = m - m' \tag{2.8}$$

and, using the splitting result [16, Lemma 2.6],

$$E_m = \lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(v + w_n) = I(v) + \lim_{n\to\infty} I(w_n). \tag{2.9}$$

We shall prove below a claim and then conclude the whole proof.

Claim. $\lim_{n\to\infty} \|w_n\|_{L^2(\mathbb{R}^N)} = 0$. In particular, $m' = m$ by (2.8).

Let $t_n := \|w_n\|^2_{L^2(\mathbb{R}^N)}$ for each $n \in \mathbb{N}^+$. If $\lim_{n\to\infty} t_n > 0$, then (2.8) gives that $m' \in (0, m)$. In view of the definition of E_{t_n} and Lemma 2.2 (v), we obtain

$$\lim_{n\to\infty} I(w_n) \geq \lim_{n\to\infty} E_{t_n} = E_{m-m'}.$$

From (2.9) and (2.1), it follows

$$E_m \geq I(w) + E_{m-m'} \geq E_{m'} + E_{m-m'} \geq \frac{m'}{m} E_m + \frac{m-m'}{m} E_m = E_m.$$

Thus necessarily $I(w) = E_{m'}$ and this shows that $E_{m'}$ is reached at $v \in S_{m'}$. But then still using (2.9) and (2.1), we obtain a contradiction:

$$E_m \geq E_{m'} + E_{m-m'} > \frac{m'}{m} E_m + \frac{m-m'}{m} E_m = E_m,$$

and so the claim is proved.

Conclusion. Clearly, $v \in S_m$ by the above claim and thus $I(v) \geq E_m$. Since the claim and Lemma 2.1 (i) imply that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(w_n)dx \leq 0, \tag{2.10}$$

we also have $\lim_{n\to\infty} I(w_n) \geq 0$. Therefore, by (2.9) we get $E_m \geq I(v)$ and hence $E_m < 0$ is reached at $v \in S_m$. □

Remark 2.3 *One can deduce further that $u_n(\cdot + y_n) \to v$ in $H^1(\mathbb{R}^N)$. Indeed, from (2.9), (2.10) and the fact that $I(v) = E_m$, it follows*

$$\|\nabla w_n\|_{L^2(\mathbb{R}^N)} \to 0 \qquad \text{as } n \to \infty.$$

Since $\lim_{n\to\infty} \|w_n\|_{L^2(\mathbb{R}^N)} = 0$, we obtain the strong convergence.

3 Sign, symmetry and monotonicity

This section is devoted to the proof of Theorem 1.4. Unless otherwise noted, for given $\mu > 0$ we use the notations

$$g_\mu(t) := -\mu t + f(t) \qquad \text{and} \qquad G_\mu(t) := -\frac{1}{2}\mu t^2 + F(t).$$

To deal with the case $N = 1$, a special treatment is required. To be more precise, we shall make use of the following classification result which is deduced by means of simple methods of ordinary differential equations.

Lemma 3.1 *Assume that $N = 1$, f is a locally Lipschitz continuous function on $\mathbb{R}$ satisfying ($f1$), and $w \in H^1(\mathbb{R})$ is a nontrivial critical point of J_μ for some $\mu > 0$. Then, when w is negative or positive somewhere, the following two statements hold respectively.*

(a) $\zeta_- := \sup\{t < 0 \mid G_\mu(t) = 0\} \in (-\infty, 0)$,

$$g_\mu(\zeta_-) < 0,$$

and after a suitable translation of the origin w satisfies

$(a1)$ $w(x) = w(-x)$ *for any* $x \in \mathbb{R}$,

$(a2)$ $w(x) < 0$ *for any* $x \in \mathbb{R}$, *(a3)* $w(0) = \zeta_-$,

(a4) $w'(x) > 0$ *for any* $x > 0$.

(b) $\zeta_+ := \inf\{t > 0 \mid G_\mu(t) = 0\} \in (0, \infty)$,

$$g_\mu(\zeta_+) > 0,$$

and after a suitable translation of the origin w *satisfies*

(b1) $w(x) = w(-x)$ *for any* $x \in \mathbb{R}$,

(b2) $w(x) > 0$ *for any* $x \in \mathbb{R}$,

(b3) $w(0) = \zeta_+$,

(b4) $w'(x) < 0$ *for any* $x > 0$.

In particular, w *is a translation of the unique solution to the initial value problem* $-u'' = g_\mu(u)$ *with* $u(0) = \zeta_-$ *(or* $u(0) = \zeta_+$*) and* $u'(0) = 0$.

Proof. By regularity $w \in C^2(\mathbb{R}, \mathbb{R})$ and thus

$$-w'' = g_\mu(w) \qquad \text{in } \mathbb{R}. \tag{3.1}$$

Since $|w(x)|$ and $|w'(x)|$ decay to zero exponentially as $|x| \to \infty$, we have

$$\frac{1}{2}|w'(x)|^2 + G_\mu(w(x)) = 0 \qquad \text{for } x \in \mathbb{R}. \tag{3.2}$$

Without loss of generality, we only consider the case when w is negative somewhere. By translating the point where w reaches its negative minimum to the origin, one may assume that $w'(0) = 0$. In view of (3.2),

$$G_\mu(w(0)) = 0 \qquad \text{and} \qquad \zeta_- > -\infty.$$

Since $(f1)$ gives that $G_\mu(t) < 0$ for any $t < 0$ close enough to the origin, we also have $\zeta_- < 0$. Now assume by contradiction that $g_\mu(\zeta_-) \geq 0$. Since $w(0) \leq \zeta_-$, there exists $x^* \in \mathbb{R}$ such that $w(x^*) = \zeta_-$. Then

$$w'(x^*) = 0 \qquad \text{and} \qquad w''(x^*) = -g_\mu(\zeta_-) \leq 0$$

via (3.2) and (3.1) respectively. If $g_\mu(\zeta_-) > 0$, then since whenever $w(x) = \zeta_-$ one also has $w'(x) = 0$ and $w''(x) < 0$, w can never go above $\zeta_- < 0$, which is impossible. On the other hand, if $g_\mu(\zeta_-) = 0$, then by uniqueness the conditions

$$w(x^*) = \zeta_- \qquad \text{and} \qquad w'(x^*) = w''(x^*) = 0$$

imply $w \equiv \zeta_-$, which is also impossible. With the desired conclusion $g_\mu(\zeta_-) < 0$ at hand, there exists $\varepsilon > 0$ such that $G_\mu(t) > 0$ for any $t \in (\zeta_- - \varepsilon, \zeta_-)$. If $w(0) < \zeta_-$, then $w(x_*) \in (\zeta_- - \varepsilon, \zeta_-)$ for some $x_* \in \mathbb{R}$ and so

$$\frac{1}{2}|w'(x_*)|^2 + G_\mu(w(x_*)) > 0.$$

This contradicts (3.2), and therefore $w(0) = \zeta_-$. Since w is the global solution of (3.1) with the initial conditions $w(0) = \zeta_-$ and $w'(0) = 0$, the rest follows from a standard adaption of some arguments in [1, Proof of Theorem 5]. □

Remark 3.2 *Even though the nonlinearity f in Lemma 3.1 is locally Lipschitz continuous, the nontrivial critical points of J_μ (if exist) are not necessarily unique up to a translation in $\mathbb{R}$ and up to a sign since we allow f to be not odd.*

In the higher dimensional case $N \geq 2$, the radial symmetry of minimizers will be obtained as a direct consequence of a general symmetry result in [24], and the proof of the monotonicity relies on Lemma 3.3 below. We remark that the first part of Lemma 3.3 is well known and the second part is a simple corollary of [2, Lemma 3.2].

Lemma 3.3 *Let v be a nonnegative measurable function defined on $\mathbb{R}^N$ such that for any $\alpha > 0$ the function $(v-\alpha)^+$ belongs to $H^1(\mathbb{R}^N)$ and has compact support, and denote by v^* the Schwarz rearrangement of v. Then*

$$\int_{\mathbb{R}^N} |\nabla v^*|^2 dx \leq \int_{\mathbb{R}^N} |\nabla v|^2 dx. \tag{3.3}$$

Moreover, if the equality in (3.3) holds then the level set

$$\chi_\alpha := \{x \in \mathbb{R}^N \mid v(x) > \alpha\}$$

is equivalent to a ball for any $\alpha \in (0, \text{ess sup}(v))$.

Proof of Theorem 1.4 By Lemma 3.1, the case $N=1$ is proved. We treat below the case $N \geq 2$. For given minimizer $v \in S_m$ of (Inf_m), we set

$$v^+ := \max\{0, v\} \qquad \text{and} \qquad v^- := \min\{0, v\}.$$

If $m^\pm := \|v^\pm\|^2_{L^2(\mathbb{R}^N)} \neq 0$, then Lemma 2.2 (iv) gives that

$$E_m = I(v) = I(v^+) + I(v^-) \geq E_{m^+} + E_{m^-} \geq \frac{m^+}{m} E_m + \frac{m^-}{m} E_m = E_m,$$

and thus $E_{m^\pm}$ is reached at $v^\pm \in S_{m^\pm}$. Using Lemma 2.2 (iv) again, we obtain a contradiction:

$$E_m \geq E_{m^+} + E_{m^-} > \frac{m^+}{m} E_m + \frac{m^-}{m} E_m = E_m.$$

Hence v has constant sign. Since any minimizer of (Inf_m) is a solution of (Q_μ) for some $\mu > 0$ and then by regularity must be of class C^1, we also deduce from [24, Theorem 2] that v is radially symmetric up to a translation in $\mathbb{R}^N$. To proceed further, without loss of generality, we may assume that $v \geq 0$ and $v(x) = \overline{v}(|x|)$ for some one variable function $\overline{v} : [0, \infty) \to [0, \infty)$. By the fact that $v(x) \to 0$ as $|x| \to \infty$, it can be seen that v is bounded and for any $\alpha > 0$ the function $(v-\alpha)^+$ belongs to $H^1(\mathbb{R}^N)$ and has compact support. Since the Schwarz rearrangement v^* satisfies

$$v^* \in S_m \qquad \text{and} \qquad \int_{\mathbb{R}^N} F(v^*) dx = \int_{\mathbb{R}^N} F(v) dx,$$

it follows from Lemma 3.3 that $E_m \leq I(v^*) \leq I(v) = E_m$ and thus

$$\int_{\mathbb{R}^N} |\nabla v^*|^2 dx = \int_{\mathbb{R}^N} |\nabla v|^2 dx.$$

By Lemma 3.3 again, for any $\alpha \in (0, \max(v))$, the level set χ_α is equivalent to a ball. We now assume by contradiction that $\overline{v}$ is not nonincreasing. Then

$$\overline{v}(r_2) > \overline{v}(r_1) > 0 \qquad \text{for some } r_2 > r_1 \geq 0.$$

Since $\overline{v}(r) \to 0$ as $r \to \infty$, there exists $r_3 > r_2$ such that $\overline{v}(r_3) = \overline{v}(r_1)$. Denoting $a := \overline{v}(r_1)$ and $b := \overline{v}(r_2)$, one may see that for any $\alpha \in (a, b)$ the level set χ_α is nonempty but not equivalent to a ball. This gives a contradiction and thus v is nonincreasing with respect to the radial variable. □

4 Least action characterization

In this section we show the least action characterization for any minimizer $v \in S_m$ of (Inf_m) by using a mountain pass characterization of nontrivial solutions of (Q_μ) with $\mu = \mu(v) > 0$. This mountain pass characterization, see Lemma 4.1 below, is the core of the proof of Theorem 1.6. It highlights the role of the L^2 mass and seems to have not been formulated before. Some of our arguments are motivated by [3, 18, 19].

Lemma 4.1 *Assume that $N \geq 1$, $\mu > 0$ and $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the conditions ($f1$) and*

$(f2)'$ *when $N \geq 3$, $\limsup_{t\to\infty} \frac{|f(t)|}{|t|^{\frac{N+2}{N-2}}} < \infty$,*
when $N = 2$, for any $\alpha > 0$

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t^2}} = 0,$$

and suppose in addition that f is locally Lipschitz continuous when $N = 1$. Then for any nontrivial critical point $w \in H^1(\mathbb{R}^N)$ of J_μ, any $\delta > 0$ and any $M > 0$, there exist a constant $T = T(w, \delta, M) > 0$ and a continuous path $\gamma : [0, T] \to H^1(\mathbb{R}^N)$ satisfying

(i) $\gamma(0) = 0$, $J_\mu(\gamma(T)) < -1$, $\max_{t\in[0,T]} J_\mu(\gamma(t)) = J_\mu(w)$;

(ii) $\gamma(\tau) = w$ for some $\tau \in (0, T)$, and

$$J_\mu(\gamma(t)) < J_\mu(w)$$

for any $t \in [0, T]$ such that $\|\gamma(t) - w\|_{L^2(\mathbb{R}^N)} \geq \delta$;

(iii) $m(t) := \|\gamma(t)\|^2_{L^2(\mathbb{R}^N)}$ is a strictly increasing continuous function with $m(T) > M$.

Proof. When $N \geq 3$ and for the given $w \in H^1(\mathbb{R}^N)$, we set

$$\gamma(t) := \begin{cases} w\left(\frac{\cdot}{t}\right), & \text{for } t > 0, \\ 0, & \text{for } t = 0. \end{cases}$$

Note that $m(t) := \|\gamma(t)\|^2_{L^2(\mathbb{R}^N)} = t^N \|w\|^2_{L^2(\mathbb{R}^N)}$ and by the Pohozaev identity

$$\begin{aligned} J_\mu(\gamma(t)) &= \frac{1}{2} t^{N-2} \int_{\mathbb{R}^N} |\nabla w|^2 dx - t^N \int_{\mathbb{R}^N} G_\mu(w) dx \\ &= \frac{1}{2}\Big(t^{N-2} - \frac{N-2}{N} t^N\Big) \int_{\mathbb{R}^N} |\nabla w|^2 dx. \end{aligned}$$

For any $\delta > 0$ and $M > 0$, we can thus choose a large constant $T = T(w, M) > 0$ such that the continuous path $\gamma : [0, T] \to H^1(\mathbb{R}^N)$ satisfies Items $(i) - (iii)$ of Lemma 4.1.

In the case of $N = 1$, without loss of generality, we only consider the situation when the given $w \in H^1(\mathbb{R})$ is negative somewhere. Then the statement (a) of Lemma 3.1 holds and we can define a negative continuous function $W : \mathbb{R} \to \mathbb{R}$ by

$$W(x) = \begin{cases} w(x), & \text{for } x \geq 0, \\ \zeta_- - x^4, & \text{for } x \in [-\varepsilon, 0), \\ \zeta_- - \varepsilon^4, & \text{for } x < -\varepsilon. \end{cases}$$

Here $\varepsilon > 0$ is a chosen small constant such that

$$\frac{1}{2}|W'(x)|^2 - G_\mu(W(x)) = 8x^6 - G_\mu(\zeta_- - x^4) < 0 \qquad \text{for } x \in [-\varepsilon, 0), \tag{4.1}$$

and it follows from $G_\mu(\zeta_-) = 0$ and $g_\mu(\zeta_-) < 0$. Setting

$$\gamma(t) := \begin{cases} W(|\cdot| - \ln t), & \text{for } t > 0, \\ 0, & \text{for } t = 0, \end{cases}$$

one may see that the path $\gamma : [0, \infty) \to H^1(\mathbb{R})$ is continuous,

$$m(t) := \|\gamma(t)\|^2_{L^2(\mathbb{R})} = \begin{cases} \|w\|^2_{L^2(\mathbb{R})} - \displaystyle\int_{\ln t}^{-\ln t} |w(x)|^2 dx, & \text{for } t \in (0,1), \\ \|w\|^2_{L^2(\mathbb{R})}, & \text{for } t = 1, \\ \|w\|^2_{L^2(\mathbb{R})} + 2\displaystyle\int_{-\ln t}^{0} |W(x)|^2 dx, & \text{for } t > 1, \end{cases}$$

and

$$J_\mu(\gamma(t)) = \begin{cases} J_\mu(w) - \displaystyle\int_{\ln t}^{-\ln t} \Big(\frac{1}{2}|w'(x)|^2 - G_\mu(w(x))\Big) dx, & \text{for } t \in (0,1), \\ J_\mu(w), & \text{for } t = 1, \\ J_\mu(w) + 2\displaystyle\int_{-\ln t}^{0} \Big(\frac{1}{2}|W'(x)|^2 - G_\mu(W(x))\Big) dx, & \text{for } t > 1. \end{cases}$$

By the fact that $G_\mu(w(x)) < 0$ for $x \neq 0$ and (4.1), we have

$$J_\mu(\gamma(t)) < J_\mu(w) \qquad \text{for } t \neq 1$$

and

$$J_\mu(\gamma(t)) < J_\mu(w) - 2G_\mu(\zeta_- - \varepsilon^4) \cdot (\ln t - \varepsilon) \to -\infty \qquad \text{as } t \to \infty.$$

Noting also that $m(t)$ is strictly increasing and $m(t) \to \infty$ as $t \to \infty$, for any $\delta > 0$ and $M > 0$ there exists a large constant $T = T(w, M) > 0$ such that $\gamma : [0, T] \to H^1(\mathbb{R})$ is a desired continuous path of Lemma 4.1 when $N = 1$.

To cope with the remaining case $N = 2$, we adapt some arguments from [3, Proposition 2]. For the given $w \in H^1(\mathbb{R}^2)$, we define $\Psi : [0, \infty) \times (0, \infty) \to \mathbb{R}$ by

$$\Psi(\theta, s) := J_\mu\left(\theta w\left(\frac{\cdot}{s}\right)\right) = \frac{1}{2}\theta^2 \int_{\mathbb{R}^2} |\nabla w|^2 dx - s^2 \int_{\mathbb{R}^2} G_\mu(\theta w) dx.$$

It can be easily seen that

$$\begin{aligned} \Psi_\theta(\theta, s) &= \theta \int_{\mathbb{R}^2} |\nabla w|^2 dx - s^2 \int_{\mathbb{R}^2} g_\mu(\theta w) w dx, \\ \Psi_s(\theta, s) &= -2s \int_{\mathbb{R}^2} G_\mu(\theta w) dx, \\ \frac{d}{d\theta} \int_{\mathbb{R}^2} G_\mu(\theta w) dx &= \int_{\mathbb{R}^2} g_\mu(\theta w) w dx. \end{aligned}$$

Since the Nehari and Pohozaev identities give respectively

$$\int_{\mathbb{R}^2} g_\mu(w) w dx = \int_{\mathbb{R}^2} |\nabla w|^2 dx \qquad \text{and} \qquad \int_{\mathbb{R}^2} G_\mu(w) dx = 0,$$

there exist two positive constants $\theta_1 < 1 < \theta_2$ such that

$$\frac{d}{d\theta} \int_{\mathbb{R}^2} G_\mu(\theta w) dx > 0 \qquad \text{for } \theta \in [\theta_1, \theta_2],$$

and thus

$$\int_{\mathbb{R}^2} G_\mu(\theta w) dx \begin{cases} < 0, & \text{for } \theta \in [\theta_1, 1), \\ = 0, & \text{for } \theta = 1, \\ > 0, & \text{for } \theta \in (1, \theta_2]. \end{cases} \tag{4.2}$$

As a direct consequence,

$$\Psi_s(\theta, s) \begin{cases} > 0, & \text{for } (\theta, s) \in [\theta_1, 1) \times (0, \infty), \\ = 0, & \text{for } (\theta, s) \in \{1\} \times (0, \infty), \\ < 0, & \text{for } (\theta, s) \in (1, \theta_2] \times (0, \infty). \end{cases} \tag{4.3}$$

On the other hand, noting that

$$\Psi_\theta(1, s) = \int_{\mathbb{R}^2} |\nabla w|^2 dx - s^2 \int_{\mathbb{R}^2} g_\mu(w) w dx = (1 - s^2) \int_{\mathbb{R}^2} |\nabla w|^2 dx,$$

for any $s \neq 1$ there exists $\vartheta_s \in (0,1)$ such that

$$\Psi_\theta(\theta,s) \begin{cases} > 0, & \text{for } (\theta,s) \in [1-\vartheta_s, 1+\vartheta_s] \times (0,1), \\ < 0, & \text{for } (\theta,s) \in [1-\vartheta_s, 1+\vartheta_s] \times (1,\infty). \end{cases} \tag{4.4}$$

Also, with at hand the continuous function

$$h(t) := \begin{cases} \dfrac{g_\mu(t)}{t}, & \text{for } t \neq 0, \\ -\mu, & \text{for } t = 0, \end{cases}$$

one may find a small constant $s^* \in (0,1)$ such that

$$\Psi_\theta(\theta,s) = \theta\left(\int_{\mathbb{R}^2} |\nabla w|^2 dx - s^2 \int_{\mathbb{R}^2} h(\theta w) w^2 dx\right) > 0 \qquad \text{for } (\theta,s) \in (0,1] \times (0,s^*]. \tag{4.5}$$

Now for any $\delta > 0$ we fix a small constant $\varepsilon = \varepsilon(\delta) > 0$ such that $1-\varepsilon > s^*$ and

$$\left\| w\left(\frac{\cdot}{s}\right) - w \right\|_{L^2(\mathbb{R}^2)} < \delta \qquad \text{for } s \in [1-\varepsilon, 1+\varepsilon],$$

and denote by $\eta(t) = (\theta(t), s(t)) : [0,\infty) \to \mathbb{R}^2$ the piecewise linear curve joining

$$\begin{aligned} (0,s^*) \to (1-\theta^*, s^*) \to (1-\theta^*, 1-\varepsilon) \to (1, 1-\varepsilon) & \\ \to (1,1) & \\ \to (1, 1+\varepsilon) & \to (1+\theta^*, 1+\varepsilon) \to (1+\theta^*, \infty). \end{aligned}$$

Here $\theta^* = \theta^*(w,\delta) \in (0,1)$ is a chosen constant satisfying

$$\theta^* \leq \min\{1-\theta_1, \theta_2 - 1, \vartheta_{1-\varepsilon}, \vartheta_{1+\varepsilon}\},$$

and each segment is horizontal or vertical. Let $0 =: t_0 < t_1 < \cdots < t_6 < t_7 := \infty$ be such that for each $k = 0,1,\cdots,7$, the element $\eta(t_k) \in \mathbb{R}^2$ is an end point of a linear segment of the piecewise linear curve η. We define

$$\gamma(t) := \theta(t) w\left(\frac{\cdot}{s(t)}\right), \qquad t \geq 0.$$

Then the function $J_\mu(\gamma(t)) = \Psi(\eta(t))$ is strictly increasing on (t_0,t_1), (t_1,t_2) and (t_2,t_3) by (4.5), (4.3) and (4.4) respectively. One may also see that $J_\mu(\gamma(t))$ is constant on (t_3,t_4), (t_4,t_5) by (4.3), and strictly decreasing on (t_5,t_6) and (t_6,t_7) via (4.4) and (4.3) respectively. Moreover, using (4.2),

$$\begin{aligned} J_\mu(\gamma(t)) &= \frac{1}{2}(1+\theta^*)^2 \int_{\mathbb{R}^2} |\nabla w|^2 dx - s^2(t) \int_{\mathbb{R}^2} G_\mu((1+\theta^*)w) dx \\ &\to -\infty \qquad \text{as } t \to \infty. \end{aligned} \tag{4.6}$$

Finally we observe that the mass function $m(t) := \|\gamma(t)\|^2_{L^2(\mathbb{R}^2)} = \theta^2(t)s^2(t)\|w\|^2_{L^2(\mathbb{R}^2)}$ is strictly increasing and

$$m(t) = (1+\theta^*)^2 s^2(t) \|w\|^2_{L^2(\mathbb{R}^2)} \to \infty \qquad \text{as } t \to \infty. \tag{4.7}$$

Since for any $M > 0$ we can deduce from (4.6) and (4.7) the existence of a large constant $T = T(w, \delta, M) > 0$ such that

$$J_\mu(\gamma(T)) < -1 \qquad \text{and} \qquad m(T) > M,$$

the continuous path $\gamma : [0, T] \to H^1(\mathbb{R}^2)$ is a desired one and this completes the proof of Lemma 4.1. □

Proof of Theorem 1.6. To prove Item (i), denoting by $w \in H^1(\mathbb{R}^N)$ an arbitrary nontrivial critical point of J_μ, we only need to show that

$$J_\mu(w) \geq J_\mu(v) = E_m + \frac{1}{2}\mu m.$$

For a fixed $\delta > 0$ and $M := m > 0$, let $\gamma : [0, T] \to H^1(\mathbb{R}^N)$ be the continuous path given by Lemma 4.1. In view of Lemma 4.1 (i) and (iii), there exists $t_0 \in (0, T)$ such that

$$\|\gamma(t_0)\|^2_{L^2(\mathbb{R}^N)} = m$$

and thus

$$\begin{aligned} J_\mu(w) = \max_{t\in[0,T]} J_\mu(\gamma(t)) &\geq J_\mu(\gamma(t_0)) \\ &= I(\gamma(t_0)) + \frac{1}{2}\mu \int_{\mathbb{R}^N} |\gamma(t_0)|^2 dx \\ &\geq E_m + \frac{1}{2}\mu m. \end{aligned}$$

We now prove Item (ii). In view of Item (i), an arbitrary least action solution $w \in H^1(\mathbb{R}^N)$ of (Q_μ) satisfies

$$J_\mu(w) = A_\mu = E_m + \frac{1}{2}\mu m. \tag{4.8}$$

Assume by contradiction that $\|w\|^2_{L^2(\mathbb{R}^N)} \neq m$. Then, for

$$\delta := \left|\sqrt{m} - \|w\|_{L^2(\mathbb{R}^N)}\right| > 0 \qquad \text{and} \qquad M := m > 0,$$

we have the continuous path $\gamma : [0, T] \to H^1(\mathbb{R}^N)$ given by Lemma 4.1. Noting that by Lemma 4.1 (iii) there exists $t_0 \in (0, T)$ such that

$$\|\gamma(t_0)\|^2_{L^2(\mathbb{R}^N)} = m \qquad \text{and} \qquad \|\gamma(t_0) - w\|_{L^2(\mathbb{R}^N)} \geq \delta,$$

it follows from Lemma 4.1 (ii) a contradiction:

$$\begin{aligned} J_\mu(w) &> J_\mu(\gamma(t_0)) \\ &= I(\gamma(t_0)) + \frac{1}{2}\mu \int_{\mathbb{R}^N} |\gamma(t_0)|^2 dx \\ &\geq E_m + \frac{1}{2}\mu m. \end{aligned}$$

Since we have proved $\|w\|^2_{L^2(\mathbb{R}^N)} = m$, it is easy to see further that $I(w) = E_m$ by (4.8). □

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