Ocneanu's trace and Starkey's rule

Meinolf Geck and Nicolas Jacon

Abstract

We give a new simple proof for the weights of Ocneanu's trace on Iwahori–Hecke algebras of type A. This trace is used in the construction of the HOMFLYPT-polynomial of knots and links (which includes the famous Jones polynomial as a special case). Our main tool is Starkey's rule concerning the character tables of Iwahori–Hecke algebras of type A.

1 Introduction

Let q be an indeterminate and $K = \mathbb{Q}(q)$ the field of rational functions in q. Let $\mathcal{H}_n(q)$ be the Iwahori–Hecke algebra associated with the symmetric group $\mathfrak{S}_n$ (of type A_{n-1}). Thus, $\mathcal{H}_n(q)$ is an associative K-algebra with a basis $\{T_w \mid w \in \mathfrak{S}_n\}$ and the multiplication is determined by the following rules.

$$\begin{aligned} T_{s_i}^2 &= qT_1 + (q-1)T_{s_i} && \text{for } 1 \leqslant i \leqslant n-1, \\ T_w T_{w'} &= T_{ww'} && \text{if } l(ww') = l(w) + l(w'). \end{aligned}$$

Here, s_i denotes the basic transposition $(i, i+1)$ and $l(w)$ denotes the length of a shortest possible expression of w as a product of the transpositions s_i. (It can be shown that $l(w)$ is the number of inversions of the permutation $w \in \mathfrak{S}_n$, that is, the number of pairs $i < j$ such that $w(i) > w(j)$.) See also the description of $\mathcal{H}_n(q)$ given by Jones [9, §4]. (We have $T_{s_i} = g_i$ in Jones' notation.)

A trace function is a K-linear map $\psi\colon \mathcal{H}_n(q) \to K$ such that $\psi(hh') = \psi(h'h)$ for all $h, h' \in \mathcal{H}_n(q)$. Now let $z \in K$. A trace function $\tau\colon \mathcal{H}_n(q) \to K$ with $\tau(T_1) = 1$ is called an *Ocneanu trace* with parameter z if $\tau(T_w T_{s_m}) = z\tau(T_w)$ for any $1 \leqslant m \leqslant n$ and any $w \in \langle s_1, \ldots, s_{m-1}\rangle$. By [9, Theorem 5.1], for any $z \in K$, there exists a unique Ocneanu trace τ with parameter z; we will therefore write $\tau = \tau_z$. (See also [6, §4] for an alternative construction of τ_z.)

Another characterisation of Ocneanu's trace (which will be used later in this paper) can be given as follows. Every trace function is uniquely determined by its values on basis elements T_w where w runs over a certain set of representatives of the various conjugacy classes of $\mathfrak{S}_n$ (see [7, 8.2.6]). Following [7, 3.1.16], these particular representatives can be described as follows. The conjugacy classes of $\mathfrak{S}_n$ are naturally parametrised by the partitions $\mu \vdash n$. If μ has non-zero parts $\mu_1, \mu_2, \ldots$, then we take $w_\mu := s_{i_1} s_{i_2} \cdots s_{i_k}$ as representative in the class labelled by μ, where $\{i_1, \ldots, i_k\}$ is the set obtained by removing the integers $\mu_1, \mu_1 + \mu_2, \ldots$ from $\{1, 2, \ldots, n\}$. (For example, if $n = 8$ and $\mu = (4, 3, 1)$, then

$w_\mu = s_1 s_2 s_3 s_5 s_6$.) The point about choosing these representatives is that w_μ has minimal length in its conjugacy class, that is, we have $l(w_\mu) \leqslant l(w)$ for any $w \in \mathfrak{S}_n$ which is conjugate to w_μ. Now, applying the defining formula for Ocneanu's trace τ_z to an element w_μ as above, we see that

$$(*) \qquad \tau_z(T_{w_\mu}) = z^{l(w_\mu)} \quad \text{for all } \mu \vdash n.$$

Conversely, if ψ is any trace function on $\mathcal{H}_n(q)$ such that $\psi(T_{w_\mu}) = z^{l(w_\mu)}$ for all $\mu \vdash n$, then we necessarily have $\psi = \tau_z$. (This follows from the above-mentioned fact that any trace function on $\mathcal{H}_n(q)$ is uniquely determined by its values on the elements T_{w_μ}.)

Now we consider the vectorspace of all trace functions on $\mathcal{H}_n(q)$. There is a distinguished basis of that vectorspace, constructed as follows. It is known that $\mathcal{H}_n(q) = \bigoplus_{\lambda \vdash n} M_\lambda$ where each M_λ is a two-sided ideal isomorphic to a full matrix algebra over K. By extending the usual matrix trace on M_λ to all of $\mathcal{H}_n(q)$ (where the extension is zero outside M_λ) we obtain a trace function χ_q^λ on $\mathcal{H}_n(q)$. The set

$$\operatorname{Irr}(\mathcal{H}_n(q)) = \{\chi_q^\lambda \mid \lambda \vdash n\}$$

is the desired basis of the space of trace functions on $\mathcal{H}_n(q)$. The elements of $\operatorname{Irr}(\mathcal{H}_n(q))$ are called the irreducible characters of $\mathcal{H}_n(q)$. See [7] for a general exposition of the theory of Iwahori–Hecke algebras and their characters.

For any $z \in K$, we now have a unique expression

$$\tau_z = \sum_{\lambda \vdash n} \omega_\lambda(q,z)\, \chi_q^\lambda \quad \text{with } \omega_\lambda(q,z) \in K,$$

where the coefficients $\omega_\lambda(q,z)$ are called the *weights* of τ_z.

The purpose of this note is to give a new proof of the following result, which is due to Ocneanu (unpublished; see [9]) and Wenzl [15, §3]. We shall identify a partition $\lambda \vdash n$ with its diagram, i.e., the set of all $(i,j) \in \mathbb{N} \times \mathbb{N}$ such that $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant \lambda_i$, where $\lambda_1 \geqslant \ldots \geqslant \lambda_l > 0$ are the non-zero parts of λ. Using this convention, we denote by $c(x) = j - i$ the content and by $h(x)$ the hook length of $x \in \lambda$ (see [12, Ex. I.1.1]); furthermore, we set $n(\lambda) = \sum_{i=1}^{l} (i-1)\lambda_i$.

Theorem 1.1 *The weights of the Ocneanu trace with parameter z are given by*

$$\omega_\lambda(q,z) = q^{n(\lambda)} \prod_{x \in \lambda} \frac{q - 1 + (q^{c(x)} - 1)z}{q^{h(x)} - 1}.$$

The proof we will give in Section 2 uses only some classical results about Schur polynomials and the characters of $\mathfrak{S}_n$ (which can be found in Macdonald's monograph [12]) and Starkey's rule (see Theorem 2.2) for the irreducible characters of $\mathcal{H}_n(q)$ (a self-contained proof of which can be found in [3]).

Remark 1.2 The above result has the following application. Consider the special case where $z = 0$. Then $(*)$ shows that $\tau_0(T_1) = 1$ and $\tau_0(T_{w_\mu}) = 0$ for all $\mu \vdash n$, $\mu \neq (1^n)$. Thus, τ_0 is the canonical symmetrizing trace on $\mathcal{H}_n(q)$ and we have

$$\omega_\lambda(q,0) = \frac{D_\lambda(q)}{P_n(q)} \qquad \text{for all } \lambda \vdash n,$$

where $P_n(q) = \prod_{i=1}^n (q^{i-1} + q^{i-2} + \cdots + q + 1)$ is the Poincaré polynomial of $\mathcal{H}_n(q)$ and $D_\lambda(q)$ denotes the *generic degree* of χ_q^λ; see [7, 9.4.5]. If we specialise q to a prime power, p^f say, then the generic degrees have a meaning in the representation theory of the general linear group $\mathrm{GL}_n(k)$ where k is the finite field with p^f elements. Setting $z = 0$ in Theorem 1.1, we obtain the formula

$$D_\lambda(q) = q^{n(\lambda)} \frac{(q-1)^n P_n(q)}{\prod_{x \in \lambda} (q^{h(x)} - 1)}.$$

This formula is originally due to Steinberg; see [7, 10.5.2] (modulo the identities concerning hook lengths in [12, I.1.1]). Thus, the weight formula for Ocneanu traces also yields a new proof for the generic degrees in type A. This deduction of the generic degrees from Theorem 1.1 was first described by Ram–Remmel [14].

Remark 1.3 Jones wrote in [9, p. 346] that there should be analogues of Ocneanu's trace for Iwahori–Hecke algebras other than those of type A. The trace given by Lambropoulou [10] was the first such analogue for Iwahori-Hecke algebras of type B. In type B there are infinitely many Markov traces; these are constructed in [5] using results of [6]. Subsequently, Lambropoulou [11] constructed analogues of Ocneanu's trace for the so-called cyclotomic algebras associated with the complex reflection groups $(\mathbb{Z}/e\mathbb{Z}) \wr \mathfrak{S}_n$ where $e \geqslant 1$. These algebras were first defined and studied by Ariki–Koike [1] and Broué–Malle [2]. If $e = 1$, one just gets the Iwahori–Hecke algebra $\mathcal{H}_n(q)$ associated with $\mathfrak{S}_n$; if $e = 2$, one gets the Iwahori–Hecke algebra of type B_n.

The generalization of Ocneanu's trace to a cyclotomic algebra depends on e parameters $z, y_1, \ldots, y_{e-1}$. The problem of determining the weights of these traces was first considered by Orellana [13]. She determined the weights for $e = 2$ and for special choices of the parameters z, y_1. Then Iancu [8] found a formula (in the case $e = 2$) which actually expresses the weights as polynomial functions in the parameters z, y_1. Furthermore, she conjectured a general weight formula for any $e \geqslant 2$. This conjecture was subsequently proved by Geck–Iancu–Malle [4]. It should be noted that both Orellana's proof and the part of the proof in [4] which is concerned with a generalization of Orellana's argument use the knowledge of the weights for Iwahori–Hecke algebras of type A, that is, the formula in Theorem 1.1.

Thus, the weights of Ocneanu's trace on $\mathcal{H}_n(q)$ play a crucial role in the determination of the weights for cyclotomic algebras. This was one of our motivations to find a new simple proof of Theorem 1.1.

2 Proof of Theorem 1.1

Let $z \in K$ and let us define a trace function $\psi_z \colon \mathcal{H}_n(q) \to K$ by the formula

$$\psi_z := \sum_{\lambda \vdash n} \Big(q^{n(\lambda)} \prod_{x \in \lambda} \frac{q-1+(q^{c(x)}-1)z}{q^{h(x)}-1} \Big) \chi_q^\lambda.$$

In order to prove Theorem 1.1, we must show that ψ_z is the Ocneanu trace with parameter z. By the remarks following $(*)$, this is equivalent to showing that $\psi_z(T_{w_\mu}) = z^{l(w_\mu)}$ for all $\mu \vdash n$. Thus, it remains to prove the following identity:

$$(1) \qquad z^{l(w_\mu)} = \sum_{\lambda \vdash n} \Big(q^{n(\lambda)} \prod_{x \in \lambda} \frac{q-1+(q^{c(x)}-1)z}{q^{h(x)}-1} \Big) \chi_q^\lambda(T_{w_\mu}) \quad \text{for all } \mu \vdash n.$$

For this purpose, we note that both sides can actually be expressed as polynomials in z with coefficients in $K = \mathbb{Q}(q)$. Thus, if we let $\mathbf{z}$ be an indeterminate over K, we have to show the following identity of polynomials in $K[\mathbf{z}]$:

$$(2) \qquad \mathbf{z}^{l(w_\mu)} = \sum_{\lambda \vdash n} \Big(q^{n(\lambda)} \prod_{x \in \lambda} \frac{q-1+(q^{c(x)}-1)\mathbf{z}}{q^{h(x)}-1} \Big) \chi_q^\lambda(T_{w_\mu}) \quad \text{for all } \mu \vdash n.$$

In order to prove such a polynomial identity, it is enough to prove it for infinitely many specialisations of the variable $\mathbf{z}$ to elements in K. Following Wenzl [15], we shall use the specialisations

$$\mathbf{z} \mapsto z_r := q^r \frac{1-q}{1-q^r} \qquad \text{for all } r \in \mathbb{N}, r \geqslant n.$$

The point about this choice is that we have the following identity:

$$q^{n(\lambda)} \prod_{x \in \lambda} \frac{q-1+(q^{c(x)}-1)z_r}{q^{h(x)}-1} = \Big(\frac{1-q}{1-q^r} \Big)^n s_\lambda(1, q, q^2, \ldots, q^{r-1}),$$

where s_λ is the Schur polynomial corresponding to λ in r variables $x_1, \ldots, x_r$; see [12, Ex. I.3.1]. Hence we must show the following identity for $\mu \vdash n$, $r \geqslant n$:

$$(3) \qquad q^{rl(w_\mu)} \Big(\frac{1-q^r}{1-q} \Big)^{l(\mu)} = \sum_{\lambda \vdash n} s_\lambda(1, q, q^2, \ldots, q^{r-1})\, \chi_q^\lambda(T_{w_\mu}).$$

In order the evaluate the right hand side, we use two basic results about the characters of $\mathfrak{S}_n$ and $\mathcal{H}_n(q)$: Frobenius' character formula and Starkey's rule. Denote by $\mathrm{Irr}(\mathfrak{S}_n) = \{\chi^\lambda \mid \lambda \vdash n\}$ the set of irreducible characters of $\mathfrak{S}_n$.

Theorem 2.1 (Frobenius' character formula; see [12, I.7]) *For any $r \geqslant n$ and any partition $\nu \vdash n$, we have*

$$\sum_{\lambda \vdash n} s_\lambda(x_1, \ldots, x_r)\, \chi^\lambda(w_\nu) = \prod_{i \geqslant 1} (x_1^{\nu_i} + \cdots + x_r^{\nu_i}),$$

where $\nu_1, \nu_2, \ldots$ are the non-zero parts of ν.

The following rule shows how χ_q^λ is determined by χ^λ. First note that the labellings are compatible in the following sense. We have $\chi_q^\lambda(T_w) \in \mathbb{Q}[q]$ for $\lambda \vdash n$ and $w \in \mathfrak{S}_n$. Then, by Tits' deformation theorem (see [7, 8.1.7]), we have

$$\chi^\lambda(w) = \chi_q^\lambda(T_w)\big|_{q=1} \qquad \text{for all } \lambda \vdash n \text{ and } w \in \mathfrak{S}_n.$$

Theorem 2.2 (Starkey's rule; see [3]) *For partitions $\mu, \nu \vdash n$, we define*

$$p_\mu^\nu(q) := \frac{|C_\nu \cap \mathfrak{S}_\mu|}{|\mathfrak{S}_\mu|}(q-1)^{-l(\mu)} \prod_{i \geqslant 1}(q^{\nu_i} - 1),$$

where $\nu_1, \nu_2, \ldots$ are the non-zero parts of ν and C_ν denotes the conjugacy class of elements of cycle type ν in $\mathfrak{S}_n$. Then we have

$$\chi_q^\lambda(T_{w_\mu}) = \sum_{\nu \vdash n} p_\mu^\nu(q)\, \chi^\lambda(w_\nu) \quad \textit{for all } \lambda, \mu \vdash n.$$

Using the above two results, we can now evaluate the right hand side of (3).

$$\begin{aligned}
\text{r.h.s. of (3)} &= \sum_{\lambda \vdash n} s_\lambda(1, q, q^2, \ldots, q^{r-1}) \sum_{\nu \vdash n} p_\mu^\nu(q)\, \chi^\lambda(w_\nu) \quad \text{by Theorem 2.2} \\
&= \sum_{\nu \vdash n} p_\mu^\nu(q) \sum_{\lambda \vdash n} s_\lambda(1, q, q^2, \ldots, q^{r-1})\, \chi^\lambda(w_\nu) \\
&= \sum_{\nu \vdash n} p_\mu^\nu(q) \prod_{i \geqslant 1}(1 + q^{\nu_i} + q^{2\nu_i} + \cdots + q^{(r-1)\nu_i}) \quad \text{by Theorem 2.1} \\
&= \sum_{\nu \vdash n} p_\mu^\nu(q) \prod_{i \geqslant 1} \frac{q^{r\nu_i} - 1}{q^{\nu_i} - 1} = \Big(\frac{1 - q^r}{1 - q}\Big)^{l(\mu)} \sum_{\nu \vdash n} p_\mu^\nu(q^r)
\end{aligned}$$

where the last equality is proved by using the defining formula for $p_\mu^\nu(q)$ and applying it to $p_\mu^\nu(q^r)$ as well. Thus, it remains to show the following identity:

$$q^{rl(w_\mu)} = \sum_{\nu \vdash n} p_\mu^\nu(q^r) \quad \text{for all } \mu \vdash n \text{ and } r \geqslant n. \tag{4}$$

For this purpose, we consider the 1-dimensional representation $\operatorname{ind}_q \colon \mathcal{H}_n(q) \to K$ given by $\operatorname{ind}_q(T_w) = q^{l(w)}$. Specialising $q \mapsto 1$ shows that ind_q corresponds to the trivial character of $\mathfrak{S}_n$. Hence, by Theorem 2.2, we have

$$\sum_{\nu \vdash n} p_\mu^\nu(q) = \operatorname{ind}_q(T_{w_\mu}) = q^{l(w_\mu)}.$$

We can do the same with q^r instead of q and, thus, obtain (4). This completes the proof of Theorem 1.1.

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Institut Girard Desargues, bat. Jean Braconnier, Université Lyon 1, 21 av Claude Bernard, F–69622 Villeurbanne cedex, France

E-mail addresses: geck@desargues.univ-lyon1.fr, jacon@desargues.univ-lyon1.fr