Ocneanu’s trace and Starkey’s rule

Meinolf Geck and Nicolas Jacon

Abstract

We give a new simple proof for the weights of Ocneanu’s trace on Iwahori–Hecke algebras of type $A$. This trace is used in the construction of the HOMFLYPT-polynomial of knots and links (which includes the famous Jones polynomial as a special case). Our main tool is Starkey’s rule concerning the character tables of Iwahori–Hecke algebras of type $A$.

1 Introduction

Let $q$ be an indeterminate and $K = \mathbb{Q}(q)$ the field of rational functions in $q$. Let $\mathcal{H}_n(q)$ be the Iwahori–Hecke algebra associated with the symmetric group $S_n$ (of type $A_{n-1}$). Thus, $\mathcal{H}_n(q)$ is an associative $K$-algebra with a basis $\{T_w \mid w \in S_n\}$ and the multiplication is determined by the following rules.

\[
T_{s_i}^2 = qT_{s_i} + (q-1)T_{s_i} \quad \text{for } 1 \leq i \leq n-1,
\]

\[
T_w T_{w'} = T_{ww'} \quad \text{if } l(ww') = l(w) + l(w').
\]

Here, $s_i$ denotes the basic transposition $(i, i+1)$ and $l(w)$ denotes the length of a shortest possible expression of $w$ as a product of the transpositions $s_i$. (It can be shown that $l(w)$ is the number of inversions of the permutation $w \in S_n$, that is, the number of pairs $i < j$ such that $w(i) > w(j)$.) See also the description of $\mathcal{H}_n(q)$ given by Jones [9, §4]. (We have $T_{s_i} = g_i$ in Jones’ notation.)

A trace function is a $K$-linear map $\psi: \mathcal{H}_n(q) \to K$ such that $\psi(hh') = \psi(h'h)$ for all $h, h' \in \mathcal{H}_n(q)$. Now let $z \in K$. A trace function $\tau: \mathcal{H}_n(q) \to K$ with $\tau(T_1) = 1$ is called an Ocneanu trace with parameter $z$ if $\tau(T_s T_{s_m}) = z\tau(T_s)$ for any $1 \leq m \leq n$ and any $w \in \langle s_1, \ldots, s_{m-1} \rangle$. By [9, Theorem 5.1], for any $z \in K$, there exists a unique Ocneanu trace $\tau$ with parameter $z$; we will therefore write $\tau = \tau_z$. (See also [6, §4] for an alternative construction of $\tau_z$.)

Another characterisation of Ocneanu’s trace (which will be used later in this paper) can be given as follows. Every trace function is uniquely determined by its values on basis elements $T_w$ where $w$ runs over a certain set of representatives of the various conjugacy classes of $S_n$ (see [7, 8.2.6]). Following [7, 3.1.16], these particular representatives can be described as follows. The conjugacy classes of $S_n$ are naturally parametrised by the partitions $\mu \vdash n$. If $\mu$ has non-zero parts $\mu_1, \mu_2, \ldots$, then we take $w_\mu := s_{i_1} s_{i_2} \cdots s_{i_k}$ as representative in the class labelled by $\mu$, where $\{i_1, \ldots, i_k\}$ is the set obtained by removing the integers $\mu_1, \mu_1 + \mu_2, \ldots$ from $\{1, 2, \ldots, n\}$. (For example, if $n = 8$ and $\mu = (4, 3, 1)$, then...
The point about choosing these representatives is that \( w_{\mu} \) has minimal length in its conjugacy class, that is, we have \( l(w_{\mu}) \leq l(w) \) for any \( w \in \mathfrak{S}_n \) which is conjugate to \( w_{\mu} \). Now, applying the defining formula for Ocneanu’s trace \( \tau_z \) to an element \( w_{\mu} \) as above, we see that

\[
\tau_z(T_{w_{\mu}}) = z^{l(w_{\mu})} \quad \text{for all } \mu \vdash n.
\]

Conversely, if \( \psi \) is any trace function on \( \mathcal{H}_n(q) \) such that \( \psi(T_{w_{\mu}}) = z^{l(w_{\mu})} \) for all \( \mu \vdash n \), then we necessarily have \( \psi = \tau_z \). (This follows from the above-mentioned fact that any trace function on \( \mathcal{H}_n(q) \) is uniquely determined by its values on the elements \( T_{w_{\mu}} \).)

Now we consider the vectorspace of all trace functions on \( \mathcal{H}_n(q) \). There is a distinguished basis of that vectorspace, constructed as follows. It is known that \( \mathcal{H}_n(q) = \bigoplus_{\lambda \vdash n} M_\lambda \) where each \( M_\lambda \) is a two-sided ideal isomorphic to a full matrix algebra over \( K \). By extending the usual matrix trace on \( M_\lambda \) to all of \( \mathcal{H}_n(q) \) (where the extension is zero outside \( M_\lambda \)) we obtain a trace function \( \chi_\lambda^u \) on \( \mathcal{H}_n(q) \). The set

\[
\text{Irr}(\mathcal{H}_n(q)) = \{ \chi_\lambda^u \mid \lambda \vdash n \}
\]

is the desired basis of the space of trace functions on \( \mathcal{H}_n(q) \). The elements of \( \text{Irr}(\mathcal{H}_n(q)) \) are called the irreducible characters of \( \mathcal{H}_n(q) \). See [7] for a general exposition of the theory of Iwahori–Hecke algebras and their characters.

For any \( z \in K \), we now have a unique expression

\[
\tau_z = \sum_{\lambda \vdash n} \omega_{\lambda}(q, z) \chi_\lambda^u \quad \text{with } \omega_{\lambda}(q, z) \in K,
\]

where the coefficients \( \omega_{\lambda}(q, z) \) are called the \textit{weights} of \( \tau_z \).

The purpose of this note is to give a new proof of the following result, which is due to Ocneanu (unpublished; see [9]) and Wenzl [15, §3]. We shall identify a partition \( \lambda \vdash n \) with its diagram, i.e., the set of all \((i, j) \in \mathbb{N} \times \mathbb{N}\) such that \( 1 \leq i \leq l \) and \( 1 \leq j \leq \lambda_i \), where \( \lambda_1 \geq \ldots \geq \lambda_l > 0 \) are the non-zero parts of \( \lambda \). Using this convention, we denote by \( c(x) = j - i \) the content and by \( h(x) \) the hook length of \( x \in \lambda \) (see [12, Ex. 1.1.1]); furthermore, we set \( n(\lambda) = \sum_{i=1}^l (i - 1)\lambda_i \).

\begin{theorem}
The weights of the Ocneanu trace with parameter \( z \) are given by

\[
\omega_{\lambda}(q, z) = q^{n(\lambda)} \prod_{x \in \lambda} \frac{q - 1 + (q^{c(x)} - 1)z}{q^{h(x)} - 1}.
\]
\end{theorem}

The proof we will give in Section 2 uses only some classical results about Schur polynomials and the characters of \( \mathfrak{S}_n \) (which can be found in Macdonald’s monograph [12]) and Starkey’s rule (see Theorem 2.2) for the irreducible characters of \( \mathcal{H}_n(q) \) (a self-contained proof of which can be found in [3]).

\begin{remark}
The above result has the following application. Consider the special case where \( z = 0 \). Then (\( \ast \)) shows that \( \tau_0(T_1) = 1 \) and \( \tau_0(T_{w_{\mu}}) = 0 \) for
all $\mu \vdash n$, $\mu \neq (1^n)$. Thus, $\tau_0$ is the canonical symmetrizing trace on $\mathcal{H}_n(q)$ and we have

$$\omega_\lambda(q,0) = \frac{D_\lambda(q)}{P_n(q)} \quad \text{for all } \lambda \vdash n,$$

where $P_n(q) = \prod_{i=1}^n (q^{-1} + q^{i-2} + \cdots + q + 1)$ is the Poincaré polynomial of $\mathcal{H}_n(q)$ and $D_\lambda(q)$ denotes the generic degree of $\chi^\lambda_\psi$; see [7, 9.4.5]. If we specialise $q$ to a prime power, $p^f$ say, then the generic degrees have a meaning in the representation theory of the general linear group $GL_n(k)$ where $k$ is the finite field with $p^f$ elements. Setting $z = 0$ in Theorem 1.1, we obtain the formula

$$D_\lambda(q) = q^{n(\lambda)} (q - 1)^n P_n(q) \prod_{x \in \lambda} (q^h(x) - 1).$$

This formula is originally due to Steinberg; see [7, 10.5.2] (modulo the identities concerning hook lengths in [12, I.1.1]). Thus, the weight formula for Ocneanu traces also yields a new proof for the generic degrees in type $A$. This deduction of the generic degrees from Theorem 1.1 was first described by Ram–Remmel [14].

**Remark 1.3** Jones wrote in [9, p. 346] that there should be analogues of Ocneanu’s trace for Iwahori–Hecke algebras other than those of type $A$. The trace given by Lambropoulou [10] was the first such analogue for Iwahori-Hecke algebras of type $B$. In type $B$ there are infinitely many Markov traces; these are constructed in [5] using results of [6]. Subsequently, Lambropoulou [11] constructed analogues of Ocneanu’s trace for the so-called cyclotomic algebras associated with the complex reflection groups $(\mathbb{Z}/e\mathbb{Z}) \wr S_n$ where $e \geq 1$. These algebras were first defined and studied by Ariki–Koike [1] and Brüe–Malle [2]. If $e = 1$, one just gets the Iwahori–Hecke algebra $\mathcal{H}_n(q)$ associated with $S_n$; if $e = 2$, one gets the Iwahori–Hecke algebra of type $B_n$.

The generalization of Ocneanu’s trace to a cyclotomic algebra depends on $e$ parameters $z, y_1, \ldots, y_{e-1}$. The problem of determining the weights of these traces was first considered by Orellana [13]. She determined the weights for $e = 2$ and for special choices of the parameters $z, y_1$. Then Iancu [8] found a formula (in the case $e = 2$) which actually expresses the weights as polynomial functions in the parameters $z, y_1$. Furthermore, she conjectured a general weight formula for any $e \geq 2$. This conjecture was subsequently proved by Geck–Iancu–Malle [4]. It should be noted that both Orellana’s proof and the part of the proof in [4] which is concerned with a generalization of Orellana’s argument use the knowledge of the weights for Iwahori–Hecke algebras of type $A$, that is, the formula in Theorem 1.1.

Thus, the weights of Ocneanu’s trace on $\mathcal{H}_n(q)$ play a crucial role in the determination of the weights for cyclotomic algebras. This was one of our motivations to find a new simple proof of Theorem 1.1.
2 Proof of Theorem 1.1

Let \( z \in K \) and let us define a trace function \( \psi_z : \mathcal{H}_n(q) \to K \) by the formula

\[
\psi_z := \sum_{\lambda \vdash n} q^{n(\lambda)} \prod_{x \in \lambda} \frac{q - 1 + (q^{e(x)} - 1)z}{q^{h(x)} - 1} \lambda_q^\lambda.
\]

In order to prove Theorem 1.1, we must show that \( \psi_z(T_{w_n}) = z^{l(w_n)} \) for all \( \mu \vdash n \). Thus, it remains to prove the following identity:

\[
(1) \quad z^{l(w_n)} = \sum_{\lambda \vdash n} q^{n(\lambda)} \prod_{x \in \lambda} \frac{q - 1 + (q^{e(x)} - 1)z}{q^{h(x)} - 1} \lambda_q^\lambda(T_{w_n}) \quad \text{for all } \mu \vdash n.
\]

For this purpose, we note that both sides can actually be expressed as polynomials in \( z \) with coefficients in \( K = \mathbb{Q}(q) \). Thus, if we let \( z \) be an indeterminate over \( K \), we have to show the following identity for \( \mu \vdash n, r \geq n \):

\[
(2) \quad z^{l(w_n)} = \sum_{\lambda \vdash n} q^{n(\lambda)} \prod_{x \in \lambda} \frac{q - 1 + (q^{e(x)} - 1)z}{q^{h(x)} - 1} \lambda_q^\lambda(T_{w_n}) \quad \text{for all } \mu \vdash n.
\]

In order to prove such a polynomial identity, it is enough to prove it for infinitely many specialisations of the variable \( z \) to elements in \( K \). Following Wenzl [15], we shall use the specialisations

\[
\mathbf{z} \mapsto z_r := q^{r} \frac{1 - q}{1 - q^r} \quad \text{for all } r \in \mathbb{N}, r \geq n.
\]

The point about this choice is that we have the following identity:

\[
q^{n(\lambda)} \prod_{x \in \lambda} \frac{q - 1 + (q^{e(x)} - 1)z_r}{q^{h(x)} - 1} = \frac{1 - q}{1 - q^r}^n s_\lambda(1, q, q^2, \ldots, q^{r-1}),
\]

where \( s_\lambda \) is the Schur polynomial corresponding to \( \lambda \) in \( r \) variables \( x_1, \ldots, x_r \); see [12, Ex. I.3.1]. Hence we must show the following identity for \( \mu \vdash n, r \geq n \):

\[
(3) \quad q^{l(w_n)} \left( \frac{1 - q^r}{1 - q} \right)^{l(\mu)} = \sum_{\lambda \vdash n} s_\lambda(1, q, q^2, \ldots, q^{r-1}) \lambda_q^\lambda(T_{w_n}).
\]

In order the evaluate the right hand side, we use two basic results about the characters of \( S_n \) and \( \mathcal{H}_n(q) \): Frobenius’ character formula and Starkey’s rule. Denote by \( \text{Irr}(S_n) = \{ \chi^\lambda \mid \lambda \vdash n \} \) the set of irreducible characters of \( S_n \).

**Theorem 2.1** (Frobenius’ character formula; see [12, I.7]) For any \( r \geq n \) and any partition \( \nu \vdash n \), we have

\[
\sum_{\lambda \vdash n} s_\lambda(x_1, \ldots, x_r) \chi^\lambda(w_n) = \prod_{i \geq 1} (x_1^{\nu_1} + \cdots + x_r^{\nu_r}),
\]

where \( \nu_1, \nu_2, \ldots \) are the non-zero parts of \( \nu \).
The following rule shows how $\chi^\lambda_\mu(T_w)$ is determined by $\chi^\lambda_\mu$. First note that the labellings are compatible in the following sense. We have $\chi^\lambda_\mu(T_w) \in \mathbb{Q}[q]$ for $\lambda \vdash n$ and $w \in S_n$. Then, by Tits’ deformation theorem (see [7, 8.1.7]), we have

$$\chi^\lambda_\mu(w) = \chi^\lambda_\mu(T_w)_{|q=1} \quad \text{for all } \lambda \vdash n \text{ and } w \in S_n.$$

**Theorem 2.2 (Starkey’s rule; see [3])** For partitions $\mu, \nu \vdash n$, we define

$$p^\mu_\nu(q) := \frac{|C_\nu \cap S_\mu|}{|S_\mu|} (q - 1)^{-l(\mu)} \prod_{i \geq 1} (q^{\nu_i} - 1),$$

where $\nu_1, \nu_2, \ldots$ are the non-zero parts of $\nu$ and $C_\nu$ denotes the conjugacy class of elements of cycle type $\nu$ in $S_n$. Then we have

$$\chi^\lambda_\mu(T_w^\mu) = \sum_{\nu \vdash n} p^\nu_\mu(q) \chi^\lambda_\nu(w_\nu) \quad \text{for all } \lambda, \mu \vdash n.$$

Using the above two results, we can now evaluate the right hand side of (3).

r.h.s. of (3) = \sum_{\lambda \vdash n} s_\lambda(1, q, q^2, \ldots, q^{r-1}) \sum_{\nu \vdash n} p^\nu_\mu(q) \chi^\lambda_\nu(w_\nu) \quad \text{by Theorem 2.2}

= \sum_{\nu \vdash n} p^\nu_\mu(q) \sum_{\lambda \vdash n} s_\lambda(1, q, q^2, \ldots, q^{r-1}) \chi^\lambda_\nu(w_\nu)

= \sum_{\nu \vdash n} p^\nu_\mu(q) \prod_{i \geq 1} (1 + q^{\nu_i} + q^{2\nu_i} + \cdots + q^{(r-1)\nu_i}) \quad \text{by Theorem 2.1}

= \sum_{\nu \vdash n} p^\nu_\mu(q) \prod_{i \geq 1} \frac{q^{\nu_i} - 1}{q^{\nu_i} - 1} = \left(\frac{1 - q^r}{1 - q}\right)^{l(\mu)} \sum_{\nu \vdash n} p^\nu_\mu(q^r)

where the last equality is proved by using the defining formula for $p^\nu_\mu(q)$ and applying it to $p^\nu_\mu(q^r)$ as well. Thus, it remains to show the following identity:

$$q^{rl(\nu_\nu)} = \sum_{\nu \vdash n} p^\nu_\mu(q^r) \quad \text{for all } \mu \vdash n \text{ and } r \geq n.$$

For this purpose, we consider the 1-dimensional representation $\text{ind}_q: \mathcal{H}_n(q) \to K$ given by $\text{ind}_q(T_w) = q^{l(w)}$. Specialising $q \mapsto 1$ shows that $\text{ind}_q$ corresponds to the trivial character of $S_n$. Hence, by Theorem 2.2, we have

$$\sum_{\nu \vdash n} p^\nu_\mu(q) = \text{ind}_q(T_w^\mu) = q^{l(w_\nu)}.$$

We can do the same with $q^r$ instead of $q$ and, thus, obtain (4). This completes the proof of Theorem 1.1.

**References**


Institut Girard Desargues, bat. Jean Braconnier, Université Lyon 1, 21 av Claude Bernard, F–69622 Villeurbanne cedex, France
E-mail addresses: geck@desargues.univ-lyon1.fr, jacon@desargues.univ-lyon1.fr