Isometries on symmetric spaces

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Content

decreasing rearrangement

Let $I = (0,1)$, $a \in (0,\infty]$, equipped with Lebesgue measure m. Let $L(I,m)$ be the space of all measurable complex-valued functions on I equipped with Lebesgue measure m i.e. functions which coincide almost everywhere are considered identical. Define $S(I,m)$ (or $S(I)$ for brevity) to be the linear subspace of $L(I,m)$ which consists of all functions x such that $m(\{t : |x(t)| > s\})$ is finite for some $s > 0$.

For $x \in S(I)$, we denote by $\mu(x)$ the decreasing rearrangement of the function $|x|$. That is,

$$\mu(t;x) = \inf\{s \geq 0 : \; m(\{|x| > s\}) \leq t\}, \quad t > 0.$$

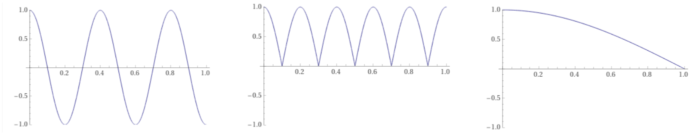

Figure 1: $f : t \mapsto \cos(5\pi t)$, $|f| : t \mapsto |\cos(5\pi t)|$, $\mu(f) : t \mapsto \cos(\frac{\pi}{2} t)$, $t \in (0,1)$

symmetric function spaces

Definition (Lindentrauss–Tzafriri, Classical Banach spaces II & Krein–Petunin–Semenov, Interpolation of linear operators)

We say that $(E(I), \|\cdot\|_E)$ is a Banach symmetric function space (or rearrangement-invariant space) on I if the following hold:

1. $E(I)$ is a subset of $S(I, m)$;
2. $(E, \|\cdot\|_E)$ is a Banach space;
3. If $x \in E$ and if $y \in S(I, m)$ are such that $\mu(y) \le \mu(x)$, then $y \in E$ and $\|y\|_E \le \|x\|_E$.

Examples: L_p-spaces, $1 \le p \le \infty$, Lorentz function spaces and Orlicz functions spaces.

For the sake of simplicity, we always assume that $\mathcal{M}$ is a semifinite von Neumann algebra on a separable Hilbert space (the separability of the Hilbert space is not necessary), equipped with a semifinite faithful normal trace τ.

The collection of all τ-measurable operators with respect to $\mathcal{M}$ is denoted by $S(\mathcal{M},\tau)$.

Definition

Let $x \in S(\mathcal{M},\tau)$. The generalised singular value function $\mu(x) : t \to \mu(t;x)$, $t > 0$, of the operator x is defined by setting

$$\mu(t;x) = \inf\left\{\|xp\|_\infty : \ p \in \mathcal{P}(\mathcal{M}), \ \tau(\mathbf{1}-p) \le t\right\}.$$

Noncommutative symmetric spaces

Let $E(0,\infty)$ be a Banach symmetric function space. The operator space $E(\mathcal{M},\tau)$ defined by

$$E(\mathcal{M},\tau) := \{x \in S(\mathcal{M},\tau) : \ \mu(x) \in E(0,\infty)\}, \ \|x\|_{E(\mathcal{M},\tau)} := \|\mu(x)\|_E$$

is a Banach space [Kalton–Sukochev, Crelle's journal, 2008].

Question

Let $E(0,\infty)$ be a separable symmetric function space on $(0,\infty)$ and let $(\mathcal{M},\tau)$ be a semifinite von Neumann algebra (on a separable Hilbert space) with a semifinite faithful normal trace τ. How can one describe the family of surjective isometries on the symmetric operator space $E(\mathcal{M},\tau)$ associated with $E(0,\infty)$?

The study of the above question has a very long history, initiated by Stefan Banach, who obtained the general form of isometries between L_p-spaces on a finite measure space in the 1930s.
Indeed, Banach proved that case of ℓ_p, $1 \leq p < \infty$, $p \neq 2$ [S. Banach, Theorie des operations lineares, Chelsea, Warsaw, 1932] and he remarked that the proof for $L_p[0,1]$ will appear in Studia Mathematica IV. However, this promised paper never appeared.

R. Fleming, J. Jamison, *Isometries on Banach spaces: function spaces*, Monographs and Surveys in Pure and Applied Mathematics, 129. Chapman & Hall/CRC, Boca Raton, FL, 2003.

R. Fleming, J. Jamison, *Isometries on Banach spaces: Vector-valued function spaces,* Vol. 2. Monographs and Surveys in Pure and Applied Mathematics, 138. Chapman & Hall/CRC, Boca Raton, FL, 2008.

Lamperti's theorem

Theorem (Lamperti, 1958, Pacific J. Math.)

Let $1 \leq p < \infty$, $p \neq 2$. *Let* $(\Omega_1, \Sigma_1, \mu_1)$ *and* $(\Omega_2, \Sigma_2, \mu_2)$ *be two* σ*-finite measure spaces.*

Then, U *is a linear isometry from* $L_p(\Omega_1, \Sigma_1, \mu_1)$ *into* $L_p(\Omega_2, \Sigma_2, \mu_2)$ *if and only if there exists a regular set isomorphism* T *from* Σ_1 *into* Σ_2 *and a function* h *defined on* Ω_2 *so that*

$$Uf(t) = h(t)T'(f(t)),\ \forall f \in L_p(\Omega_1, \Sigma_1, \mu_1)$$

where h *satisfies that* $\int_{T(A)} |h|^p d\mu_2 = \mu_1(A)$ *for each* $A \in \Sigma_1$ *and* T' *is the unique linear transformation* T' *from* Σ_1*-measurable functions into* Σ_2*-measurable functions induced by the set isomorphism* T.

Zaidenberg's theorem

Lamperti's result was later extended by Lumer [TAMS, 1961; BAMS, 1962 & Ann. Inst. Fourier, 1963] and Zaidenberg [Soviet Math. Dokl. 1976 & 1977; Inst. Fourier (Grenoble), 1995] by the hermitian operator method (we will explain it later).

Theorem (Zaidenberg, Soviet Math. Dokl., 1977)

Let $E_1(\Omega_1, \Sigma_1, \mu_1)$ and $E_2(\Omega_2, \Sigma_2, \mu_2)$ be two complex symmetric function spaces over atomless σ-finite measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$. If the norm on E_1 is not proportional to the norm on $L_2(\Omega_1, \Sigma_1, \mu_1)$, then any surjective isometry T between complex symmetric function spaces $E_1(\Omega_1, \Sigma_1, \mu_1)$ and $E_2(\Omega_2, \Sigma_2, \mu_2)$ must be of the elementary form

$$(Tf)(t) = h(t)(T_1 f)(t), \; f \in E_1, \tag{1}$$

where T_1 is the operator induced by a regular set isomorphism from Ω_1 onto Ω_2 and h is a measurable function on Ω_2.

Arazy's theorem

Theorem (Arazy, Math. Z., 1985)

Let E be a separable complex-linear symmetric sequence space different from ℓ_2. Then, every isometry on E is of the form

$$Vx = (\lambda(k)x(\pi(k)))_{k=1}^{\infty},$$

where $\lambda(k)$'s are unimodular scalars and π is a permutation of positive integers.

Kadison's theorem

For the case when $L_\infty(0,\infty)$, the question posed at the beginning of the talk was answered by Kadison.

Theorem (Kadison, Ann. of Math., 1951)

Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras. Then, there exists a surjective isometry T from $\mathcal{M}$ onto $\mathcal{N}$ if and only if $\mathcal{M}$ and $\mathcal{N}$ are Jordan $*$-isomorphic. In particular,

$$T(x) = w \cdot J(x),\ \forall x \in \mathcal{M},$$

where w is a unitary element in $\mathcal{N}$ and J is a Jordan $*$-isomorphism (i.e., $J : \mathcal{M} \to \mathcal{N}$ is a bijective, normal, $*$-preserving linear map which satisfies that $J(xy + yx) = J(xy) + J(yx)$, $x, y \in \mathcal{M}$).

In general, $J(xy) \neq J(x)J(y)$. However, for any Jordan $*$-isomorphism J from $\mathcal{M}$ onto $\mathcal{N}$, there exists a central projection in $\mathcal{N}$ such that $z \cdot J(\cdot)$ is a $*$-homomorphism (i.e., $zJ(xy) = zJ(x)J(y)$) and $(\mathbf{1} - z)J(\cdot)$ is a $*$-anti-homomorphism (i.e. $(\mathbf{1} - z)J(xy) = (\mathbf{1} - z)J(y)J(x)$) on $\mathcal{M}$ [Størmer, TAMS, 1965].

Isometries on noncommutative L_p-spaces

After the non-commutative L_p-spaces were introduced in the 1950s, the description of isometries between noncommutative L_p-spaces was investigated by Broise [Ann. Sci. Ec. Norm. Super. 1966], Russo [BAMS, 1968], Arazy [Israel. J. Math, 1975] and Tam [TAMS, 1979], etc. Finally, using the noncommutative Clarkson inequality, Yeadon obtained the general form of isometries from a noncommutative L_p-space into another [Math. Proc. Cambridge Philos. Soc. 1981] (in the semifinite setting).

Theorem (Yeadon, 1981)

Let $1 \leq p < \infty$, $p \neq 2$. Let $L_p(\mathcal{M}, \tau)$ and $L_p(\mathcal{N}, \tau)$ be two noncommutative L_p-spaces (in the semifinite setting).
Then, a continuous linear mapping $T : L_p(\mathcal{M}, \tau) \to_{into} L_p(\mathcal{N}, \nu)$ is an isometry if and only if there exists a partial isometry $w \in \mathcal{N}$, an unbounded positive self-adjoint operator B affiliated with $\mathcal{N}$ and a normal injective Jordan $*$-homomorphism from $\mathcal{M}$ onto a weakly closed $*$-subalgebra of $\mathcal{N}$ such that

1. $w^*w = J(\mathbf{1}_{\mathcal{M}}) = \mathrm{supp}(B)$;
2. *every spectral projection of B commutes with $J(x)$, $x \in \mathcal{M}$;*
3. $\tau(x) = \nu(B^p J(x))$ *for all $x \in \mathcal{M}$, $x \geq 0$;*
4. $$T(x) = w \cdot B \cdot J(x),$$
   for all $x \in L_p(\mathcal{M}, \tau) \cap \mathcal{M}$.

Remark

Isometries between noncommutative L_p-spaces associated to two (not necessarily semifinite) von Neumann algebras have been obtained by Watanabe [JOT, 1992] and Sherman [JFA, 2005] (relies on the condition characterizing when the noncommutative Clarkson inequality becomes an equality, see [Kosaki, TAMS, 1984] and [Raynaud–Xu, JFA, 2003]).

Sourour's theorem

Theorem (Sourour, JFA, 1981)

Let J be a separable symmetric ideal in $B(\mathcal{H})$, whose norm is not proportional to the norm on the Schatten class C_2. A continuous linear operator T on J is an isometry if and only if there are unitary operators U and V on $\mathcal{H}$ such that

$$T(X) = UXV \text{ or } T(X) = UX^T V, \ \forall X \in J.$$

Here, X^T denotes the transpose of X with respect to a fixed orthonormal basis of $\mathcal{H}$.

The above theorem for the special case when $J = C_p$, $1 \le p < \infty$, $p \neq 2$, was obtained by Arazy [Israel J. Math., 1975] by the Clarkson inequality.

generalization of Sourour's theorem

After that, several authors tried to extend Sourour's theorem to the semifinite setting.
However, Sourour's approach strongly relies on the matrix representation of compact operators on a separable Hilbert space, which is not applicable for general von Neumann algebras.

1. Using the extreme points approach, Chilin, Medzhitov and Sukochev characterized the isometries on Lorentz spaces $\Lambda_\psi(\mathcal{M}, \tau)$ associated with a finite von Neumann algebra $\mathcal{M}$ [Math. Z. 1989 & Dokl. Acad. Nauk UzSSR, 1988];
2. Under some conditions imposed on the isometries (positivity, disjointness-preserving), several results were obtained, see e.g. [Sukochev–Veksler, IEOT, 2019], [Huang–Sukochev-Zanin, JFA, 2020] and [de Jager–Conradie, Positivity, 2020].
3. Adopting Sourour's techniques, Sukochev [IEOT, 1996] obtained the description of isometries on separable symmetric operator spaces affiliated with the hyperfinite II_1 factor $\mathcal{R}$.

Sourour's theorem for the Hyperfinite II_1 factor

Theorem (Sukochev, IEOT, 1996)

Let $\mathcal{R}$ be the hyperfinite II_1 factor and let $E(\mathcal{R})$ be a separable symmetric space associated with $\mathcal{R}$. Assume that $\|\cdot\|_{E(\mathcal{R})}$ and $\|\cdot\|_{L_2(\mathcal{R})}$ are not proportional. A continuous linear operator T on $E(\mathcal{R})$ is an isometry if and only if

$$T(x) = uJ(x), \; x \in \mathcal{R} \cap E(\mathcal{R}),$$

where u is a unitary operator in $\mathcal{R}$ and J is a normal trace-preserving $*$-isomorphism (or anti-$*$-isomorphism) on $\mathcal{R}$.

Hermitian operators

The notion of hermitian operators on a Banach space was formulated by Lumer [TAMS, 1961] in his seminal paper, for the purpose of extending Hilbert space type arguments to Banach spaces. This notion plays an important role in different fields such as operator theory on Banach spaces, matrix theory, optimal control theory and computer science. There are several equivalent definitions for a bounded linear operator T on a Banach space X to be hermitian [Theorem 5.2.6, Fleming–Jamison, Isometries on Banach spaces, 2003]. The following definition does not involve the so-called semi-inner-product in the sense of Lumer.

Definition

A bounded linear operator $T : X \to X$ is said to be hermitian if e^{itT} (i.e., $I + \sum_{n=1}^{\infty} \frac{(itT)^n}{n!}$) is an isometry on X for each $t \in \mathbb{R}$.

General form of hermitian operators I

1. The set of all hermitian operators T on a complex symmetric function space E (on $(0,1)$ or $(0,\infty)$, other than L_2) coincides with the set of all operators of multiplication by bounded real functions h, i.e.,

$$T(x) = hx,$$

   for every $x \in E$ [Zaidenberg, 1977].

2. The set of all hermitian operators T on a complex separable symmetric sequence space E (other than ℓ_2) coincides with the set of all operators of multiplication by bounded real sequence [Arazy, 1985], that is,

$$T(x) = hx, \; x \in E.$$

General form of hermitian operators II

3. Let J be a separable symmetric ideal in $B(\mathcal{H})$, which is not C_2. Then, a bounded linear operator T on J is hermitian if and only if there are bounded self-adjoint operators a and b on $\mathcal{H}$ such that [Sourour, 1981]
$$T(x) = ax + xb, \; x \in J.$$
4. Paterson and Sinclair [JLMS, 1972] (see also [Sinclair, PAMS, 1970]) studied the hermitian operators on C^*-algebras. In particular, any hermitian operator T on a von Neumann algebra $\mathcal{M}$ is of the form
$$T(x) = ax + xb, \; x \in \mathcal{M},$$
where a, b are self-adjoint elements in $\mathcal{M}$.
5. The general form of hermitian operators was unknown even for noncommutative L_p-spaces.

Sketch of Zaidenberg's proof

Theorem

Let $E_1(\Omega_1, \Sigma_1, \mu_1)$ and $E_2(\Omega_2, \Sigma_2, \mu_2)$ be complex symmetric function spaces over the atomless σ-finite measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$. If the norm on E_1 is not proportional to the norm on $L_2(\Omega_1, \Sigma_1, \mu_1)$, then any surjective isometry T between two complex symmetric function spaces $E_1(\Omega_1, \Sigma_1, \mu_1)$ and $E_2(\Omega_2, \Sigma_2, \mu_2)$ must be of the elementary form

$$(Tf)(t) = h(t)(T_1 f)(t), \ f \in E_1, \tag{2}$$

where T_1 is the operator induced by a regular set isomorphism from Ω_1 onto Ω_2 and h is a measurable function on Ω_2.

Proof.

1. Let M_a be the multiplication operator by a real function $a \in L_\infty(\Omega_1)$ on $E_1(\Omega_1)$, i.e., $M_a(x) = ax$. In particular, M_a is a hermitian operator on $E_1(\Omega_1)$.
2. By the fact that TM_aT^{-1} is hermitian on $E_2(\Omega_2)$ (see e.g. [Fleming–Jamision, 1989]).
   Therefore, $TM_aT^{-1} = M_{J(a)}$, where $J(a) \in L_\infty(\Omega_2)$. It is easy to verify that J is a linear operator from $L_\infty(\Omega_1)$ onto $L_\infty(\Omega_2)$ satisfies that
   $J(a^2) = TM_{a^2}T^{-1} = TM_aT^{-1}TM_aT^{-1} = M_{J(a)}M_{J(a)} = M_{J(a)J(a)}$.
   We claim that this J induces a regular set isomorphism from Ω_1 onto Ω_2 and therefore induces T_1 as required.
3. For the sake of simplicity, we assume that Ω_1 is finite. Then, we have

$$T(a) = TM_aT^{-1}T(\mathbf{1}) = J(a)T(\mathbf{1}).$$

Hence, $h = T(\mathbf{1})$.

From now on, let $E(\mathcal{M}, \tau)$ be a separable symmetric space on an atomless semifinite von Neumann algebra (or an atomic von Neumann algebra with all atoms having the same trace) $\mathcal{M}$ equipped with a semifinite faithful normal trace τ. Assume that $\|\cdot\|_E$ is not proportional to $\|\cdot\|_2$.

Theorem (H.–Sukochev)

A bounded operator T on $E(\mathcal{M}, \tau)$ is a hermitian operator on $E(\mathcal{M}, \tau)$ if and only if there exist (bounded) self-adjoint operators a and b in $\mathcal{M}$ such that

$$Tx = ax + xb, \ x \in E(\mathcal{M}, \tau).$$

Idea of the proof.

Show that the restriction of T on $E(\mathcal{M}, \tau) \cap \mathcal{M}$ (the bounded part) can be extended to the ideal $C_0(\mathcal{M}, \tau)$ of τ-compact operators in $\mathcal{M}$, which is a C^*-algebra. Show that T is also a Hermitian operator on $C_0(\mathcal{M}, \tau)$. Then, we can apply Paterson and Sinclair's result. □

Theorem (H.–Sukochev)

If $T : E(\mathcal{M}_1, \tau_1) \to F(\mathcal{M}_2, \tau_2)$ is a surjective isometry, then there exist two sequences of elements $A_i \in F(\mathcal{M}_2, \tau_2)$, disjointly supported from the left and $B_i \in F(\mathcal{M}_2, \tau_2)$, disjointly supported from the right, a surjective Jordan $*$-isomorphism $J : \mathcal{M}_1 \to \mathcal{M}_2$ and a central projection $z \in \mathcal{M}_2$ such that

$$T(x) = \|\cdot\|_F - \lim_{n\to\infty} \sum_{i=1}^{n} J(x)A_i z + B_i J(x)(\mathbf{1} - z), \ \forall x \in E(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1.$$

Corollary

If the trace τ is finite, then there exist elements $A, B \in F(\mathcal{M}_2, \tau_2)$ such that

$$T(x) = J(x)Az + BJ(x)(\mathbf{1} - z), \ \forall x \in E(\mathcal{M}_1, \tau_1) \cap \mathcal{M}_1.$$

Remarks

1. The case for real function spaces and complex function spaces are quite different. The hermitian operator approach works only for the complex case. Braverman and Semenov (1974 & 1975) described the general form of isometries on real symmetric sequence spaces; Kalton and Randrianantoanina (1994) obtained the general form of isometries T on a real symmetric function space $E(0,1)$ (which is not isometrically equal to L_2, and has the Fatou property or order-continuous norm):

$$Tf(s) = a(s)f(\sigma(s)), a.e.$$

for any $f \in E(0,1)$, where a is a non-vanishing real Borel function and σ is an invertible Borel map from $[0,1]$ into $[0,1]$.
It is interesting to describe the isometries on the real part of a symmetric operator space $E(\mathcal{M}, \tau)$.

2. The symmetric spaces are required to be separable in our main theorem, while Zaidenberg's theorem does not require the separability of the function spaces. So, it will be interesting to ask whether we can remove the separability imposed on the symmetric operator spaces in our theorem.