VARIOUS SLICING INDICES ON BANACH SPACES

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Abstract. We give a short and direct proof for the computation of the Szlenk index of the $C(K)$ spaces, when K is a countable compact space and determine their Lavrientiev indices. We also compute the Szlenk index of certain $C(\alpha)$ spaces, where α is an uncountable ordinal. Finally, we show that if the Szlenk index of a Banach space is ω (first infinite ordinal), then its weak*-dentability index is at most ω^2 and that this estimate is optimal.

1. Introduction - Notation

We start with the definition of the Szlenk derivation and the Szlenk index that have been first introduced in [23] and used there to show that there is no universal space for the class of separable reflexive Banach spaces. So consider a real Banach space X and K a weak*-compact subset of X^*. For $\varepsilon > 0$ we let $\mathcal{V}$ be the set of all relatively weak*-open subsets V of K such that the norm diameter of V is less than ε and $s_\varepsilon K = K \setminus \cup\{V : V \in \mathcal{V}\}$. Then we define inductively $s_\varepsilon^\alpha K$ for any ordinal α by $s_\varepsilon^{\alpha+1} K = s_\varepsilon(s_\varepsilon^\alpha K)$ and $s_\varepsilon^\alpha K = \cap_{\beta<\alpha} s_\varepsilon^\beta K$ if α is a limit ordinal. We denote by B_{X^*} the closed unit ball of X^*. We then define $\mathrm{Sz}(X,\varepsilon)$ to be the least ordinal α so that $s_\varepsilon^\alpha B_{X^*} = \emptyset$, if such an ordinal exists. Otherwise we write $\mathrm{Sz}(X,\varepsilon) = \infty$. The *Szlenk index* of X is finally defined by $\mathrm{Sz}(X) = \sup_{\varepsilon>0} \mathrm{Sz}(X,\varepsilon)$.

Note that our definition is in general different from the definition of W. Szlenk, but equivalent to it, as soon as X is a separable Banach space which does not contain any isomorphic copy of $\ell_1(\mathbb{N})$ (see [13]).

We also introduce an alternative *convex Szlenk index.* If K is weak*-compact and convex we may define $c_\varepsilon K = \overline{\mathrm{co}}^* s_\varepsilon K$ (namely, the weak*-closed convex hull of $s_\varepsilon K$). Then $\mathrm{Cz}(X,\varepsilon)$ and $\mathrm{Cz}(X)$ are defined as before, using instead the derivation c_ε.

Finally, if K is weak*-compact and convex, we call weak*-slice of K any non empty set of the form $S = \{x^* \in K,\ x^*(x) > t\}$, where $x \in X$ and $t \in \mathbb{R}$. Then we denote by $\mathcal{S}$ the set of all weak*-slices of K of norm diameter less than ε and $d_\varepsilon K = K \setminus \cup\{S : S \in \mathcal{S}\}$. From this derivation, we define similarly the *weak**-*dentability indices* of X that we denote $\mathrm{Dz}(X,\varepsilon)$ and $\mathrm{Dz}(X)$.

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We recall now standard facts about ordinals. For that purpose, we follow the notation of [19]. An ordinal α is identified with the set of ordinals β such that $\beta < \alpha$, ω is the first infinite ordinal and ω_1 is the first uncountable ordinal. For an ordinal α, we denote $\alpha+ = \alpha + 1$. We always consider that the sets of ordinals are topological spaces equipped with the order topology. Then, for any ordinal α, $C(\alpha+)$ is the space of all real valued continuous functions on $[0, \alpha]$ equipped with the supremum norm and $C_0(\alpha) = \{f \in C(\alpha+),\ f(\alpha) = 0\}$. Note that for any infinite α, $C(\alpha+)$ is isomorphic to $C_0(\alpha)$. Through the natural isometries, we will identify the dual space of $C(\alpha+)$ to $\ell_1([0, \alpha])$ and the dual space of $C_0(\alpha)$ to $\ell_1([0, \alpha))$. For α and β countable ordinals, we set $e_\alpha(\beta) = 1$ if $\alpha = \beta$ and 0 otherwise.

The isomorphic classification of the spaces $C(\alpha+)$, for α countable, is due to C. Bessaga and A. Pełczyński [3]. Subsequently, C. Samuel [20] performed the exact computation of $\mathrm{Sz}(C(\alpha+))$, for α countable, which implied that the Szlenk index perfectly determines the isomorphic classes of the separable $C(\alpha+)$ spaces. It is underlined in the survey paper of H.P. Rosenthal [19] that Samuel's proof is rather involved and relies on a deep result of D.E. Alspach and Y. Benyamini [1], but also contains more information. It is also suggested that one should look for a more direct approach. We propose such a proof in section 3. We also compute the Lavrientiev index of $C(\alpha+)$. Finally, we show that the Szlenk index does not distinguish the isomorphic classes of the non separable $C(\alpha+)$ spaces.

Let us now recall that it follows from the classical theory of Asplund spaces (see for instance [7] and references therein) that for a Banach space X, each of the following conditions: $\mathrm{Dz}(X) \neq \infty$, $\mathrm{Cz}(X) \neq \infty$ and $\mathrm{Sz}(X) \neq \infty$ is equivalent to X being an Asplund space. In particular, if X is a separable Banach space, each of the conditions $\mathrm{Dz}(X) < \omega_1$, $\mathrm{Cz}(X) < \omega_1$ and $\mathrm{Sz}(X) < \omega_1$ is equivalent to the separability of X^*. In [15] it is shown, using an approach from descriptive set theory due to B. Bossard (see [5] and [6]), that there is a universal function $\psi : \omega_1 \to \omega_1$, such that if X is an Asplund space with $\mathrm{Sz}(X) < \omega_1$, then $\mathrm{Dz}(X) \leq \psi(\mathrm{Sz}(X))$. In section 4, we look for a concrete expression of ψ and give its first interesting value, namely $\psi(\omega) = \omega^2$. Our argument relies in part on a deep result of Knaust, Odell and Schlumprecht ([12]) on the linear structure of spaces with Szlenk index at most ω.

2. Elementary properties of the slicing indices

Our first proposition can be found in [15], but this phenomenon was first observed for Lavrientiev indices by A. Sersouri in [22].

Proposition 2.1. *Let X be a Banach space. If $s_\varepsilon^{\omega^\alpha}(B_{X^*}) \neq \emptyset$, for some ordinal α and $\varepsilon > 0$, then $Sz(X) \geq \omega^{\alpha+1}$. In particular, if X is an Asplund space, then $Sz(X) = \omega^\alpha$, for some ordinal α.*

Next, we prove the following simple but useful fact.

Proposition 2.2. *Let X be a Banach space and α an ordinal. Assume that*

$$\forall \varepsilon > 0 \ \ \exists \delta(\varepsilon) > 0 \ \ s_\varepsilon^\alpha(B_{X^*}) \subset (1-\delta(\varepsilon))B_{X^*}.$$

Then

$$Sz(X) \leq \alpha.\omega$$

Proof. Let $\varepsilon > 0$. An easy homogeneity argument shows that for any $n \in \mathbb{N}$:

$$s_\varepsilon^{\alpha.n}(B_{X^*}) \subset (1-\delta(\varepsilon))^n B_{X^*}.$$

Consequently, there exists an integer N so that $s_\varepsilon^{\alpha.N}(B_{X^*}) \subset \frac{\varepsilon}{3}B_{X^*}$ and therefore $s_\varepsilon^{\alpha.N+1}(B_{X^*}) = \emptyset$. This finishes the proof. □

Remark 2.3. The analogues of Propositions 2.1 and 2.2 are also true for the weak*-dentability index and the convex Szlenk index.

Proposition 2.4. *For any Banach space X,*

$$Sz(X \oplus X) = Sz(X).$$

Proof. It is clearly enough to show that $\mathrm{Sz}(X \oplus X) \leq \mathrm{Sz}(X)$. We may also assume that $X \oplus X$ is equipped with the norm $\|(x, x')\| = \|x\| + \|x'\|$. First, we show that for any A and B weak*-compact subsets of X^* and for any $\varepsilon > 0$,

$$s_\varepsilon(A \times B) \subset (A \times s_\varepsilon(B)) \cup (s_\varepsilon(A) \times B). \tag{2.1}$$

Indeed, let $(x^*, y^*) \notin (A \times s_\varepsilon(B)) \cup (s_\varepsilon(A) \times B)$. We need to prove that $(x^*, y^*) \notin s_\varepsilon(A \times B)$ and thus may assume that $(x^*, y^*) \in A \times B$. Thus there exist U and V weak*-open subsets of X^* containing respectively x^* and y^* such that $U \cap A$ and $V \cap B$ have diameter less than ε. Then $W = U \times V$ is a weak*-open subset of $(X \oplus_1 X)^* = X^* \oplus_\infty X^*$, containing (x^*, y^*) and so that the diameter of $W \cap (A \times B)$ is less than ε. So $(x^*, y^*) \notin s_\varepsilon(A \times B)$.
On the other hand, a straightforward transfinite induction yields that for any C and D weak*-compact subsets of $X^* \times X^*$

$$\forall \varepsilon > 0 \ \ \forall \alpha \ \ s_\varepsilon^\alpha(C \cup D) \subset (s_\varepsilon^\alpha(C) \cup s_\varepsilon^\alpha(D)). \tag{2.2}$$

The next step is to show by transfinite induction that for any A and B weak*-compact subsets of X^*

$$\forall \varepsilon > 0 \ \ \forall \alpha \geq 0 \ \ s_\varepsilon^{\omega^\alpha}(A \times B) \subset (A \times s_\varepsilon^{\omega^\alpha}(B)) \cup (s_\varepsilon^{\omega^\alpha}(A) \times B). \tag{2.3}$$

The case $\alpha = 0$ is given by (2.1). Suppose now that the above statement is true for any $\beta < \alpha$. If α is a limit ordinal, then it is clearly also true for α. So let us assume that $\alpha = \beta + 1$ and that the statement is true for β. Then, it follows from an iterated application of (2.2) that

$$(2.4)\qquad \forall n\in\mathbb{N}\ \ s_\varepsilon^{\omega^\beta.n}(A\times B)\subset\bigcup_{k=0}^{n}\big(s_\varepsilon^{\omega^\beta.k}(A)\times s_\varepsilon^{\omega^\beta.(n-k)}(B)\big).$$

Therefore for any $(x^*,y^*)\in s_\varepsilon^{\omega^{\beta+1}}(B_{X^*}\times B_{X^*})$, we have

$$\forall n\in\mathbb{N}\ \ \exists k(n)\le n\ \ x^*\in s_\varepsilon^{\omega^\beta.k(n)}(B_{X^*})\text{ and }y^*\in s_\varepsilon^{\omega^\beta.(n-k(n))}(B_{X^*}).$$

If $(k(n))_n$ is unbounded, then $x^*\in s_\varepsilon^{\omega^{\beta+1}}(B_{X^*})$. Otherwise, $(n-k(n))_n$ is unbounded and $y^*\in s_\varepsilon^{\omega^{\beta+1}}(B_{X^*})$. This finishes the inductive proof of (2.3)

Finally, we conclude the proof of Proposition 2.4 by combining (2.3) and Proposition 2.1

□

3. A direct computation of Sz((C(K)) for K countable compact topological space

We shall need in this section the following Lemma, which is the easy part of the fundamental classification result of Bessaga and Pełczyński (Lemma 1 of [3]).

Lemma 3.1. *Let α and β be two ordinals so that $\omega\le\alpha<\omega_1$ and $\alpha\le\beta<\alpha^\omega$. Then $C(\alpha+)$ is isomorphic to $C(\beta+)$.*

We will now give a new and direct proof of the following theorem, due to C. Samuel [20].

Theorem 3.2. *For any $0\le\alpha<\omega_1$,*

$$Sz(C(\omega^{\omega^\alpha}+))=\omega^{\alpha+1}.$$

Proof. Showing the inequality $\mathrm{Sz}(C(\omega^{\omega^\alpha}+))\ge\omega^{\alpha+1}$ is the easy part of the proof. Indeed, using the fact that the set $(e_\gamma)_{\gamma\le\beta}$ is 2-separated for the norm of $\ell_1([0,\beta])$ and w^*-homeomorphic to $[0,\beta]$, we get that for any $\beta<\omega_1$, $\mathrm{Sz}(C(\omega^\beta+),1)>\beta$ (see [19] for details). Then Lemma 3.1 implies that for any n in $\mathbb{N}$, $C(\omega^{\omega^\alpha}+)$ is isomorphic to $C(\omega^{\omega^\alpha.n}+)$ and therefore $\mathrm{Sz}(C(\omega^{\omega^\alpha}+))>\omega^\alpha.n$, which yields the desired inequality. Note that Proposition 2.1 also allows to conclude that $\mathrm{Sz}(C(\omega^{\omega^\alpha}+))\ge\omega^{\alpha+1}$.

So we now concentrate on the converse inequality. For a fixed $0\le\alpha<\omega_1$, we denote $Z=\ell_1([0,\omega^{\omega^\alpha}))$ equipped with the weak*-topology induced by $C_0(\omega^{\omega^\alpha})$. Then, for all $\gamma<\omega^{\omega^\alpha}$, we set $Z_\gamma=\ell_1([0,\gamma])$ equipped with the weak*-topology induced by $C(\gamma+)$ and P_γ the canonical projection from Z onto Z_γ. The following Lemma is the crucial step of our argument (in this statement, the Szlenk derived sets are meant with the weak*-topologies described above for Z and Z_γ).

Lemma 3.3. *Let $\alpha < \omega_1$, $\gamma < \omega^{\omega^\alpha}$, $\beta < \omega_1$ and $\varepsilon > 0$.*
If $z \in s^\beta_{3\varepsilon}(B_Z)$ and $\|P_\gamma z\| > 1-\varepsilon$, then $P_\gamma z \in s^\beta_\varepsilon(B_{Z_\gamma})$.

Proof. We will use a transfinite induction on β. The statement is trivially true for $\beta = 0$. Assume it is true for any $\mu < \beta$. If β is a limit ordinal, then clearly, it is also true for β. So assume $\beta = \mu+1$ and let $z \in B_Z$ such that $\|P_\gamma z\| > 1-\varepsilon$ and $P_\gamma z \notin s^\beta_\varepsilon(B_{Z_\gamma})$. We need to show that $z \notin s^\beta_{3\varepsilon}(B_Z)$, so we may assume that $z \in s^\mu_{3\varepsilon}(B_Z)$ and therefore that $P_\gamma z \in s^\mu_\varepsilon(B_{Z_\gamma})$. Then, there is a weak*-open subset V of Z_γ containing $P_\gamma z$ such that $d =$diam$(V \cap s^\mu_\varepsilon(B_{Z_\gamma})) < \varepsilon$. We may assume that

$$V = \bigcap_{i=1}^n \{x \in Z_\gamma,\ f_i(x) > \alpha_i\},$$

where $\alpha_i \in \mathbb{R}$ and $f_i \in C(\gamma+)$. Since $\|P_\gamma z\| > 1-\varepsilon$, we may also assume that $\|f_1\| = 1$ and $\alpha_1 > 1-\varepsilon$, which implies that $V \cap (1-\varepsilon)B_{Z_\gamma} = \emptyset$.
We now define functions $g_i \in C_0(\omega^{\omega^\alpha})$ by $g_i = f_i$ on $[1,\gamma]$ and $g_i = 0$ on $(\gamma, \omega^{\omega^\alpha})$. Then we consider the weak*-open subset of Z:

$$U = \bigcap_{i=1}^n \{y \in Z,\ g_i(y) > \alpha_i\}.$$

It is clear that $z \in U \cap s^\mu_{3\varepsilon}(B_Z)$. For any $y \in U \cap s^\mu_{3\varepsilon}(B_Z)$, $P_\gamma y \in V$, so $\|P_\gamma y\| > 1-\varepsilon$ and by the induction hypothesis $P_\gamma y \in V \cap s^\mu_\varepsilon(B_{Z_\gamma})$. Therefore for all $y, y' \in U \cap s^\mu_{3\varepsilon}(B_Z)$, $\|P_\gamma y - P_\gamma y'\| \le d < \varepsilon$. Since moreover $\|P_\gamma y\| > 1-\varepsilon$ and $\|P_\gamma y'\| > 1-\varepsilon$, we have that $\|y - y'\| \le d + 2\varepsilon < 3\varepsilon$. This shows that $z \notin s^\beta_{3\varepsilon}(B_Z)$ and finishes our induction.

□

In order to conclude the proof of Theorem 3.2, it is enough to show that

$$\forall 0 \le \alpha < \omega_1 \ \ \forall \gamma < \omega^{\omega^\alpha} \ \ \forall \varepsilon > 0 \quad s^{\omega^\alpha}_\varepsilon(B_{Z_\gamma}) = \emptyset. \tag{3.5}$$

This will be done by transfinite induction on α. If $\alpha = 0$, then for any $\gamma < \omega$, Z_γ is finite dimensional and therefore $s_\varepsilon(B_{Z_\gamma}) = \emptyset$. So the statement is true for $\alpha = 0$. It also passes easily to limit ordinals. So assume now that it is true for α. Then Lemma 3.3 implies that

$$\forall \varepsilon > 0 \ \ s^{\omega^\alpha}_\varepsilon(B_Z) \subset (1 - \frac{\varepsilon}{3})B_Z, \tag{3.6}$$

where $Z = \ell_1([0, \omega^{\omega^\alpha}))$ is equipped with the weak*-topology induced by $C_0(\omega^{\omega^\alpha})$. It now follows from (3.6) and Proposition 2.2 that

$$\forall \varepsilon > 0 \ \ s^{\omega^{\alpha+1}}_\varepsilon(B_Z) = \emptyset \tag{3.7}$$

Now, Lemma 3.1 implies that for any $\omega^{\omega^\alpha} \leq \gamma < \omega^{\omega^{\alpha+1}}$, $C(\gamma+)$ is isomorphic to $C(\omega^{\omega^\alpha}+)$ and therefore to $C_0(\omega^{\omega^\alpha})$. So $s_\varepsilon^{\omega^{\alpha+1}}(B_{Z_\gamma}) = \emptyset$, for any $\varepsilon > 0$ and any $\gamma < \omega^{\omega^{\alpha+1}}$. This finishes our induction.

□

It should be noted that the isomorphic classes of the separable $C(\alpha+)$ spaces are also determined by other ordinal indices. For instance, D.E. Alspach, R. Judd and E. Odell studied in [2] the ordinal index $I(X)$, introduced by J. Bourgain in [4], which measures the presence of ℓ_1 in a separable Banach space X. Among other thing they proved that

$$\forall 0 \leq \alpha < \omega_1 \quad I(C(\omega^{\omega^\alpha}+)) = \omega^{1+\alpha+1}.$$

We will now add a remark on the Lavrientiev index of the $C(\alpha+)$ spaces. If (M,d) is a compact metrizable space, the functions of first Baire class from M into $\mathbb{R}$ can be classified with the help of different ordinal indices: the separation, oscillation and convergence indices. The separation index was introduced by M. Lavrientiev [17] and a thorough study of these three indices was done by A. Kechris and A. Louveau [11]. We will concentrate on the oscillation index. For $f : M \to \mathbb{R}$ and $\varepsilon > 0$, we define the derivation

$$b_{f,\varepsilon}(M) = M \setminus \cup\{V :\ V \text{ is an open subset of } M \text{ and } \operatorname{diam}(f(V)) < \varepsilon\}.$$

Then the oscillation indices of f, $\beta(f,\varepsilon)$ and $\beta(f) = \sup_{\varepsilon>0} \beta(f,\varepsilon)$ are defined in the usual way. If X is a separable Banach space, B_{X^*} equipped with the weak*-topology is a compact metrizable space on which we can compute the oscillation index $\beta(x^{**})$ for any $x^{**} \in X^{**}$. Then we define the Lavrientiev index of X by:

$$\beta(X) = \sup_{x^{**} \in X^{**}} \beta(x^{**}).$$

Clearly, $\beta(X) = 1$ if and only if X is reflexive. On the other hand $\beta(X) < \omega_1$ if and only if X does not contain any isomorphic copy of ℓ_1 (this is an improvement due to J. Bourgain [4] of the celebrated result of E. Odell and H.P. Rosenthal [18]). We shall now indicate how $\beta(C(K))$ can be computed. We wish to thank the referee for pointing out an incomplete argument in the first proof of this result.

Proposition 3.4. *For any countable compact space K*

$$\beta(C(K)) = Sz(C(K)).$$

Proof. It is clear that for any Banach space X, $\beta(X) \leq \mathrm{Sz}(X)$. So, in view of Theorem 3.2 and of the version of Proposition 2.1 for the index $\beta(X)$ due to A. Sersouri [22], it is enough to find $x^{**} \in C(K)^{**} = \ell_\infty(K)$ such that $b^{\omega^\alpha}_{x^{**},1}(B_{C(K)^*}) \neq \emptyset$, whenever K is a countable compact space whose Cantor derived set $K^{(\omega^\alpha)} \neq \emptyset$. So assume that K is countable compact and $K^{(\omega^\alpha)} \neq \emptyset$ and denote by L_α the set of limit ordinals less than ω^α. Then consider

$$x^{**} = \sum_{\beta \in L_\alpha} \sum_{0 \leq k < \omega} (-1)^k \mathbb{1}_{K^{(\beta+k)} \setminus K^{(\beta+k+1)}} \quad \in \ell_\infty(K).$$

Recall that $e_t \in C(K)^*$ is the evaluation at $t \in K$ and denote by M the closed unit ball of $C(K)^*$ equipped with the weak*-topology. We will show by transfinite induction that

$$\forall \gamma < \omega^\alpha, \ \ \{e_t, \ t \in K^{(\gamma)} \setminus K^{(\gamma+1)}\} \subset b^\gamma_{x^{**},1}(M). \tag{3.8}$$

The statement is clearly true for $\gamma = 0$.
Assume that it is true for γ and consider $t \in K^{(\gamma+1)} \setminus K^{(\gamma+2)}$. Since the isolated points of a countable compact space F are dense in F, there exists a sequence (t_n) in $K^{(\gamma)} \setminus K^{(\gamma+1)}$ converging to t. This implies that (e_{t_n}) is weak*-converging to e_t. But x^{**} is built in such a way that, for all $n \in \mathbb{N}$, $|x^{**}(e_{t_n} - e_t)| = 2$. Since, by induction hypothesis, (e_{t_n}) is included in $b^\gamma_{x^{**},1}(M)$, we get that $e_t \in b^{\gamma+1}_{x^{**},1}(M)$.
Let now γ be a limit ordinal and assume our statement true for all $\beta < \gamma$. Let $t \in K^{(\gamma)} \setminus K^{(\gamma+1)}$. We now fix $\beta < \gamma$. Using the density of the isolated points of $K^{(\beta)}$, we deduce the existence of a sequence (t_n) in $K^{(\beta)} \setminus K^{(\beta+1)}$ converging to t. By our induction hypothesis we obtain that (e_{t_n}) is included in $b^\beta_{x^{**},1}(M)$. This later set being weak*-closed, we have that $e_t \in b^\beta_{x^{**},1}(M)$. Since, this is true for any $\beta < \gamma$, we finally get that $e_t \in b^\gamma_{x^{**},1}(M)$. This finishes our induction.
It follows from (3.8) that for any $\gamma < \omega^\alpha$, $b^\gamma_{x^{**},1}(M) \neq \emptyset$ and by weak*-compactness that $b^{\omega^\alpha}_{x^{**},1}(M) \neq \emptyset$.

□

We conclude this section with a few remarks on the spaces $C(\alpha+)$, when α is a simple uncountable ordinal. First we obtain

Proposition 3.5.

$$Sz(C(\omega_1+)) = \omega_1.\omega$$

Proof. For any $\alpha < \omega_1$, $\mathrm{Sz}(C(\omega^\alpha+), 1) > \alpha$ and $C(\omega^\alpha+)$ embeds isometrically in $C(\omega_1+)$, so $\mathrm{Sz}(C(\omega_1+), 1) \geq \omega_1$. Since ω_1 is a limit ordinal, we actually obtain, using weak*-compactness, that $\mathrm{Sz}(C(\omega_1+), 1) > \omega_1$. Then it follows from Proposition 2.1 that $\mathrm{Sz}(C(\omega_1+)) \geq \omega_1.\omega$. On the other hand, the techniques of Lemma 3.3 yield similarly that $\mathrm{Sz}(C(\omega_1+)) \leq \omega_1.\omega$. □

Corollary 3.6. *For any $\omega_1 \leq \alpha < \omega_1.\omega$*

$$Sz(C(\alpha+)) = \omega_1.\omega$$

Proof. For any $\omega_1 \leq \alpha < \omega_1.\omega$, $C(\omega_1+)$ embeds in $C(\alpha+)$ and $C(\alpha+)$ embeds in some finite sum $C(\omega_1+) \oplus ... \oplus C(\omega_1+)$. Then Propositions 2.4 and 3.5 imply that

$$\mathrm{Sz}(C(\omega_1+)) = \omega_1.\omega = \mathrm{Sz}(C(\alpha+)).$$

□

Remark 3.7. Z. Semadeni [21] proved that for $\omega_1 \leq \alpha < \beta < \omega_1.\omega$, $C(\alpha+)$ and $C(\beta+)$ are isomorphic if and only if $\omega_1.n \leq \alpha < \beta < \omega_1.(n+1)$ for some integer n. So, unlike in the separable case, the Szlenk index does not distinguish the isomorphic classes for the non separable $C(\alpha+)$ spaces.

4. Comparing the weak*-dentability index and the Szlenk index

The main result of this section is the following.

Theorem 4.1. *Let X be a Banach space. If $Sz(X) \leq \omega$, then $Dz(X) \leq \omega^2$.*

This estimate is optimal. More precisely

Corollary 4.2. *Let X be a Banach space which is not superreflexive and such that $Sz(X) \leq \omega$. Then $Dz(X) = \omega^2$. In particular $Dz(c_0) = \omega^2$.*

Proof. This a direct consequence of Theorem 4.1, the analogue of Proposition 2.1 for Dz, and the following standard fact: $\mathrm{Dz}(X) \leq \omega$ if and only if X is superreflexive (see [14] or [10]). □

We shall need the following finite dimensional result.

Proposition 4.3. *Let X be a finite dimensional normed space, D be a closed convex subset of X with non empty interior and C be a closed bounded convex subset of X strictly containing D. Then, for any $\delta > 0$, there is a sequence $(H_i)_{i=1}^\infty$ of open half spaces in X such that*

$$C \setminus \bigcup_{i=1}^{\infty} H_i = D \quad \text{and} \quad \forall k \geq 1 \quad \mathrm{diam}[(C \setminus \bigcup_{i=1}^{k} H_i) \cap H_{k+1})] < \delta.$$

Proof. Let μ be the Haar measure on X.

Lemma 4.4. *Let B be a closed bounded convex subset of X such that $D \subsetneq B$. Then for any $\delta > 0$, there is an open half space H in X satisfying*

$$\overline{H} \cap D = \emptyset,\ \mathrm{diam}(B \cap \overline{H}) < \delta \text{ and } \mu(H \cap B) > 0.$$

Proof. Since X is finite dimensional, B is the closed convex hull of its strongly exposed points (see [8] and references therein). So there exists $x \in B \setminus D$ which is strongly exposed in B. Consequently, there is an open half space H in X such that $x \in H$, $\overline{H} \cap D = \emptyset$ and $\mathrm{diam}(\overline{H} \cap B) < \delta$. Since D has non empty interior, so does $H \cap B$. □

End of proof of Proposition 4.3. We set $B_0 = C$ and, using Lemma 4.4, we build by induction a sequence (B_n) such that $B_{n+1} = B_n \setminus H_{n+1}$, where H_{n+1} is an open half space so that

$$\overline{H_{n+1}} \cap D = \emptyset \text{ and } \operatorname{diam}(\overline{H_{n+1}} \cap B_n) < \delta \tag{4.9}$$

and also such that $2\mu(H_{n+1} \cap B_n)$ is greater than the supremum of $\mu(H \cap B_n)$ over all open half spaces H satisfying $\overline{H} \cap D = \emptyset$ and $\operatorname{diam}(\overline{H} \cap B_n) < \delta$.
If this process stops after n steps, then $C \setminus \bigcup_{i=1}^n H_i = D$, and it is enough to set $H_i = H_n$ for all $n \geq i$ to get the desired conclusion. So let us assume that this process does not end. Let $B = \bigcap_{n=0}^\infty B_n$. We only need to show that $B = D$. If not, then Lemma 4.4 insures the existence of an open half space H so that

$$\overline{H} \cap D = \emptyset, \ \operatorname{diam}(B \cap \overline{H}) < \delta \text{ and } \mu(H \cap B) > 0. \tag{4.10}$$

By compactness, we get that for n large enough, $\operatorname{diam}(B_n \cap \overline{H}) < \delta$.
Besides, $\sum_{n=1}^\infty \mu(H_{n+1} \cap B_n) < \infty$. Therefore, for n big enough,

$$2\mu(H_{n+1} \cap B_n) < \mu(H \cap B) \leq \mu(H \cap B_n),$$

which is in contradiction with the way the sequence (H_n) was constructed. So $B = D$.

□

Proof of Theorem 4.1. Since the conditions $\mathrm{Sz}(X) \leq \omega$ and $\mathrm{Dz}(X) \leq \omega^2$ are separably determined (see [15]), we may assume that X is separable. Then we can use a fundamental result of H. Knaust, E. Odell and T. Schlumprecht [12] , which asserts that there is a dual Banach space $Z = Y^*$, so that X^* embeds for the norm and weak* topologies into Y^* and such that Y^* admits a boundedly complete finite dimensional decomposition $(F_n)_{n=1}^\infty$ satisfying the following estimate, for some $p \in [1, +\infty)$: for every block basic sequence $(z_j)_{j=1}^J$, with respect to the finite dimensional decomposition (F_n)

$$\|\sum_{j=1}^J z_j\|^p \geq \sum_{j=1}^J \|z_j\|^p. \tag{4.11}$$

Thus, it is enough to show that $\mathrm{Dz}(Y) \leq \omega^2$.
We denote by $(E_n)_{n=1}^\infty$, the shrinking finite dimensional decomposition of Y, whose dual decomposition is $(F_n)_{n=1}^\infty$. For $N \in \mathbb{N}$, we set $Z_N = F_1 \oplus ... \oplus F_N$ and P_N the projection from X onto Z_N whose kernel is $\oplus \sum_{n=N+1}^\infty F_n$. Let now $\varepsilon > 0$ and $(H_i)_{i=1}^\infty$ be the family of open half spaces given by Proposition 4.3, for $X = Z_N$, $C = B_{Z_N}$, $D = (1-\varepsilon^p)^{\frac{1}{p}} B_{Z_N}$ and some δ in $(0, \varepsilon)$. We denote as before, $B_0 = C$ and $B_k = C \setminus \bigcup_{i=1}^k H_i$, for $k \geq 1$. Then we have the following analogue of Lemma 3.3

Lemma 4.5. *Let $k \in \mathbb{N}$. If $z \in d_{3\varepsilon}^k(B_Z)$ and $\|P_N z\|^p > 1 - \varepsilon^p$, then $P_N z \in B_k$.*

Proof. The proof will be done by induction on k. The statement is clearly true for $k=0$, so assume it is satisfied for some $k \geq 0$. Let $z \in B_Z$ such that $\|P_N z\|^p > 1-\varepsilon^p$ and $P_N z \notin B_{k+1}$. We need to show that $z \notin d_{3\varepsilon}^{k+1}(B_Z)$. So we may assume that $z \in d_{3\varepsilon}^{k}(B_Z)$ and therefore, by induction hypothesis, that $P_N z \in B_k$. Hence, by the proof of Proposition 4.3:

$$P_N z \in H_{k+1} \cap B_k,\ \overline{H_{k+1}} \cap (1-\varepsilon^p)^{\frac{1}{p}} B_{Z_N} = \emptyset, \text{ and } \mathrm{diam}(\overline{H_{k+1}} \cap B_k) < \delta \tag{4.12}$$

The set H_{k+1} can be written $H_{k+1} = \{x \in Z_N,\ f(x) > \alpha\}$, where $\alpha \in \mathbb{R}$ and $f \in Z_N^*$. We can write $f = (f_1, .., f_N)$ in the decomposition $(E_1, .., E_N)$ of $Z_N^* = E_1 \oplus .. \oplus E_N$. Now we define $g = (f_1, .., f_N, 0, .., 0, ..)$ in the decomposition $(E_n)_{n=1}^{\infty}$ of Y and $G_{k+1} = \{x \in Z,\ g(x) > \alpha\}$. Then $z \in G_{k+1} \cap d_{3\varepsilon}^{k}(B_Z)$. Moreover, for any $x \in G_{k+1} \cap d_{3\varepsilon}^{k}(B_Z)$, $P_N x \in H_{k+1}$, so $\|P_N x\|^p > 1-\varepsilon^p$ and it follows from the induction hypothesis that $P_N x \in B_k$. Thus, for all $x, x' \in G_{k+1} \cap d_{3\varepsilon}^{k}(B_Z)$, $\|P_N x - P_N x'\| \leq \delta$. On the other hand, $\|P_N x\|^p > 1-\varepsilon^p$ and $\|P_N x'\|^p > 1-\varepsilon^p$. So it follows from (4.11) that $\|x - P_N x\| < \varepsilon$ and $\|x' - P_N x'\| < \varepsilon$. Therefore $\mathrm{diam}(G_{k+1} \cap d_{3\varepsilon}^{k}(B_Z)) \leq \delta + 2\varepsilon < 3\varepsilon$ and $z \notin d_{3\varepsilon}^{k+1}(B_Z)$.

□

End of proof of Theorem 4.1. It follows now from Lemma 4.5 that

$$\forall \varepsilon > 0 \ \ d_{\varepsilon}^{\omega}(B_Z) \subset (1 - (\frac{\varepsilon}{3})^p)^{\frac{1}{p}} B_Z.$$

Finally, the analogue, for the weak*-dentability index, of Proposition 2.2 yields that $\mathrm{Dz}(Y) \leq \omega^2$.

□

Remark 4.6. If we denote, for a countable ordinal α:

$$\psi(\alpha) = \sup_{\mathrm{Sz}(X) \leq \alpha} \mathrm{Dz}(X).$$

It is clear that $\psi(1) = \omega$ and we just showed that $\psi(\omega) = \omega^2$.
The values of $\psi(\omega^\alpha)$, for $\alpha \geq 2$ are not known. However, there is an uncountable set $S \subset [1, \omega_1)$ such that ψ is the identity on S. The proof of this fact, which relies on the so-called "pressing down Lemma" can be found in [16].

We finish this section by explaining how a general comparison of $\mathrm{Cz}(X)$ and $\mathrm{Dz}(X)$ is given by a recent work of F. García, L. Oncina, J. Orihuela and S. Troyanski [9]. First recall that for a bounded subset B of a metric space X, the Kuratowski index of non compactness of B, denoted by $\alpha(B)$ is defined to be the infimum of all $\varepsilon > 0$ such that B can be covered by a finite union of balls of diameter less than ε. Now, for a Banach space X, we define a new derivation as follows: if K is a weak*-compact and convex subset of X^* and $\varepsilon > 0$, we set $\mathcal{T}$ the set of all weak*-slices S of K so that $\alpha(S) < \varepsilon$. Then

$k_\varepsilon K = K \setminus \cup\{S : S \in \mathcal{T}\}$. Finally, using this derivation, we define in the usual way the indices $\mathrm{Kz}(X, \varepsilon)$ and $\mathrm{Kz}(X)$ that we call *weak***-Kuratowski index of* X. The following result is due to F. García, L. Oncina, J. Orihuela and S. Troyanski [9] (see Proposition 7 and the details of its proof).

Theorem 4.7. *For any Banach space* X*,*

$$Dz(X) \leq \omega^\omega.Kz(X).$$

Then we have

Proposition 4.8. *For any Banach space* X*:*

$$Cz(X) = Kz(X) \ \text{ and therefore } \ Dz(X) \leq \omega^\omega.Cz(X).$$

Proof. It is enough to show that for any $\varepsilon > 0$ and any convex weak*-compact subset K of X^*:

$$c_{4\varepsilon}(K) \subset k_{2\varepsilon}(K) \subset c_\varepsilon(K).$$

Let $x^* \in K \setminus c_\varepsilon(K)$. Since $c_\varepsilon(K)$ is convex and weak*-closed, the Hahn-Banach theorem insures the existence of a weak*-slice S of K such that $x^* \in S$ and $\overline{S}^* \cap c_\varepsilon(K) = \emptyset$. Now, for any $y^* \in \overline{S}^*$, $y^* \in K \setminus c_\varepsilon(K) \subset K \setminus s_\varepsilon(K)$ and therefore, we can pick a weak*-neighborhood V_{y^*} of y^* such that the norm diameter of $V_{y^*} \cap K$ is less than ε. Since $\overline{S}^*$ is weak*-compact, it can be covered by a finite collection $V_{y_1^*}, .., V_{y_n^*}$ and therefore by a finite family of balls of diameter less than 2ε. Thus $x^* \in K \setminus k_{2\varepsilon}(K)$, and the second inclusion is proved.
Let $x^* \in K \setminus k_{2\varepsilon}(K)$. There exist a weak*-open slice S of K and closed balls $B_1, .., B_n$ of X^* with diameter less than 2ε such that $x^* \in S$ and $S \subset \cup_{i=1}^n B_i$. For $y^* \in S$, we set $I = \{i, \ y^* \in B_i\}$. Then $S \setminus \cup_{i \notin I} B_i$ is a weak*-open subset of K containing y^* and included in $\cup_{i \in I} B_i$. Since y^* belongs to all B_i for $i \in I$, the diameter of $\cup_{i \in I} B_i$ is at most 4ε. Thus $y^* \notin s_{4\varepsilon}(K)$ and $s_{4\varepsilon}(K) \subset K \setminus S$. Since $K \setminus S$ is convex and weak*-closed, we also have $c_{4\varepsilon}(K) \subset K \setminus S$, and therefore $x^* \in K \setminus c_{4\varepsilon}(K)$, which ends the proof of the first inclusion.

□

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