ON FUNCTIONAL CALCULUS PROPERTIES OF RITT OPERATORS

FLORENCE LANCIEN AND CHRISTIAN LE MERDY

Abstract. We compare various functional calculus properties of Ritt operators. We show the existence of a Ritt operator $T\colon X\to X$ on some Banach space X with the following property: T has a bounded $\mathcal{H}^\infty$ functional calculus with respect to the unit disc $\mathbb{D}$ (that is, T is polynomially bounded) but T does not have any bounded $\mathcal{H}^\infty$ functional calculus with respect to a Stolz domain of $\mathbb{D}$ with vertex at 1. Also we show that for an R-Ritt operator, the unconditional Ritt condition of Kalton-Portal is equivalent to the existence of a bounded $\mathcal{H}^\infty$ functional calculus with respect to such a Stolz domain.

2000 Mathematics Subject Classification : 47A60.

1. Introduction

Ritt operators on Banach spaces have a specific $\mathcal{H}^\infty$ functional calculus which was formally introduced in [11]. This functional calculus is related to various classical notions playing a role in the harmonic analysis of single operators, such as square functions, maximal inequalities, multipliers and dilation properties, see in particular the above mentioned paper and [1, 2, 12]. The purpose of the present paper is to compare the $\mathcal{H}^\infty$ functional calculus of Ritt operators to two closely related notions, namely polynomial boundedness and the unconditional Ritt condition from [9].

Let $\mathbb{D}=\{z\in\mathbb{C}\,:\,|z|<1\}$ be the open unit disc of the complex field, let X be a (complex) Banach space and recall that a bounded operator $T\colon X\to X$ is called polynomially bounded if there exists a constant $K\geq 0$ such that

$$\|P(T)\|\,\leq K\sup\bigl\{|P(z)|\,:\,z\in\mathbb{D}\bigr\}$$

for any polynomial P. We say that T is a Ritt operator provided that the spectrum of T is included in $\overline{\mathbb{D}}$ and the set

$$\bigl\{(\lambda-1)R(\lambda,T)\;:\;|\lambda|>1\bigr\} \tag{1.1}$$

is bounded. (Here $R(\lambda,T)=(\lambda-T)^{-1}$ denotes the resolvent operator.) For any $\gamma\in\bigl(0,\frac{\pi}{2}\bigr)$, let B_γ be the open Stolz domain defined as the interior of the convex hull of 1 and the disc $D(0,\sin\gamma)$, see Figure 1 below.

It is well-known that the spectrum of any Ritt operator T is included in the closure $\overline{B_\gamma}$ of one of those Stolz domains. Following [11], we say that T has a bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus if there is a constant $K\geq 0$ such that

$$\|P(T)\|\,\leq K\sup\bigl\{|P(z)|\,:\,z\in B_\gamma\bigr\} \tag{1.2}$$

Date: January 22, 2013.

The second named author is supported by the research program ANR 2011 BS01 008 01.

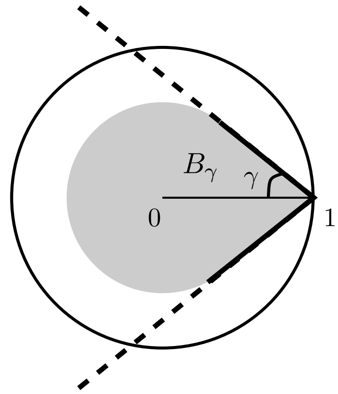

Figure 1.

for any polynomial P. Since $B_\gamma \subset \mathbb{D}$, it is plain that this property implies polynomial boundedness. It was shown in [11] that the converse holds true on Hilbert spaces. Our main result asserts that this does not remain true on all Banach spaces. We will exhibit a Banach space X and a Ritt operator $T\colon X \to X$ which is polynomially bounded but has no bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus. This will be achieved in Section 3 (see Theorem 3.2). This example is obtained by first developing and then exploiting a construction of Kalton concerning sectorial operators [8]. Section 2 is devoted to preliminary results and to the main features of Kalton's example.

Following [9] we say that T satisfies the unconditional Ritt condition if there exists a constant $K \geq 0$ such that

$$\Big\|\sum_{k\geq 1} a_k\big(T^k - T^{k-1}\big)\Big\| \leq K\sup\big\{|a_k| \,:\, k\geq 1\big\} \tag{1.3}$$

for any finite sequence $(a_k)_{k\geq 1}$ of complex numbers. This property is stronger than the Ritt condition [9, Prop. 4.3] and it is easy to check that if T admits a bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus for some $\gamma < \frac{\pi}{2}$, then T satisfies the unconditional Ritt condition (see Lemma 4.1 below). We do not know if the converse holds true. However we will show in Section 4 that if T is R-Ritt and satisfies the unconditional Ritt condition, then it admits a bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus for some $\gamma < \frac{\pi}{2}$. As a consequence we generalize [9, Thm. 4.7] by showing that on a large class of Banach spaces, the unconditional Ritt condition is equivalent to certain square function estimates for R-Ritt operators.

2. Sectorial operators and Kalton's example

Let X be a Banach space and let $A : D(A) \to X$ be a closed operator with dense domain $D(A) \subset X$. We let $\sigma(A)$ denote the spectrum of A and whenever λ belongs to the resolvent set $\mathbb{C} \setminus \sigma(A)$, we let $R(\lambda, A) = (\lambda - A)^{-1}$ denote the corresponding resolvent operator.

For any $\omega \in (0,\pi)$, we let $\Sigma_\omega = \{z \in \mathbb{C}^* \,:\, |\mathrm{Arg}(z)| < \omega\}$. We also set $\Sigma_0 = (0,\infty)$ for convenience. We recall that by definition, A is sectorial if there exists an angle ω such that

$\sigma(A) \subset \overline{\Sigma_\omega}$ and for any $\nu \in (\omega, \pi)$ the set

$$\big\{\lambda R(\lambda, A)\ :\ \lambda \in \mathbb{C} \setminus \overline{\Sigma_\nu}\,\big\} \tag{2.4}$$

is bounded. The smallest $\omega \in [0, \pi)$ with this property is called the sectorialy angle of A.

We will need a few facts about $\mathcal{H}^\infty$ functional calculus for sectorial operators that we now recall. For backgound and complements, we refer the reader to [6, 7, 13].

Let A be a sectorial operator with sectorialy angle $\omega \geq 0$. One can naturally define a bounded operator $F(A)$ for any rational function F with nonpositive degree and poles outside $\sigma(A)$. Let $\phi \geq \omega$. The operator A is said to admit a bounded $\mathcal{H}^\infty(\Sigma_\phi)$ functional calculus if there exists a constant K such that for all functions F as above,

$$\|F(A)\| \leq K \sup\big\{|F(z)|\ :\ z \in \Sigma_\phi\big\}. \tag{2.5}$$

In that case, if μ denotes the infimum of all angles ϕ for which such an estimate holds, then A is said to admit a bounded $\mathcal{H}^\infty$ functional calculus of type μ.

Note that the above definition makes sense even for $\phi = \omega$, which is important for our purpose (see Proposition 2.2 below). If $\phi > \omega$ and A has dense range, it follows from [6, 13] that when the estimate (2.5) holds true on rational functions, then the homomorphism $F \mapsto F(A)$ naturally extends to a bounded operator on $\mathcal{H}^\infty(\Sigma_\phi)$, the Banach algebra of all bounded analytic functions on Σ_ϕ. In particular for $s \in \mathbb{R}$, the image of the function $z \mapsto z^{is}$ under this homomorphism coincides with the classical imaginary power A^{is} of A. These imaginary powers hence satisfy the estimate

$$\|A^{is}\| \leq K e^{\phi|s|}, \qquad s \in \mathbb{R},$$

when (2.5) holds true.

On a Hilbert space, a well known result of McIntosh [13] asserts that if A is a sectorial operator with sectoriality angle ω which admits bounded imaginary powers or a bounded $\mathcal{H}^\infty(\Sigma_\phi)$ functionnal calculus for some $\phi > \omega$, then it has a bounded $\mathcal{H}^\infty(\Sigma_\phi)$ functionnal calculus for any $\phi > \omega$. That is, its $\mathcal{H}^\infty$ functional calculus type coincides with its sectoriality angle.

However on general Banach spaces, this property can fail. Indeed in [8] Kalton constructs, for any $\theta \in (0, \pi)$, a Banach space X_θ and a sectorial operator A on X_θ with sectoriality angle 0, which admits a bounded $\mathcal{H}^\infty$ functional calculus of type θ.

The construction is as follows. On the classical space $L^2(\mathbb{R})$, consider the norms $\|.\|_\theta$ defined by

$$\|f\|_\theta^2 = \int_{\mathbb{R}} e^{-2\theta|\xi|} |\widehat{f}(\xi)|^2\, d\xi. \tag{2.6}$$

Obviously $\|.\|_0$ is the usual L^2-norm and $\|\cdot\|_\theta$ is a smaller norm. For any $\theta \in (0, \pi)$, we let H_θ denote the completion of $L^2(\mathbb{R})$ for the norm $\|\cdot\|_\theta$; this is a Hilbert space.

Let A be the multiplication operator on $L^2(\mathbb{R})$ defined by

$$Af(x) = e^{-x} f(x).$$

In the sequel we will keep the same notation to denote various extensions of A on some spaces containing $L^2(\mathbb{R})$ as a dense subspace. Note that for any $\phi > 0$ and any $F \in \mathcal{H}^\infty(\Sigma_\phi)$, $F(A)$ is the multiplication operator associated to $x \mapsto F(e^{-x})$.

According to [8], A extends to a sectorial operator on H_θ with a bounded $\mathcal{H}^\infty$ functional calculus of type θ. This (non-trivial) fact follows from the following observations. First, for any $f \in L^2(\mathbb{R})$, we have $A^{is}f(x) = e^{-isx}f(x)$, hence

$$\widehat{A^{is}f}(\xi) = \widehat{f}(\xi + s) \tag{2.7}$$

for any s, ξ in $\mathbb{R}$. Second, using the definition of $\|\cdot\|_\theta$, this implies that

$$\|A^{is}\|_{H_\theta \to H_\theta} = e^{\theta|s|}, \qquad s \in \mathbb{R}. \tag{2.8}$$

This equality implies, by the above mentioned result of McIntosh, that the operator A on H_θ admits a bounded $\mathcal{H}^\infty(\Sigma_\phi)$ functional calculus for all $\phi > \theta$.

The next step is to construct a new completion X_θ of $L^2(\mathbb{R})$ on which A has similar $\mathcal{H}^\infty$ functional calculus properties but a 'better' sectoriality angle. We will point out some important elements of this construction. Consider a new norm on $L^2(\mathbb{R})$ by letting

$$\|f\|_{X_\theta} = \sup_{a\in\mathbb{R}} \|f\chi_{(-\infty,a)}\|_\theta. \tag{2.9}$$

Then let X_θ be the completion of $L^2(\mathbb{R})$ for this norm. Clearly for any $f \in L^2(\mathbb{R})$, we have

$$\|f\|_\theta \leq \|f\|_{X_\theta} \leq \|f\|_0.$$

Thus $L^2(\mathbb{R}) \subset X_\theta \subset H_\theta$ with contractive embeddings. Note that contrary to H_θ, X_θ is not a Hilbert space. Again A extends to a sectorial operator on X_θ. A key fact is that on this new space, the sectoriality angle of A is equal to 0. This is a consequence of the following computation. For any $f \in L^2(\mathbb{R})$ and any $\lambda \in \mathbb{C} \setminus \mathbb{R}_+$,

$$(\lambda - e^{-x})^{-1}f(x) = \int_{\mathbb{R}} \frac{\lambda e^{-t}}{(\lambda - e^{-t})^2}\, f(x)\, \chi_{(-\infty,t)}(x)\, dt \tag{2.10}$$

for any $x \in \mathbb{R}$. If we let $\psi = \arg\lambda$, this implies

$$\|\lambda R(\lambda, A)f\|_\theta \leq \|f\|_{X_\theta} \int_0^\infty |s - e^{i\psi}|^{-2}\, ds.$$

Applying this with $f\chi_{(-\infty,a)}$ instead of f, we deduce a uniform estimate $\|\lambda R(\lambda, A)\|_{X_\theta \to X_\theta} \leq K_\psi$, which yields the desired sectoriality property.

If $m \in L^\infty(\mathbb{R})$ is such that the multiplication operator $f \mapsto mf$ is bounded on H_θ with norm less than C_m, then the same holds true on X_θ, since

$$\|mf\|_{X_\theta} = \sup_{a\in\mathbb{R}} \|mf\chi_{(-\infty,a)}\|_\theta \leq C_m\|f\|_{X_\theta}.$$

Since $F(A)$ is such a multiplication operator for any $F \in \mathcal{H}^\infty(\Sigma_\phi)$, we derive the following.

Lemma 2.1. *If A admits a bounded $\mathcal{H}^\infty(\Sigma_\phi)$ functional calculus on H_θ, then it admits a bounded $\mathcal{H}^\infty(\Sigma_\phi)$ functional calculus on X_θ as well.*

Finally, and this is the most difficult part of [8], it turns out that the imaginary powers of A have the same norms on X_θ and on H_θ, namely

$$\|A^{is}\|_{X_\theta \to X_\theta} = \|A^{is}\|_{H_\theta \to H_\theta} = e^{\theta|s|} \tag{2.11}$$

for any $s \in \mathbb{R}$. Combining with Lemma 2.1, this implies that on X_θ, the operator A admits a bounded $\mathcal{H}^\infty(\Sigma_\phi)$ functional calculus for any $\phi > \theta$ but cannot have a bounded $\mathcal{H}^\infty(\Sigma_\phi)$ functional calculus for some $\phi < \theta$.

We finally consider the case $\phi = \theta$, which is not treated in [8] but is important for our purpose. This requires a new ingredient, namely the next statement which is implicit in [11].

Proposition 2.2. *Let A be a sectorial operator with dense range on some Hilbert space H, assume that A admits bounded imaginary powers and that for some $\theta \in (0, \pi)$, they satisfy an exact estimate $\|A^{is}\| \leq e^{\theta|s|}$ for any $s \in \mathbb{R}$. Then A has a bounded $\mathcal{H}^\infty(\Sigma_\theta)$ functional calculus.*

Proof. Let iU be the generator of the c_0-semigroup $(A^{is})_{s\geq 0}$. Our assumption ensures that it both satisfies

$$\|e^{s(iU-\theta)}\| \leq 1 \qquad \text{and} \qquad \|e^{s(-iU-\theta)}\| \leq 1$$

for any $s \geq 0$. This means that $iU - \theta$ and $-iU - \theta$ both generate contractive semigroups on H. Thus for all $h \in D(U)$, one has

$$\mathrm{Re}\big\langle(\theta + iU)h, h\big\rangle \geq 0 \qquad \text{and} \qquad \mathrm{Re}\big\langle(\theta - iU)h, h\big\rangle \geq 0.$$

Hence the numerical range of U lies in the closed band $\Omega = \{z \in \mathbb{C} : |\mathrm{Im} z| \leq \theta\}$. By [5, Thm. 1], this implies the existence of a constant $K > 0$ such that

$$\|G(U)\| \leq K \sup\big\{|G(w)| : w \in \Omega\big\} \tag{2.12}$$

for any rational function G bounded on Ω. The argument in [5] can be extended to more general functions. It is observed in [11] that in particular, it applies to all functions G of the form $G(w) = F(e^w)$, where F is a rational function with negative degree and poles off $\overline{\Sigma_\theta}$ and in this case, $G(U) = F(A)$. In this situation, $\sup\{|G(w)| : w \in \Omega\}$ coincides with $\sup\{|F(z)| : z \in \Sigma_\theta\}$. Hence we deduce from (2.12) that A admits a bounded $\mathcal{H}^\infty(\Sigma_\theta)$ functional calculus. □

According to (2.8), the above proposition applies to Kalton's operator A on H_θ. Hence the latter admits a bounded $\mathcal{H}^\infty(\Sigma_\theta)$ functional calculus. Applying Lemma 2.1, we deduce that the operator A constructed above on X_θ has a bounded $\mathcal{H}^\infty(\Sigma_\phi)$ functional calculus for all $\phi \geq \theta$ (not only for $\phi > \theta$).

3. Main result

Our main purpose is to prove Theorem 3.2 below. We first need to modify Kalton's example discussed in the previous section. Roughly speaking we need a similar example with the additional property that the the operator should be bounded. We will get a more precise result.

We consider the restriction B of A on $L^2(\mathbb{R}_+)$. More explicitly, $B\colon L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is the bounded operator defined by

$$Bf(x) = e^{-x} f(x), \qquad f \in L^2(\mathbb{R}_+).$$

Then we let H_θ^+ be the completion of $L^2(\mathbb{R}_+)$ for the norm $\|\cdot\|_\theta$ defined by (2.6), we let X_θ^+ be the completion of $L^2(\mathbb{R}_+)$ for the norm $\|\cdot\|_{X_\theta}$ defined by (2.9) and we consider extensions of B to those spaces, as was done in Section 2. Of course X_θ^+ is a closed subspace of X_θ and the operator B on X_θ^+ is the restriction of the operator A on X_θ. Thus for any $\phi \in (0,\pi)$ and any appropriate $F \in \mathcal{H}^\infty(\Sigma_\phi)$, we have $F(B) = F(A)_{|X_\theta^+ \to X_\theta^+}$, and hence

$$\|F(B)\|_{X_\theta^+ \to X_\theta^+} \leq \|F(A)\|_{X_\theta \to X_\theta}. \tag{3.13}$$

Similar comments apply for H_θ and H_θ^+.

Proposition 3.1. *On the Banach space X_θ^+, the operator B is sectorial, its sectoriality angle is equal to 0, its spectrum $\sigma(B)$ lies in $[0,1]$, it admits a bounded $\mathcal{H}^\infty(\Sigma_\phi)$ functional calculus for all $\phi \geq \theta$, and*

$$\|B^{is}\|_{X_\theta^+ \to X_\theta^+} = e^{\theta|s|}, \qquad s \in \mathbb{R}. \tag{3.14}$$

Proof. It is clear from (3.13) and results established for A in Section 2 that on X_θ^+, B is sectorial with a sectoriality angle equal to 0, and it admits a bounded $\mathcal{H}^\infty(\Sigma_\phi)$ functional calculus for all $\phi \geq \theta$.

To show the spectral inclusion $\sigma(B) \subset [0,1]$, consider $\lambda \in \mathbb{C} \setminus [0,1]$. As in (2.10), we have

$$(\lambda - e^{-x})^{-1} f(x) = \int_0^\infty \frac{e^{-t}}{(\lambda - e^{-t})^2}\, f(x)\, \chi_{(-\infty,t)}(x)\, dt$$

for any $f \in L^2(\mathbb{R}_+)$ and any $x \geq 0$. Note that contrary to (2.10), integration is now taken on $(0,\infty)$. We can therefore deduce that

$$\|(\lambda - B)^{-1} f\|_{X_\theta} \leq \|f\|_{X_\theta} \int_0^\infty \frac{e^{-t}}{|\lambda - e^{-t}|^2}\, dt$$

for any $f \in L^2(\mathbb{R}_+)$, which ensures that $\lambda - B$ is invertible on X_θ^+.

It remains to prove (3.14). We will establish it by appealing to (2.11) and by showing that for any $s \in \mathbb{R}$,

$$\|B^{is}\|_{X_\theta^+ \to X_\theta^+} = \|A^{is}\|_{X_\theta \to X_\theta}.$$

Let us start with a simple observation. Let τ_a denote the translation operator defined by $\tau_a f(x) = f(x-a)$. Then for any $f \in L^2(\mathbb{R})$ and for any $a \in \mathbb{R}$, we have $\widehat{\tau_a f}(\xi) = e^{-ia\xi} \widehat{f}(\xi)$ for any $\xi \in \mathbb{R}$. Looking at the definition (2.6), we deduce that

$$\|\tau_a f\|_\theta = \|f\|_\theta. \tag{3.15}$$

For any $t \in \mathbb{R}$, we have $\chi_{(-\infty,t)} \tau_a f = \tau_a\big(\chi_{(-\infty,t-a)} f\big)$ hence we immediately deduce that

$$\|\tau_a f\|_{X_\theta} = \|f\|_{X_\theta}. \tag{3.16}$$

Now take a function f in $L^2(\mathbb{R})$ with bounded support included in some compact interval $[-M, M]$. Given any $t \in \mathbb{R}$, we have

$$\begin{aligned} \|\chi_{(-\infty,t)} A^{is} f\|_\theta &= \|\tau_M(\chi_{(-\infty,t)} A^{is} f)\|_\theta \\ &= \|\chi_{(-\infty,t+M)} \tau_M(A^{is} f)\|_\theta \\ &\leq \|\tau_M(A^{is} f)\|_{X_\theta} \end{aligned}$$

by (3.15). Further, $A^{is} f(x) = e^{-isx} f(x)$ hence $\big[\tau_M(A^{is} f)\big](x) = e^{isM} A^{is}(\tau_M f)(x)$ for any real x. Thus

$$\|\tau_M(A^{is} f)\|_{X_\theta} = \|A^{is}(\tau_M f))\|_{X_\theta}.$$

Since $\tau_M f$ has support in $\mathbb{R}_+$, we derive that

$$\|\tau_M(A^{is}f)\|_{X_\theta} \leq \|B^{is}\|_{X_\theta^+ \to X_\theta^+} \|\tau_M f\|_{X_\theta}.$$

According to (3.16) and the preceding inequalities, we deduce that

$$\|\chi_{(-\infty,t)} A^{is} f\|_\theta \leq \|B^{is}\|_{X_\theta^+ \to X_\theta^+} \|f\|_{X_\theta}.$$

Taking the supremum over $t \in \mathbb{R}$, one obtains $\|A^{is}f\|_{X_\theta} \leq \|B^{is}\|_{X_\theta^+ \to X_\theta^+} \|f\|_{X_\theta}$. Hence

$$\|A^{is}\|_{X_\theta \to X_\theta} \leq \|B^{is}\|_{X_\theta^+ \to X_\theta^+}.$$

The reverse inequality is clear, see (3.13). □

We now turn to Ritt operators. Recall the definition of a bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus from Section 1 (see also [11]).

Theorem 3.2. *There exists a Ritt operator T on a Banach space X which is polynomially bounded but admits no bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus for any $\gamma < \frac{\pi}{2}$.*

Proof. We take for X the Banach space $X_{\frac{\pi}{2}}^+$ considered above and we let $B\colon X \to X$ be the operator considered in Proposition 3.1. Then we let

$$T = (I - B)(I + B)^{-1}.$$

We note that $z \mapsto \frac{1-z}{1+z}$ maps $\Sigma_{\frac{\pi}{2}}$ onto $\mathbb{D}$ and $[0,1]$ into itself. Thus

$$\sigma(T) \subset [0,1].$$

To show that T is a Ritt operator, we consider $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. One can write $\lambda = \frac{1-z}{1+z}$ with $z \notin \overline{\Sigma_{\frac{\pi}{2}}}$. It is easy to check that

$$(\lambda - 1)(\lambda - T)^{-1} = z(z - B)^{-1}(I + B).$$

Since the sectorial angle of B is 0, the set $\big\{z(z-B)^{-1} \,:\, z \notin \overline{\Sigma_{\frac{\pi}{2}}}\big\}$ is bounded. Since B is bounded, we deduce that the set defined in (1.1) is bounded.

The fact that B has a bounded $\mathcal{H}^\infty(\Sigma_{\frac{\pi}{2}})$ functional calculus on X implies that T is polynomially bounded. Indeed if P is a polynomial, then $P(T) = F(B)$ for the rational function F defined by $F(z) = P\big(\frac{1-z}{1+z}\big)$. Hence for some constant K, we have

$$\|P(T)\| = \|f(B)\| \leq K \sup\big\{|F(z)| : z \in \Sigma_{\frac{\pi}{2}}\big\},$$

and moreover,

$$\sup\big\{|F(z)| : z \in \Sigma_{\frac{\pi}{2}}\big\} = \sup\big\{|P(w)| : w \in \mathbb{D}\big\}.$$

Now assume that T has a bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus for some $\gamma < \frac{\pi}{2}$. Consider the auxiliary operator

$$C = I - T = 2B(I + B)^{-1}.$$

By [11, Prop. 4.1], C is a sectorial operator which admits a bounded $\mathcal{H}^\infty(\Sigma_\theta)$ for some $\theta \in (0, \frac{\pi}{2})$. Thus there exists a constant $K > 0$ such that

$$\|C^{is}\| \leq K e^{\theta|s|}, \qquad s \in \mathbb{R}.$$

Further $\sigma(I+B)\subset[1,2]$. Thus $I+B$ is bounded and invertible and hence it admits a bounded $\mathcal{H}^\infty$ functional calculus of any type. Thus for any $\theta'>0$. there exists $K'>0$ such that

$$\|(I+B)^{is}\|\leq K'e^{\theta'|s|}.$$

Since B and C commute, we have

$$B^{is}=2^{-is}C^{is}(I+B)^{is},$$

hence

$$\|B^{is}\|\leq KK^{'}e^{(\theta+\theta')|s|}$$

for any $s\in\mathbb{R}$. Applying this with θ' small enough so that $\theta+\theta'<\frac{\pi}{2}$, this contradicts (3.14) on $X^{+}_{\frac{\pi}{2}}$. ☐

Remark 3.3. A Ritt operator T on Banach space X is called R-Ritt if the bounded set in (1.1) is actually R-bounded. That notion was introduced in [3], in relation with the study of discrete maximal regularity, see also [4, 9, 11, 14]. Background and references on R-boundedness can also be found in the latter references.

The existence of Ritt operators which are not R-Ritt goes back to Portal [14]. According to [11, Prop. 7.6], a polynomially bounded R-Ritt operator has a bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus for some $\gamma<\frac{\pi}{2}$. Thus the operator T constructed in Theorem 3.2 is a Ritt operator which is not R-Ritt. This example is of a different nature than the ones from [14].

4. Unconditional Ritt operators

We now investigate the links between the unconditional Ritt condition and the $\mathcal{H}^\infty$ functional calculus. It is observed in [9] that the unconditional Ritt condition (1.3) is equivalent to the existence of a constant $K>0$ such that

$$\sum_{k\geq1}\bigl|\bigl\langle\bigl(T^k-T^{k-1}\bigr)x,y\bigr\rangle\bigr|\,\leq\,K\|x\|\|y\|,\qquad x\in X,y\in X^*. \tag{4.17}$$

Moreover it is stronger than the Ritt condition. We will now show that the unconditional Ritt condition is weaker than the existence of a bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus for some $\gamma<\frac{\pi}{2}$.

Lemma 4.1. *If T admits a bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus for some $\gamma<\frac{\pi}{2}$, then T satisfies the unconditional Ritt condition.*

Proof. Assume that T admits a bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus for some $\gamma<\frac{\pi}{2}$. Consider a finite sequence $(a_k)_{k\geq1}$. Since

$$\sum_{k\geq1}a_k\bigl(T^k-T^{k-1}\bigr)=P(T)$$

for the polynomial P defined by $P(z)=\sum_{k\geq1}a_k\bigl(z^k-z^{k-1}\bigr)$, (1.2) implies that

$$\Bigl\|\sum_{k\geq1}a_k\bigl(T^k-T^{k-1}\bigr)\Bigr\|\leq K\sup\bigl\{|P(z)|\,:\,z\in B_\gamma\bigr\}.$$

Now we have

$$|P(z)| \le \sup_{k\ge 1} |a_k| \sum_{k\ge 1} |z^k - z^{k-1}| = \sup_{k\ge 1} |a_k| \left(\frac{|z-1|}{1-|z|}\right).$$

Since $z \mapsto \frac{|z-1|}{1-|z|}$ is bounded on B_γ, this implies the unconditional Ritt condition (1.3). □

We now show a partial converse. See Remark 3.3 for the notion of R-Ritt operator. We will use the companion notion of R-sectorial operator. We recall that a sectorial operator A on Banach space is called R-sectorial if there exists an angle ω such that $\sigma(A) \subset \overline{\Sigma_\omega}$ and for any $\nu \in (\omega, \pi)$ the set (2.4) is R-bounded. In accordance with terminology in Section 2, the smallest $\omega \in [0, \pi)$ with this property will be called the R-sectorialy angle of A. We refer the reader to [3, 4, 10, 11] and the references therein for information on R-sectoriality.

Theorem 4.2. *Let T be an R-Ritt operator which satisfies the unconditional Ritt condition, then it admits a bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus for some $\gamma < \frac{\pi}{2}$.*

Proof. We consider the operator

$$C = I - T.$$

According to [3, Thm. 1.1] and its proof, the assumption that T is R-Ritt implies that C is R-sectorial, with an R-sectoriality angle $< \frac{\pi}{2}$. On the other hand the unconditional Ritt condition (1.3) for T implies the so-called L_1-condition for C :

$$\int_0^\infty \bigl|\bigl\langle Ce^{-tC}x, y\bigr\rangle\bigr| \,\frac{dt}{t} \le K\|x\|\|y\|, \qquad x \in X, y \in X^*.$$

Indeed for any $t > 0$,

$$Ce^{-tC} = (I-T)e^{-t}e^{tT} = \sum_{n\ge 0} (I-T)e^{-t}\,\frac{t^n T^n}{n!}.$$

Thus for any $x \in X$ and $y \in X^*$, we have

$$\bigl\langle Ce^{-tC}x, y\bigr\rangle = \sum_{n\ge 0} e^{-t}\,\frac{t^n}{n!}\,\bigl\langle (I-T)T^n x, y\bigr\rangle.$$

This implies, using (4.17), that

$$\begin{aligned}\int_0^\infty \bigl|\bigl\langle Ce^{-tC}x, y\bigr\rangle\bigr| \,\frac{dt}{t} &\le \sum_{n\ge 0} \frac{1}{n!} \int_0^\infty \bigl|\bigl\langle (I-T)T^n x, y\bigr|\bigr\rangle\, e^{-t}t^{n-1}\,dt \\ &= \sum_{n\ge 0} \bigl|\bigl\langle (I-T)T^n x, y\bigr\rangle\bigr| \\ &\le K\|x\|\|y\|.\end{aligned}$$

Now by results of [6, Section 4], the L_1-condition implies that C admits a bounded $\mathcal{H}^\infty(\Sigma_\theta)$ functional calculus for all $\theta > \frac{\pi}{2}$. Since C is R-sectorial with an R-sectoriality angle $< \frac{\pi}{2}$, it follows from [10, Prop. 5.1] that C actually admits a bounded $\mathcal{H}^\infty(\Sigma_\theta)$ functional calculus for some $\theta < \frac{\pi}{2}$. By [11, Prop. 4.1], this is equivalent to the fact that T has a bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus for some $\gamma < \frac{\pi}{2}$. □

It is shown in [9, Thm. 4.7] that when X is a Hilbert space, the unconditional Ritt condition is equivalent to certain square function estimates. We can now extend that result to L^p-spaces. In the next statement, we let $p' = p/(p-1)$ denote the conjugate number of p.

Corollary 4.3. *Let Ω be a measure space, let $1 < p < \infty$ and let $T\colon L^p(\Omega) \to L^p(\Omega)$ be a power bounded operator. The following assertions are equivalent.*

(i) *T is R-Ritt and satisfies the unconditional Ritt condition.*

(ii) *There exists a constant $C > 0$ such that*

$$\Big\| \Big(\sum_{k=1}^{\infty} k \left| T^k(x) - T^{k-1}(x) \right|^2 \Big)^{\frac{1}{2}} \Big\|_p \leq C \|x\| \tag{4.18}$$

for any $x \in L^p(\Omega)$ and

$$\Big\| \Big(\sum_{k=1}^{\infty} k \left| T^{*k}(y) - T^{*(k-1)}(y) \right|^2 \Big)^{\frac{1}{2}} \Big\|_{p'} \leq C \|y\| \tag{4.19}$$

for any $y \in L^{p'}(\Omega)$.

Proof. If the square function estimates in (ii) hold true, then T is an R-Ritt operator by [11, Thm. 5.3]. Further T has a bounded $\mathcal{H}^\infty(B_\gamma)$ functional calculus for some $\gamma < \frac{\pi}{2}$, by [11, Thm. 1.1]. Hence Lemma 4.1 ensures that T satisfies the unconditional Ritt condition. The converse assertion that (i) implies (ii) is obtained by combining Theorem 4.2 and [11, Thm. 1.1]. □

It is clear from [11] that Corollary 4.3 holds as well on reflexive Banach lattices with finite cotype. Further generalizations hold true on more Banach spaces, using the abstract square functions introduced and discussed in [11], to which we refer for more information. Combining the results from that paper with Theorem 4.2, one obtains that when X has finite cotype and $T\colon X \to X$ is an R-Ritt operator, then T satisfies the unconditional Ritt condition if and only if T and T^* admit square function estimates.

References

[1] C. Arhancet, *Unconditionality, Fourier multipliers and Schur multipliers*, Colloquium Mathematicum 127, 17-37 (2012).

[2] C. Arhancet, and C. Le Merdy, *Dilation of Ritt operators on L^p-spaces*, Preprint 2011, arXiv:1106.1513.

[3] S. Blunck, *Maximal regularity of discrete and continuous time evolution equations*, Studia Math. 146 (2001), no. 2, 157-176.

[4] S. Blunck, *Analyticity and discrete maximal regularity on L_p-spaces*, J. Funct. Anal. 183 (2001), 211-230.

[5] M. Crouzeix, B. Delyon, *Some estimates for analytic functions of band or sectorial operators*, Arch. Math. 81 (2003), 559-566.

[6] M. Cowling, I. Doust, A. McIntosh, and A. Yagi, *Banach space operators with a bounded H^∞ functional calculus*, J. Aust. Math. Soc., Ser. A 60 (1996), 51-89.

[7] M. Haase, *The functional calculus for sectorial operators*, Operator Theory: Advances and Applications, 169, Birkhäuser Verlag, Basel, 2006. xiv+392 pp.

[8] N. J. Kalton, *A remark on sectorial operators with an H^∞-calculus*, pp. 91-99 in "Trends in Banach spaces and operator theory", Contemp. Math. 321, Amer. Math. Soc., Providence, RI, 2003.

[9] N. J. Kalton, and P. Portal, *Remarks on ℓ_1 and ℓ_∞-maximal regularity for power bounded operators*, J. Aust. Math. Soc. 84 (2008), 345-365.

[10] N. J. Kalton, and L. Weis, *The H^∞ functional calculus and sums of closed operators*, Math. Ann. 321 (2001), 319-345.

[11] C. Le Merdy, *H^∞ functional calculus and square function estimates for Ritt operators*, Preprint 2011, arXiv:1202.0768.

[12] C. Le Merdy, and Q. Xu, *Maximal theorems and square functions for analytic operators on L^p-spaces*, J. London Math. Soc., to appear (arXiv:1011.1360).

[13] A. McIntosh, *Operators which have a bounded H^∞ functional calculus*, Proc. CMA Canberra 14 (1986), 210-231.

[14] P. Portal, *Discrete time analytic semigroups and the geometry of Banach spaces*, Semigroup Forum 67 (2003), 125-144.

Université de Franche-Comté, Laboratoire de Mathématiques UMR 6623, 16 route de Gray, 25030 Besançon Cedex, FRANCE.

E-mail address: florence.lancien@univ-fcomte.fr

Université de Franche-Comté, Laboratoire de Mathématiques UMR 6623, 16 route de Gray, 25030 Besançon Cedex, FRANCE.

E-mail address: christian.lemerdy@univ-fcomte.fr