Exact controllability of a string submitted to a unilateral constraint

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Abstract

We consider the null controllability of a homogeneous linear string of length one submitted to a lower time dependent obstacle $\{\psi(t)\}_{(0\leq t\leq T)}$ at the extremity $x=1$. The Dirichlet control acts on the other extremity $x=0$. The string is modelled by the wave equation $y''-y_{xx}=0$ in $(t,x)\in(0,T)\times(0,1)$ while the obstacle is modelled by the Signorini's conditions $y(t,1)\geq\psi(t), y_x(t,1)\geq 0, y_x(t,1)(y(t,1)-\psi(t))=0$ in $(0,T)$. In the framework of the characteristic method which reduces the study to the analysis on $(0,T)\times\{1\}$, we show that the controllability of any initial condition (y^0,y^1) in a subset of $H^1(0,1)\times L^2(0,1)$ holds for any $T>2$. Two distinct approachs are used. We first introduce a penalized system in y_ϵ, $\epsilon>0$ transforming the Signorini's condition into the simpler one $y_{\epsilon,x}(t,1)=\epsilon^{-1}[y_\epsilon(t,1)-\psi(t)]$ and then constructs explicitly a family of controls $\{u_\epsilon\}_{\epsilon>0}$ in $H^1(0,T)$, uniformly bounded with respect to ϵ. A more direct approach based on differential inequation theory then leads to a similar positive conclusion. Numerical experiments complete the study.

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1 Introduction

Let $T>0$ and $Q_T=(0,T)\times(0,1)$. We consider the following system

$$\begin{cases} y''-y_{xx}=0, & (t,x)\in Q_T, \\ y(t,0)=u(t), & t\in(0,T), \\ y(t,1)\geq\psi(t),\ y_x(t,1)\geq 0,\ (y(t,1)-\psi(t))y_x(t,1)=0, & t\in(0,T), \\ y(0,x)=y^0(x), \quad y'(0,x)=y^1(x), & x\in(0,1) \end{cases} \tag{1.1}$$

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where the symbol $'$ denotes the derivative with respect to the variable t. System (1.1) models the vibration of a homogeneous and linear string of length one in the time interval $(0,T)$ submitted to an initial excitation (y^0,y^1) at time $t=0$. On the left extremity $x=0$ acts a control function $u(t)$, whereas on the right extremity $x=1$, the string is limited by a lower time dependent obstacle so that $y(1,t)\geq\psi(t)$ for all $t>0$. When the rod touches the obstacle, its reaction can be only upward, so that $y_x(1,t)\geq 0$ on the set $\{t\,:\,y(1,t)=\psi(t)\}$. When the rod does not touch the obstacle, the right end is free so that $y_x(1,t)=0$ on the set $\{t\,:\,y(1,t)>\psi(t)\}$. These usual conditions which permit to describe the presence of the obstacle are called *unilateral Signorini conditions* (see for instance [3]).

Various papers have been devoted to the existence and uniqueness of a solution of the boundary obstacle problem for the wave equation. Among all of these, we mention [4, 5] (whose idea is used in our present work) and [11].

We investigate in this work the exact boundary controllability of the non linear system (1.1) stated as follows: for any T fixed large enough and any (y^0,y^1) in a given space, does there exist a Dirichlet control $u\in H^1(0,T)$ which drives the corresponding solution of (1.1) to rest, i.e.

$$y(T)=y'(T)=0,\quad \text{in}\quad (0,1)? \tag{1.2}$$

More precisely, we will prove the following:

Theorem 1.1 *Let $T\in(2,3)$. For any $\left(y^0,y^1\right)\in H^1(0,1)\times L^2(0,1)$ and $\psi\in H^1(0,T)$ with*

$$\psi(0)\leq y^0(1),\quad \psi(T)\leq 0, \tag{1.3}$$

there exists $u\in H^1(0,T)$ such that (1.1) admits a unique solution $y\in C\left([0,T],H^1(0,1)\right)\cap C^1\left([0,T],L^2(0,1)\right)$ satisfying $y(T)=y'(T)=0$ in $(0,1)$.

To our knowledge, the study of the exact controllability when a unilateral constraint is involved has not been studied so far. In the different context of stabilization, we mention the contribution [8] where the authors prove the exponential decay of the energy associated with the solution of a damped wave equation submitted to a boundary obstacle.

We address this nonlinear controllability problem, in a constructive way, using the characteristic method. This permits to compute the behavior of the solution ϕ of the wave equation submitted to the initial condition (ϕ^0,ϕ^1) and the boundary $\phi(t,0)=u(t)$, $\phi(t,1)=f(t)$ for any $u,f\in L^2(0,T)$ and then to compute explicitly the Dirichlet-to-Neumann map A defined by $A(\phi^0,\phi^0,u,f)=\phi_x(\cdot,1)$ (see Section 2). At the right extremity $x=1$ of the string, the Signorini conditions then become the following ordinary differential inequations in $(0,T)$

$$\begin{cases} f-\psi\geq 0, & t\in(0,T)\\ A_c(\phi^0,\phi^1,u,f)\geq 0, & t\in(0,T)\\ (f-\psi)A_c(\phi^0,\phi^1,u,f)=0, & t\in(0,T)\end{cases} \tag{1.4}$$

The idea is then to find a control $u=u(f,\phi^0,\phi^1)$ for ϕ such that $\phi(\cdot,1)=f$ and f solution of (1.4). u is then a control for y solution of (1.1). In section 4, by the way of general results for differential inequation, we describe a class of such controls $u\in H^1(0,T)$, assuming that T is strictly greater than 2. In Section 3, we obtain alternatively the controllability result using a penalty method, classical in contact mechanics, which consists in relaxing the Signorini inequations by the equation $y_{\epsilon,x}(\cdot,1)=\epsilon^{-1}[y_\epsilon(\cdot,1)-\psi]^-$ in $(0,T)$. $\epsilon>0$ denotes the penalized parameter. Following the previous fixed point argument, we then construct a class of couple (u_ϵ,f_ϵ) solution of

$$A_c(\phi^0,\phi^1,u_\epsilon,f_\epsilon)=\epsilon^{-1}[f_\epsilon-\psi]^{-1},\quad t\in(0,T) \tag{1.5}$$

uniformly bounded with respect to ϵ (Section 3.1). This property then permits to the pass to the limit and obtain a control for (1.1). Section 5 presents some numerical applications in aggreement with the theoretical part while Section 6 concludes with some related extensions and open problems.

2 The Control Dirichlet-to-Neumann map of a linear system

Let $T>0$, $Q_T=(0,T)\times(0,1)$ and consider the following system:

$$\begin{cases} \phi''-\phi_{xx}=0, & (t,x)\in Q_T,\\ \phi(t,0)=u(t), & t\in(0,T),\\ \phi(t,1)=f(t), & t\in(0,T),\\ \phi(0,x)=\phi^0(x),\quad \phi'(0,x)=\phi^1(x), & x\in(0,1)\end{cases} \tag{2.6}$$

where $u\in L^2(0,T)$ is a control function and $f\in L^2(0,T)$ is given. The following result is well-known (see, for instance, [6]).

Proposition 2.1 • *If* $\left(\left(\phi^0,\phi^1\right),(u,f)\right)\in\left(L^2(0,1)\times H^{-1}(0,1)\right)\times L^2(0,T)^2$ *then there exists a unique solution* ϕ *of (2.6) such that* $\phi\in C\left([0,T],L^2(0,1)\right)\cap C^1\left([0,T],H^{-1}(0,1)\right)$ *and a positive constant* C *such that*

$$\|(\phi(t),\phi'(t))\|_{L^2(0,1)\times H^{-1}(0,1)}\leq C\left(\left\|\left(\phi^0,\phi^1\right)\right\|_{L^2(0,1)\times H^{-1}(0,1)}+\|(u,f)\|_{L^2(0,T)^2}\right).$$

- *If*

$$\left(\left(\phi^0,\phi^1\right),(u,f)\right)\in H^1(0,1)\times L^2(0,1)\times H^1(0,T)^2 \tag{2.7}$$

with the compatibility conditions

$$u(0)=\phi^0(0),\ \ f(0)=\phi^0(1) \tag{2.8}$$

then there exists a unique solution ϕ *of (2.6) such that*

$$\phi\in C\left([0,T],H^1(0,1)\right)\cap C^1\left([0,T],L^2(0,1)\right).$$

and

$$\|(\phi(t),\phi'(t))\|_{H^1(0,1)\times L^2(0,1)}\leq C\left(\left\|\left(\phi^0,\phi^1\right)\right\|_{H^1(0,1)\times L^2(0,1)}+\|(u,f)\|_{H^1(0,T)^2}\right).$$

In this paper, we work with the space

$$\mathbb{H}=\left\{\left(\left(\phi^0,\phi^1\right),(u,f)\right)\in H^1(0,1)\times L^2(0,1)\times H^1(0,T)^2,\ u(0)=\phi^0(0),\ f(0)=\phi^0(1)\right\}.$$

Given $\left(\phi^0,\phi^1,f\right)$, our aim is to find a family of explicit controls u for which the solution ϕ of (2.6) satisfies $\phi(T)=\phi'(T)=0$ in $(0,1)$. Setting

$$p=\phi'-\phi_x,\ \ q=\phi'+\phi_x, \tag{2.9}$$

leads to the hyperbolic linear system

$$\begin{cases} p'+p_x=q'-q_x=0, & (t,x)\in Q_T,\\ (p+q)(\cdot,0)=2u', & t\in(0,T),\\ (p+q)(\cdot,1)=2f' & t\in(0,T),\\ p^0=\phi^1-\phi^0_x,\ q^0=\phi^1+\phi^0_x, & x\in(0,1).\end{cases} \tag{2.10}$$

If $((p^0,q^0),(u,f)) \in L^2(0,1)^2 \times H^1(0,T)^2$ system (2.10) admits a unique generalized solution $(p,q) \in C([0,T], L^2(0,1)^2)$ (see for instance [9, Theorem 3.1, p. 650]). In view of (2.9), this solution corresponds to a solution ϕ of (2.6) satisfying

$$\phi \in C([0,T], H^1(0,1)) \cap C^1([0,T], L^2(0,1))$$

associated with data $((\phi^0,\phi^1),(u,f)) \in \mathbb{H}$.

Proposition 2.2 *Let $T \in (2,3)$ and assume that $((\phi^0,\phi^1),(u,f)) \in \mathbb{H}$. Then the solution (p,q) of (2.10) satisfies $(p,q)(T) = 0$ in $(0,1)$ if and only if $((\phi^0,\phi^1),(u,f))$ satisfies*

$$\begin{cases} u'(t) = f'(t+1) + \dfrac{1}{2}q^0(t) & \text{if } T-2 < t < 1 \\ u'(t) = f'(t+1) + f'(t-1) - \dfrac{1}{2}p^0(2-t) & \text{if } 1 < t < T-1 \\ u'(t) = f'(t-1) - \dfrac{1}{2}p^0(2-t) & \text{if } T-1 < t < 2 \\ u'(t) + u'(t-2) = f'(t-1) + \dfrac{1}{2}q^0(t-2) & \text{if } 2 < t < T. \end{cases} \tag{2.11}$$

Proof. Solving system (2.10) using the characteristics method gives the expressions:

$$p(t,x) = \begin{cases} p^0(x-t) & \text{if } 0 < t < x < 1 \\ 2u'(t-x) - q^0(t-x) & \text{if } 0 < x < t < 1+x \\ 2u'(t-x) - 2f'(t-x-1) + p^0(2-t+x) & \text{if } 1+x < t < 2+x \\ 2u'(t-x) + 2u'(t-x-2) - 2f'(t-x-1) - q^0(t-x-2) & \text{if } 2+x < t < 3 \end{cases} \tag{2.12}$$

and

$$q(t,x) = \begin{cases} q^0(x+t) & \text{if } 0 < t < 1-x \\ 2f'(t+x-1) - p^0(2-t-x) & \text{if } 1-x < t < 2-x \\ 2f'(x+t-1) - 2u'(t+x-2) + q^0(t+x-2) & \text{if } 2-x < t < 3-x \\ 2f'(t+x-1) - 2u'(t+x-2) + 2f'(t+x-3) - p^0(4-t-x) & \text{if } 3-x < t < 3 \end{cases} \tag{2.13}$$

It follows that

$$p(T,x) = \begin{cases} 2u'(T-x) - 2f'(T-x-1) + p^0(x-T+2) & \text{if } 0 < T-2 < x < 1 \\ 2u'(T-x) + 2u'(T-x-2) + -2f'(T-x-1) - q^0(T-x-2) & \text{if } 0 < x < T-2 < 1 \end{cases}$$

and

$$q(T,x) = \begin{cases} -2u'(x+T-2) + 2f'(x+T-1) + q^0(x+T-2) & \text{if } 0 < x < 3-T \\ -2u'(T+x-2) + 2f'(T+x-1) & \\ \qquad + 2f'(T+x-3) - p^0(4-T-x) & \text{if } 0 < 3-T < x < 1. \end{cases}$$

Consequently, $(p,q)(T) = 0$ in $(0,1)$ if and only if u satisfies

$$\begin{cases} u'(T-x) = f'(T-x-1) - \dfrac{1}{2}p^0(x-T+2) & \text{if } T-2 < x < 1 \\ u'(T-x) = -u'(T-x-2) + f'(T-x-1) + \dfrac{1}{2}q^0(T-x-2) & \text{if } 0 < x < T-2 \\ u'(x+T-2) = f'(x+T-1) + \dfrac{1}{2}q^0(x+T-2) & \text{if } 0 < x < 3-T \\ u'(x+T-2) = f'(T+x-1) + f'(T+x-3) - \dfrac{1}{2}p^0(4-T-x) & \text{if } 3-T < x < 1 \end{cases}$$

which turns to be equivalent to (2.11). ■

Remark 2.1 *We point out that in Proposition 2.2, the values of the control functions u are not prescribed on $(0,T-2)$. Consequently, there exists an infinite number of such control functions. The fact that u is "free" in $(0,T-2)$ plays a crucial role in the sequel.* ■

We then set

$$\mathbb{H}_c = \left\{ \left(\left(\phi^0, \phi^1\right), (u, f) \right) \in \mathbb{H} \text{ satisfying } (2.11) \right\}.$$

Corollary 2.1 *Let $T \in (2,3)$ and $\left(\left(\phi^0, \phi^1\right), (u, f) \right) \in \mathbb{H}_c$. Then the solution ϕ of (2.6) satisfies $\phi(T) = \phi'(T) = 0$ on $(0,1)$ if and only if*

$$u(T) = f(T) = 0.$$

Proof. From Proposition 2.2 and (2.9), it follows that

$$\phi'(T,x) = \phi_x(T,x) = 0, \quad x \in (0,1).$$

Thus, in particular

$$\phi(T,x) = \phi(T,0) = \phi(T,1), \quad x \in (0,1).$$

It follows that, in order to get $\phi(T) = 0$ on $(0,1)$, it is necessary and sufficient to choose $u(T) = f(T) = 0$. ■

In the sequel, we use the space

$$\mathbb{H}_c^r = \left\{ \left(\left(\phi^0, \phi^1\right), (u, f) \right) \in \mathbb{H}_c, \ u(T) = f(T) = 0 \right\}.$$

Definition 2.1 *The* ***Dirichlet-to-Neumann map*** *associated with the system (2.6) is the application $A : \mathbb{H} \to L^2(0,T)$ defined by*

$$A(\phi^0, \phi^1, u, f) = \phi_x(.,1)$$

where ϕ is the solution of (2.6) associated with $\left(\phi^0, \phi^1, u, f\right)$. The ***Control Dirichlet-to-Neumann map*** *is the application*

$$A_c = A_{|\mathbb{H}_c^r}.$$

The following lemma gives a characterization of these two maps.

Lemma 2.1 *Let $T \in (2,3)$ and $(\phi^0, \phi^1, u, f) \in \mathbb{H}$. Then*

$$A(\phi^0, \phi^1, u, f)(t) = \begin{cases} f'(t) - p^0(1-t) & 0 < t < 1 \\ f'(t) - 2u'(t-1) - q^0(t-1) & 1 < t < 2 \\ f'(t) + 2f'(t-2) - 2u'(t-1) - p^0(3-t) & 2 < t < T \end{cases} \tag{2.14}$$

As a consequence of (2.11), if $(\phi^0, \phi^1, u, f) \in \mathbb{H}_c^r$, then

$$A_c(\phi^0, \phi^1, u, f)(t) = \begin{cases} f'(t) - p^0(1-t) & 0 < t < 1 \\ f'(t) - 2u'(t-1) - q^0(t-1) & 1 < t < T-1 \\ -f'(t) & T-1 < t < T \end{cases} \tag{2.15}$$

where $p^0 = \phi^1 - \phi^0_x$ and $q^0 = \phi^1 + \phi^0_x$.

Proof. Taking into account the expressions of $p = \phi' - \phi_x$ and $q = \phi' + \phi_x$ derived in (2.12) and (2.13), we get:

$$\phi_x(t,1) = \frac{q(t,1) - p(t,1)}{2},$$

which leads to (2.14). The expression (2.15) is obtained using (2.11). ■

Remark 2.2 *Note that the expression of $A_c(\phi^0, \phi^1, u, f)$ in (2.15) involves only the part of u defined on $(0, T-2)$, i.e. the "free" part of u.* ■

Remark 2.3 *Clearly, $A_c \in \mathcal{L}\left(\mathbb{H}_c^r; L^2(0,T)\right)$. If moreover*

$$(u, f, \phi^0, \phi^1) \in H^2(0,T) \times H^2(0,T) \times H^2(0,1) \times H^1(0,1)$$

with the compatibility conditions

$$u'(0) = -\phi_x^0(0), \;\; f'(T-1) = u'(T-2) + \frac{1}{2}\left(\phi_x^0(T-2) + \phi^1(T-2)\right)$$

then $A_c(\phi^0, \phi^1, u, f) \in H^1(0,T)$. From (2.15) other regularity results can be easily derived. ■

3 A penalty method

We are going to prove Theorem 1.1 by introducing the penalized problem:

$$\begin{cases} y_\epsilon'' - y_{\epsilon,xx} = 0 & (t,x) \in Q_T, \\ y_\epsilon(t,0) = u_\epsilon(t) & t \in (0,T), \\ y_{\epsilon,x}(t,1) = \epsilon^{-1}[y_\epsilon(t,1) - \psi(t)]^- & t \in (0,T), \\ y_\epsilon(0,x) = y^0(x), \;\; y_\epsilon'(0,x) = y^1(x) & x \in (0,1). \end{cases} \tag{3.16}$$

where $[\cdot]^-$ denotes the negative part so that $[y_\epsilon(t,1) - \psi(t)]^- = -\min(0, y_\epsilon(t,1) - \psi(t))$ and $\epsilon > 0$. In a first step, we prove that for any $\left(y^0, y^1\right) \in H^1(0,1) \times L^2(0,1)$ and $\psi \in H^1(0,T)$ satisfying (1.3) there exists a family of controls $u_\epsilon \in H^1(0,T)$ such that the solution of (3.16) satisfies $y_\epsilon(T) = y_\epsilon'(T) = 0$ on $(0,1)$. At this level, we make use once again of the Dirichlet-to-Neumann map.

In a second step, we provide some estimates on u_ϵ and y_ϵ which will allow to pass to the limit in ϵ in order to obtain a solution of (1.1) satisfying $y(T) = y'(T) = 0$ on $(0,1)$.

Let $T \in (2,3)$ and $\left(\phi^0, \phi^1, u, f\right) \in \mathbb{H}_c^r$, $\psi \in H^1(0,T)$ with $\psi(T) \leq 0$. The associated solution ϕ of (2.6), as we have previously noted, satisfies $\phi(T) = \phi'(T) = 0$ on $(0,1)$. This solution ϕ is a solution of (3.16) satisfying $\phi(T) = \phi'(T) = 0$ on $(0,1)$ if and only if, for any $\epsilon > 0$, there exists f such that

$$A_c(\phi^0, \phi^1, u, f) = \epsilon^{-1}[f - \psi]^-, \;\; (0,T). \tag{3.17}$$

Let us define the space $H_r^1(1,T) = \{f \in H^1(1,T) \;:\; f(T) = 0\}$. Using (2.15), problem (3.17) becomes the following nonlinear control problem: given $T \in (2,3)$,

$$\begin{cases} \texttt{find } f \in H_r^1(1,T) \texttt{ and } u \in H^1(0,T-2) \texttt{ such that} \\ f(0) = \phi^0(1), u(0) = \phi^0(1) \\ f'(t) = \begin{cases} \epsilon^{-1}[f(t) - \psi(t)]^- + p^0(1-t) & t \in (0,1) \\ \epsilon^{-1}[f(t) - \psi(t)]^- + 2u'(t-1) + q^0(t-1) & t \in (1, T-1) \\ -\epsilon^{-1}[f(t) - \psi(t)]^- & t \in (T-1, T) \end{cases} \end{cases} \tag{3.18}$$

3.1 Existence and uniform bounds for solutions of the penalized problem

Lemma 3.1 *For any $\epsilon > 0$, problem (3.18) admits an infinite number of solutions (f_ϵ, u_ϵ).*

Proof. On $[0,1]$, let $l_\epsilon \in H^1(0,1)$ be the unique solution of

$$\begin{cases} f'(t) = \epsilon^{-1}[f(t)-\psi(t)]^- + p^0(1-t), \quad t \in (0,1) \\ f(0) = \phi^0(1). \end{cases} \tag{3.19}$$

and set $f_{1,\epsilon} = l_\epsilon(1)$. Similarly, on $(T-1,T)$, let r_ϵ be the unique solution of the backward problem

$$\begin{cases} f'(t) = -\epsilon^{-1}[f(t)-\psi(t)]^-, \quad t \in (T-1,T) \\ f(T) = 0. \end{cases} \tag{3.20}$$

and set $f_{T-1,\epsilon} = r_\epsilon(T-1)$.
We then consider the following nonlinear control problem :

$$\begin{cases} \texttt{find } u_\epsilon \in H^1(0,T-2) \texttt{ and } f \in H^1(1,T-1) \texttt{ such that} & \\ f'(t) = \epsilon^{-1}[f(t)-\psi(t)]^- + 2u_\epsilon'(t-1) + q^0(t-1), & t \in (1,T-1) \\ f(1) = f_{1,\epsilon}, \quad f(T-1) = f_{T-1,\epsilon}. & \end{cases} \tag{3.21}$$

which consists to find a control u_ϵ steering the solution of the differential equation

$$f'(t) = \frac{1}{\epsilon}[f(t)-\psi(t)]^- + 2u_\epsilon'(t-1) + q^0(t-1), \ t \in (1,T-1)$$

from the initial data $f_{1,\epsilon}$ to the final data $f_{T-1,\epsilon}$. We proceed as follows : we first consider the linear control problem

$$\begin{cases} \texttt{find } v_\epsilon \in H^1(0,T-2) \texttt{ and } \theta_\epsilon \in H^1(1,T-1) \texttt{ such that} & \\ \theta_\epsilon'(t) = 2v_\epsilon'(t-1), & t \in (1,T-1) \\ \theta_\epsilon(1) = f_{1,\epsilon}, \ \theta_\epsilon(T-1) = f_{T-1,\epsilon} & \end{cases} \tag{3.22}$$

Clearly, any $v_\epsilon \in H^1(0,T-2)$ satisfying

$$v_\epsilon(T-2) = v_\epsilon(0) + \frac{1}{2}\left(f_{T-1,\epsilon} - f_{1,\epsilon}\right) \tag{3.23}$$

(where $v_\epsilon(0) \in \mathbb{R}$ is arbitrary) gives a solution of (3.22). Without loss of generality, we take $v_\epsilon(0) = 0$. Now let us choose

$$\begin{cases} \theta_\epsilon(t) = 2v_\epsilon(t-1) + f_{1,\epsilon} & \\ 2u_\epsilon'(t-1) = 2v_\epsilon'(t-1) - \frac{1}{\epsilon}[2v_\epsilon(t-1) + f_{1,\epsilon} - \psi(t)]^- - q^0(t-1) & t \in (1,T-1) \end{cases} \tag{3.24}$$

It is straightforward that the couple $(\theta_\epsilon, u_\epsilon)$ defined by formulas (3.24) satisfies (3.21).

Thus, a family of solutions solution (f_ϵ, u_ϵ) to problem (3.18) is constructed if we take

$$f_\epsilon = \begin{cases} l_\epsilon & [0,1] \\ \theta_\epsilon & [1,T-1] \\ r_\epsilon & [T-1,T] \end{cases} \tag{3.25}$$

where l_ϵ, r_ϵ and $(\theta_\epsilon, u_\epsilon)$ are given by (3.19), (3.20) and (3.21) respectively. ■

In the following step, we prove that the sequence (u_ϵ, f_ϵ) may be chosen uniformly bounded with respect to ϵ in $H^1(0,T)$. From now on, C denotes a strictly positive constant that may varies from line to line but is independent on ϵ.

Lemma 3.2 *There exists a constant* $C = C(\phi^0, \phi^1, \psi) > 0$ *such that for any* $\epsilon > 0$*, the solution* f_ϵ *of (3.18) and satisfies the estimate:*

$$(f_\epsilon(t))^2 \leq C \quad t \in [0,1] \cup [T-1, T]\,. \tag{3.26}$$

If moreover $\phi^0(1) - \psi(0) \geq 0$*, then:*

$$\begin{aligned} \frac{1}{\epsilon}\left([f\epsilon(t) - \psi(t)]^-\right)^2 &\leq C, \quad t \in [0,1] \cup [T-1, T] && (3.27)\\ \int_0^t f_\epsilon'^2(s)ds &\leq C, \ t \in [0,1] && (3.28)\\ \int_t^T f_\epsilon'^2(s)ds &\leq C, \ t \in [T-1, T]\,. && (3.29) \end{aligned}$$

Proof. We set $h_\epsilon = f_\epsilon - \psi$ so that problem (3.19) writes

$$\left\{ \begin{array}{l} h_\epsilon'(t) = \frac{1}{\epsilon} h_\epsilon^-(t) + p^0(1-t) - \psi'(t), \quad t \in (0,1) \\ h_\epsilon(0) = \phi^0(1) - \psi(0). \end{array} \right. \tag{3.30}$$

Multiplying this equation by h_ϵ and integrating over $(0,t)$ for $t < 1$, we get

$$\begin{aligned} h_\epsilon^2(t) &= h_\epsilon^2(0) - \frac{2}{\epsilon}\int_0^t \left[h_\epsilon^-(s)\right]^2 ds + 2\int_0^t \left(p^0(1-s) - \psi'(s)\right) h_\epsilon(s) ds \\ &\leq \left(h_\epsilon^2(0) + \|p^0\|^2_{L^2(0,1)} + \|\psi'\|^2_{L^2(0,1)}\right) + \int_0^t h_\epsilon^2(s) ds. \end{aligned}$$

From Gronwall's lemma, we deduce that

$$h_\epsilon^2(t) \leq C\left(\left(\phi^0(1) - \psi(0)\right)^2 + \|p^0\|^2_{L^2(0,1)} + \|\psi'\|^2_{L^2(0,1)}\right).$$

and then

$$f_\epsilon^2(t) \leq C\left(\left(\phi^0(1) - \psi(0)\right)^2 + \|p^0\|^2_{L^2(0,1)} + \|\psi\|^2_{H^1(0,1)}\right), \ t \in (0,1)\,. \tag{3.31}$$

Similarly, problem (3.20) writes

$$\left\{ \begin{array}{l} h_\epsilon'(t) = -\frac{1}{\epsilon} h^-(t) - \psi'(t), \quad t \in (T-1, T) \\ h_\epsilon(T) = -\psi(T). \end{array} \right.$$

Multiplying as previously this equation by h_ϵ and integrating over (t, T) for $t \in (T-1, T)$, the same arguments lead to the estimate

$$h_\epsilon^2(t) \leq C\left(\psi^2(T) + \|\psi'\|^2_{L^2(T-1,T)}\right).$$

This implies

$$f_\epsilon^2(t) \leq C\left(\psi^2(T) + \|\psi\|^2_{H^1(T-1,T)}\right), \ t \in (T-1, T)\,. \tag{3.32}$$

Estimates (3.31) and (3.32) prove the first part of the lemma.

We now multiply the equation of h_ϵ in (3.30) by h_ϵ' and integrate over $(0,t)$ with $t \in (0,1)$, we get

$$\int_0^t \left(h_\epsilon'(s)\right)^2 ds + \frac{1}{\epsilon}\left(h_\epsilon^-(t)\right)^2 = \frac{1}{2\epsilon}\left(h_\epsilon^-(0)\right)^2 + \int_0^t \left(p^0(1-s) - \psi'(s)\right) h_\epsilon'(s) ds.$$

If we assume that $h_\epsilon(0) = \phi^0(1) - \psi(0) \geq 0$, then $h_\epsilon^-(0) = 0$ and Cauchy-Schwartz inequality imply

$$\int_0^t \left(h_\epsilon'(s)\right)^2 ds + \frac{1}{\epsilon}\left(h_\epsilon^-(t)\right)^2 \leq \int_0^t \left(p^0(1-s) - \psi'(s)\right)^2 ds, \quad t \in (0,1)\,.$$

From this last inequality, it follows that

$$\frac{1}{2}\int_0^t \left(f_\epsilon'(s)\right)^2 ds + \frac{1}{\epsilon}\left([f_\epsilon(t) - \psi(t)]^-\right)^2 \leq \int_0^t [\psi'(s)]^2 ds + \int_0^t \left(p^0(1-s) - \psi'(s)\right)^2 ds, \quad t \in (0,1)\,.$$

With the same argument, we get on $(T-1,T)$:

$$\int_t^T \left(f_\epsilon'(s)\right)^2 ds + \frac{1}{\epsilon}\left([f_\epsilon(t) - \psi(t)]^-\right)^2 \leq C\,\|\psi\|^2_{H^1(T-1,T)}\,, \quad t \in (T-1,T)\,.$$

This ends the proof. ■

Remark 3.1 *From (3.28) and (3.19) (resp. (3.29) and (3.20)) it can be deduced that*

$$\frac{1}{\epsilon^2}\int_0^t \left([f_\epsilon(s) - \psi(s)]^-\right)^2 ds \leq C, \;\; t \in [0,1]$$

(resp.

$$\frac{1}{\epsilon^2}\int_t^T \left([f_\epsilon(s) - \psi(s)]^-\right)^2 ds \leq C, \;\; t \in [T-1,T]).$$

The next step is to prove that the estimates (3.26)-(3.29) hold true on $(1, T-1)$.

Lemma 3.3 *In (3.24), there exists a sequence $(v_\epsilon)_{\epsilon>0} \subset H^1(0,T-2)$ such that*

$$\|f_\epsilon\|_{H^1(1,T-1)} \leq C, \quad \|u_\epsilon\|_{H^1(1,T-1)} \leq C.$$

Proof. Fix $\epsilon > 0$ sufficiently small. Let us recall first that $f_\epsilon = \theta_\epsilon$ on $(1,T-1)$ and that in (3.24), $v_\epsilon \in H^1(0,T-2)$ is an arbitrary function satisfying $v_\epsilon(0) = 0$ and $2v_\epsilon(T-2) = f_{T-1,\epsilon} - f_{1,\epsilon}$. This amounts to say that $f_\epsilon \in H^1(1,T-1)$ is an arbitrary function satisfying $f_\epsilon(1) = f_{1,\epsilon}$ and $f_\epsilon(T-1) = f_{T-1,\epsilon}$. The idea behind the following construction of f_ϵ is to choose a function joining the points $(1, f_{1,\epsilon})$ and $(1+\epsilon, \psi(1+\epsilon))$ (resp. $(T-1-\epsilon, \psi(T-1-\epsilon))$ and $(T-1, f_{T-1}, \epsilon))$ on $[1, 1+\epsilon]$ (resp. on $[T-1-\epsilon, T-1]$) in such a way that the integrals $\frac{1}{\epsilon^2}\int_1^{1+\epsilon}\left([f_\epsilon(t) - \psi(t)]^-\right)^2 ds$ and $\frac{1}{\epsilon^2}\int_{T-1-\epsilon}^{T-1}\left([f_\epsilon(t) - \psi(t)]^-\right)^2 ds$ remain uniformly bounded with respect to ϵ. On $[1+\epsilon, T-1-\epsilon]$, it is sufficient to take $f_\epsilon = \psi$. One possible choice is the following:

- If $\psi(1) - f_{1,\epsilon} < 0$ and $\psi(T-1) - f_{T-1,\epsilon} < 0$, we choose any function $f_\epsilon \in H^1(1,T-1)$ satisfying $f_\epsilon(1) = f_{1,\epsilon}$ and $f_\epsilon(T-1) = f_{T-1,\epsilon}$. For instance:

$$f_\epsilon(t) = \max\left(\frac{f_{T-1,\epsilon} - f_{1,\epsilon}}{T-2}(t-1) + f_{1,\epsilon}, \psi(t)\right), \;\; t \in (1, T-1)\,.$$

- If $\psi(1) - f_{1,\epsilon} \geq 0$ and $\psi(T-1) - f_{T-1,\epsilon} \geq 0$,

$$f_\epsilon(t) = \begin{cases} \psi(t) + (\psi(1) - f_{1,\epsilon})\left(\frac{1}{\epsilon}(t-1) - 1\right) & t \in [1, 1+\epsilon] \\ \psi(t) & t \in [1+\epsilon, T-1-\epsilon] \\ \psi(t) + (\psi(T-1) - f_{T-1,\epsilon})(-1 + \frac{T-1-t}{\epsilon}) & t \in [T-1-\epsilon, T-1] \end{cases}\,. \tag{3.33}$$

- If $\psi(1)-f_{1,\epsilon}\geq 0$ and $\psi(T-1)-f_{T-1,\epsilon}<0$,

$$f_\epsilon(t)=\begin{cases}\psi(t)+(\psi(1)-f_{1,\epsilon})\left(\frac{1}{\epsilon}(t-1)-1\right) & t\in[1,1+\epsilon]\\ \psi(t) & t\in\left[1+\epsilon,\frac{T}{2}\right]\\ \max\left(\frac{f_{T-1,\epsilon}-\psi(\frac{T}{2})}{\frac{T}{2}-1}(t-\frac{T}{2})+\psi(\frac{T}{2}),\psi(t)\right) & t\in\left[\frac{T}{2},T-1\right]\end{cases}$$

- If $\psi(1)-f_{1,\epsilon}<0$ and $\psi(T-1)-f_{T-1,\epsilon}\geq 0$,

$$f_\epsilon(t)=\begin{cases}\max\left(\frac{\psi(\frac{T}{2})-f_{1,\epsilon}}{\frac{T}{2}-1}(t-1)+f_{1,\epsilon},\psi(t)\right) & t\in\left[1,\frac{T}{2}\right].\\ \psi(t) & t\in\left[\frac{T}{2},T-1-\epsilon\right]\\ \psi(t)+(\psi(T-1)-f_{T-1,\epsilon})(-1+\frac{T-1-t}{\epsilon}) & t\in[T-1-\epsilon,T-1]\end{cases}$$

By construction $f_\epsilon\in H^1(1,T-1)$ and satisfies $f_\epsilon(1)=f_{1,\epsilon},\ f_\epsilon(T-1)=f_{T-1,\epsilon}$ in all cases. Note moreover that from (3.24), one has:

$$2u'_\epsilon(t-1)=f'_\epsilon(t)-\frac{1}{\epsilon}[f_\epsilon(t)-\psi(t)]^- - q^0(t-1),\ t\in(1,T-1).$$

so that it is sufficient to find uniform estimates with respect to ϵ for f_ϵ.

For the first case, note that $f_\epsilon\geq\psi$ on $(1,T-1)$ and that from (3.26), $|f_{T-1,\epsilon}|$ and $|f_{1,\epsilon}|$ are uniformly bounded with respect to ϵ. It is straightforward that this implies uniform bounds with respect to ϵ for $\|f_\epsilon\|_{H^1(0,T)}$ and $\int_0^T\left(\frac{[f_\epsilon(t)-\psi(t)]}{\epsilon}\right)^2dt=0$.

Assume now that $\psi(1)-f_{1,\epsilon}\geq 0$. Then

$$\begin{aligned}\int_1^{1+\epsilon}|f_\epsilon(t)|^2\,dt &= \int_1^{1+\epsilon}\left|\psi(t)+(\psi(1)-f_{1,\epsilon})\left(\frac{1}{\epsilon}(t-1)-1\right)\right|^2dt\\ &\leq C\left(\int_1^{1+\epsilon}|\psi(t)|^2\,dt+(\psi(1)-f_{1,\epsilon})^2\int_1^{1+\epsilon}\left|\left(\frac{1}{\epsilon}(t-1)-1\right)\right|^2dt\right)\\ &\leq C\left(\int_1^{1+\epsilon}|\psi(t)|^2\,dt+\epsilon\,(\psi(1)-f_{1,\epsilon})^2\right)\end{aligned}$$

On the other hand

$$\begin{aligned}\int_1^{1+\epsilon}|f'_\epsilon(t)|^2\,dt &= \int_1^{1+\epsilon}\left|\psi'(t)+\frac{1}{\epsilon}(\psi(1)-f_{1,\epsilon})\right|^2dt\\ &\leq C\left(\int_1^{1+\epsilon}|\psi'(t)|^2\,dt+\frac{|(\psi(1)-f_{1,\epsilon})|^2}{\epsilon}\right).\end{aligned}$$

These two last inequalities together with (3.27) give

$$\|f_\epsilon\|_{H^1(1,1+\epsilon)}\leq C.$$

To prove a similar estimate for u_ϵ on $(1,1+\epsilon)$, we just need to estimate $\int_1^{1+\epsilon}\left(\frac{[f_\epsilon(t)-\psi(t)]^-}{\epsilon}\right)^2dt$.

But, from (3.33), we get

$$\begin{aligned}\int_1^{1+\epsilon}\left(\frac{[f(t)-\psi(t)]^-}{\epsilon}\right)^2 &= \frac{(\psi(1)-f_{1,\epsilon})^2}{\epsilon^2}\int_1^{1+\epsilon}\left(1-\frac{1}{\epsilon}(t-1)\right)\right)^2dt\\ &\leq \frac{(\psi(1)-f_{1,\epsilon})^2}{\epsilon}\leq C\end{aligned}$$

thanks again to (3.27). Thus

$$\int_1^{1+\epsilon} |u_\epsilon'(t)|^2 \, dt \le C.$$

The same arguments on $(T-1-\epsilon, T-1)$ with $\psi(T-1) - f_{T-1,\epsilon} \ge 0$ give the estimates

$$\|f_\epsilon\|_{H^1(T-1-\epsilon,T-1)} \le C, \quad \int_{T-1-\epsilon}^{T-1} |u_\epsilon'(t)|^2 \, dt \le C.$$

The other situations are easier to treat. This ends the proof of the lemma. ∎

As a summary, we have proved:

Corollary 3.1 *Let $T \in (2,3)$ and $(\phi^0, \phi^1) \in H^1(0,1) \times L^2(0,1)$, $\psi \in H^1(0,T)$ with $\psi(T) \le 0$ and assume that $\phi^0(1) - \psi(0) \ge 0$. Then problem (3.18) admits a sequence (u_ϵ, f_ϵ) of solutions such that*

$$\begin{aligned} f_\epsilon^2(t) &\le C, \qquad t \in [0,T] \\ \|(f_\epsilon, u_\epsilon)\|_{H^1(0,T)} &\le C \\ \int_0^T \left(\frac{[f_\epsilon(t) - \psi(t)]^-}{\epsilon} \right)^2 dt &\le C. \end{aligned}$$

3.2 Limit of the family controls $\{u_\epsilon\}_{\epsilon>0}$ - Proof of the main result

The aim of this section is to obtain a solution of problem (1.1) satisfying (1.2) by passing to the limit in problem (3.18).

Thanks to Corollary 3.1, we may extract from the sequence (f_ϵ, u_ϵ) a subsequence, still denoted by (f_ϵ, u_ϵ), such that, as ϵ goes to zero,

$$\begin{array}{ll} (f_\epsilon(t), u_\epsilon(t)) \to (f(t), u(t)), & t \in [0,T] \\ (f_\epsilon, u_\epsilon) \rightharpoonup (f, u), & \texttt{in } \left(H_r^1(0,T)\right)^2 \texttt{ weak} \\ A_c(u_\epsilon, f_\epsilon, \phi^0, \phi^1) \rightharpoonup A_c(u, f, \phi^0, \phi^1) & \texttt{in } L^2(0,T) \texttt{ weak} \\ \dfrac{[f_\epsilon - \psi]^-}{\epsilon} \rightharpoonup \mu & \texttt{in } L^2(0,T) \texttt{ weak} \end{array}$$

with (f, u) satisfying (2.11) on $(T-2, T)$.

Recalling that problem (3.18) is equivalent to problem (3.17), it follows that

$$A_c(u, f, \phi^0, \phi^1) = \mu \tag{3.34}$$

On the other hand, since

$$[f_\epsilon(t) - \psi(t)]^- \to 0, \quad t \in [0,T] \texttt{ as } \epsilon \to 0$$

it follows that

$$f - \psi \ge 0, \texttt{ on } [0,T]. \tag{3.35}$$

We now prove that $(f - \psi) A_c(u, f, \phi^0, \phi^1) = 0$ on $(0,T)$. (3.17) implies that

$$(f_\epsilon - \psi) A_c(u_\epsilon, f_\epsilon, \phi^0, \phi^1) = (f_\epsilon - \psi) \frac{[f_\epsilon - \psi]^-}{\epsilon} = -\frac{\left([f_\epsilon - \psi]^-\right)^2}{\epsilon}$$

so that, using corollary (3.1), we get

$$\frac{[f_\epsilon - \psi]^-}{\sqrt{\epsilon}} \to 0 \ \text{ in } \ L^2(0,T) \text{ as } \epsilon \to 0$$

On the other hand, as ϵ goes to zero

$$\begin{array}{rcl} f_\epsilon - \psi & \to & f - \psi \ \text{ in } \ L^2(0,T), \\ A_c(u_\epsilon, f_\epsilon, \phi^0, \phi^1) & \rightharpoonup & A_c(u, f, \phi^0, \phi^1) \text{ in } L^2(0,T) \text{ weak.} \end{array}$$

Thus

$$(f_\epsilon - \psi)\, A_c(u_\epsilon, f_\epsilon, \phi^0, \phi^1) \rightharpoonup (f - \psi)\, A_c(u, f, \phi^0, \phi^1) \text{ in } L^2(0,T) \text{ weak as } \epsilon \to 0$$

and

$$(f - \psi)\, A_c(u, f, \phi^0, \phi^1) = 0, \quad \text{in } (0,T).$$

It remains to prove that the solution ϕ_ϵ of (2.6) associated with the data $\left(u_\epsilon, f_\epsilon, \phi^0, \phi^1\right)$ converges to the solution ϕ of (2.6) associated with the data $\left(u, f, \phi^0, \phi^1\right)$. But by linearity, $\phi - \phi_\epsilon$ is the solution of (2.6) associated with $(u - u_\epsilon, f - f_\epsilon, 0, 0)$. Thus by the first part of proposition 2.1, we get

$$\lim_{\epsilon \to 0} \|(\phi - \phi_\epsilon, \phi' - \phi'_\epsilon)\|_{L^2 \times H^{-1}(0,T)} = 0.$$

Therefore ϕ is a solution of our control problem.

4 A direct solution for the control problem

We now proceed to give a direct proof of Theorem 1.1. Let $T \in (2,3)$, $(\phi^0, \phi^1, u, f) \in \mathbb{H}_c^r$ and $\psi \in H^1(0,T)$ with the conditions

$$\psi(0) \le \phi^0(1), \quad \psi(T) \le 0.$$

Let ϕ be the associated solution of (2.6). From Corollary 2.1, we know that $\phi(T) = \phi'(T) = 0$ on $(0,1)$. Fixing $\left(\phi^0, \phi^1\right)$, a solution of (1.1) is obtained if (and only if) we can find (u, f) such that $(\phi^0, \phi^1, u, f) \in \mathbb{H}_c^r$ and solves the problem

$$\left\{ \begin{array}{ll} f - \psi \ge 0, & (0,T) \\ A_c(\phi^0, \phi^1, u, f) \ge 0, & (0,T) \\ (f - \psi)\, A_c(\phi^0, \phi^1, u, f) = 0, & (0,T) \\ f(0) = \phi^0(1), \ f(T) = 0. & \end{array} \right. \tag{4.36}$$

Taking into account (2.15), problem (4.36) is decomposed into two parts.

- On $(0, T-1)$, problem (4.36) writes:

$$\left\{ \begin{array}{l} f - \psi \ge 0, \\ f' - v \ge 0, \\ (f - \psi)\,(f' - v) = 0 \\ f(0) = \phi^0(1) \end{array} \right., \quad (0, T-1), \tag{4.37}$$

where

$$v(t) = \left\{ \begin{array}{ll} p^0(1-t) & 0 < t < 1 \\ 2u'(t-1) - q^0(t-1) & 1 < t < T-1 \end{array} \right., \tag{4.38}$$

- On $(T-1,T)$:

$$\left\{\begin{array}{l} f-\psi \geq 0, \\ f' \leq 0, \\ (f-\psi)\, f' = 0, \\ f(T) = 0 \end{array}\right. , \quad (T-1,T), \tag{4.39}$$

We solve separately the problems (4.37) and (4.39) using the following result which is for instance a consequence of [1] :

Lemma 4.1 *Let $h \in H^1(0,T)$ and $\theta_0 \geq h(0)$. Then the function*

$$\theta(t) = \max\left(\theta_0, \sup_{0\leq s\leq t} h(s)\right), \, t \in [0,T[$$

belongs to $H^1(0,T)$ and is the unique solution of the problem

$$\left\{\begin{array}{ll} \theta \geq h & \text{in } (0,T) \\ \theta' \geq 0 & \text{in } (0,T) \\ \theta'\,(\theta - h) = 0 & \text{in } (0,T) \\ \theta(0) = \theta_0. & \end{array}\right. \tag{4.40}$$

Using this lemma and the notation $[f]^+ = \max(0,f)$, we get:

Proposition 4.1 *Let $v \in L^2(0,T-1)$ defined by (4.38) and $V(t) = \int_0^t v(s)ds$. Then the unique solution of (4.37) is given by*

$$f(t) = V(t) + \max\left(\phi^0(1), \sup_{0\leq s\leq t}\left(\psi(s) - V(s)\right)\right), \; t \in (0,T-1)\,. \tag{4.41}$$

The unique solution of (4.39) is given by

$$f(t) = \left[\sup_{t\leq s\leq T} \psi(s)\right]^+, \; t \in (T-1,T). \tag{4.42}$$

Proof. In $(0,T-1)$, let us set

$$V(t) = \int_0^t v(s)ds, \quad \theta(t) = f(t) - V(t), \; h(t) = \psi(t) - V(t) \tag{4.43}$$

so that system (4.37) transforms into (4.40) with $\theta_0 = \phi^0(1)$. From Lemma 4.1, it follows that the unique solution of (4.37) in $H^1(0,T-1)$ is given by:

$$f(t) = V(t) + \max\left(\phi^0(1), \sup_{0\leq s\leq t}\left(\psi(s) - V(s)\right)\right), \, t \in (0,T-1)\,.$$

Similarly, in $(T-1,T)$, let us set $\delta(t) = f(T-t)$ and $g(t) = \psi(T-t)$ for $t \in (0,1)$ so that (4.39) transforms into the following system:

$$\left\{\begin{array}{ll} \delta \geq g & \text{in } (0,1) \\ \delta' \geq 0 & \text{in } (0,1) \\ \delta'\,(\delta - g) = 0 & \text{in } (0,1) \end{array}\right.$$

which (again as a consequence of Lemma 4.1), since by assumption $\delta(0) = f(T) = 0$ and $g(0) = \psi(T) \leq 0$, has a unique solution in $H^1(0,1)$ given by

$$\delta(t) = \max\left(\delta(0), \sup_{0\leq s\leq t} g(s)\right).$$

In other words,

$$f(T-t) = \max\left(0, \sup_{0\leq s\leq t} \psi(T-s)\right), \ 0 < t < 1,$$

or equivalently (4.42). ■

Proposition 4.2 *There exists u such that the function f given by (4.41) and (4.42) belongs to $H^1(0,T)$.*

Proof. To get a function $f \in H^1(0,T)$, we have to ensure the continuity of f at $t = T-1$:

$$\lim_{t\to(T-1)^-} f(t) = \lim_{t\to(T-1)^+} f(t). \tag{4.44}$$

But from

$$\lim_{t\to(T-1)^-} f(t) = V(T-1) + \max\left(\phi^0(1), \sup_{0\leq s\leq T-1} (\psi(s) - V(s))\right)$$

and

$$\lim_{t\to(T-1)^+} f(t) = \left[\sup_{T-1\leq s\leq T} \psi(s)\right]^+,$$

we are led to solve the following problem: find $u \in H^1(0,T-2)$ such that

$$V(T-1) + \max\left(\phi^0(1), \sup_{0\leq s\leq T-1} (\psi(s) - V(s))\right) = \left[\sup_{T-1\leq s\leq T} \psi(s)\right]^+. \tag{4.45}$$

Note that the number $\Lambda = \left[\sup_{T-1\leq s\leq T} \psi(s)\right]^+$ does not depend on u and that from (4.38)

$$V(t) = \begin{cases} \int_{1-t}^1 p^0(s)ds, & 0 \leq t \leq 1 \\ \int_0^1 p^0(s)ds + 2\left(u(t-1) - u(0)\right) - \int_0^{t-1} q^0(s)ds, & 1 \leq t \leq T-1 \end{cases}$$

and in particular

$$V(T-1) = 2\left(u(T-2) - u(0)\right) + \int_0^1 p^0(s)ds - \int_0^{T-2} q^0(s)ds.$$

Let us note $A = \max(\phi^0(1), \sup_{0\leq s\leq 1}(\psi(s) - V(s)))$ and look for a control u such that $\sup_{1\leq s\leq T-1}(\psi(s) - V(s)) \geq A$ i.e. such that, for all $s \in (1, T-1)$,

$$2(u(s-1) - u(0)) \geq \psi(s) - \int_0^1 p^0(y)dy - A + \int_0^{s-1} q^0(y)dy \equiv g(s). \tag{4.46}$$

We check that $g(1) \leq 0$ from the definition of A and Λ. The continuity condition (4.45) then becomes

$$2(u(T-2) - u(0)) = \Lambda - \int_0^1 p^0(y)dy - A + \int_0^{T-2} q^0(y)dy \equiv B.$$

compatible with (4.46) since we compute $B - g(T-1) \geq 0$. We then choose $u(s-1)$ and $u(0)$ such that $2(u(s-1) - u(0)) = g(s) + G(s)$ where $G(s)$ is a corrector function - linear positive - with $g(1) + G(1) = 0$ and $g(T-1) + G(T-1) = B$. From the condition $u(0) = \phi^0(0)$, this permits to fixe the control u in $(0, T-2)$ as follows :

$$u(s) = \phi^0(0) + \frac{1}{2}\bigg(g(s+1) + G(s+1)\bigg), \quad 0 \leq s \leq T-2. \tag{4.47}$$

This ends the proof. ■

Proposition 4.1 and 4.2 then prove Theorem 1.1.

5 Numerical illustration

We illustrate our controllability results with some simple applications corresponding to the numerical value $T = 2.2$ and the initial data

$$(y^0(x), y^1(x)) = \bigg(x(1 - \frac{x}{2}), -3x\bigg), \quad x \in (0,1) \tag{5.48}$$

which ensure an impact at the right extremity $x = 1$ for some $t \in (0,T)$ if the function ψ is large enough. We consider the constant case $\psi(t) = L$, $L \leq 0$ and the time dependent case with $\psi(t) = \sin(n\pi t/T)/5$ for some $n \in \mathbb{N}$.

5.1 The penalty method

ϵ, T, y^0, y^1 and the obstacle function ψ being given, the process associated with the penalized approach is as follows: the function $f(t) = y_\epsilon(t,1)$ is first computed on $(0,T)$ by solving the nonlinear ordinary differential equations 3.19 and 3.20 using the explicit Euler scheme. This then leads to the control function $u_\epsilon = y_\epsilon(t,1)$ from (3.24). The control on the time interval $(T-2,T)$ is then compute by solving the system (2.11). Once the displacement y_ϵ is known at the two extremities, the controlled displacement y_ϵ, solution of the partial differential equation (3.16) on Q_T is finally obtained using a P_1 (finite element) approximation in space and the leapfrog scheme for the time derivative. In the case where the obstacle behavior is not known a priori, specific approximations are necessary (we refer to [2, 12] where accurate and consistent schemes preserving the energy are proposed). The set $Q_T = (0,T) \times (0,1)$ is discretized with a uniform grid with $h = dt = 1/1000$.

Figures 1, 2 and Table 5.1 report some results obtained in the constant case $\psi(t) = -1/10$ on $(0,T)$ with $\epsilon = 1/200$. As expected, the penalized approach ensure a small interpenetration of the string below the obstacle, in the sense that during a time interval, the quantity $y_\epsilon(1,t) - \psi(t)$ is strictly negative, but remains of order $-\epsilon$. We also check that the control u_ϵ remains uniformly bounded with respect to ϵ. Figures 3, 4 and Table 5.1 reports similar results in the time dependent case $\psi(t) = \sin(2\pi t/T)/5$.

5.2 Direct method

For the direct method, the process is as follows: the control u is first computed on $(0, T-2)$ with the formulae (4.47) which permits to compute the function v on $(0, T-1)$ defined by (4.38), then $V(s) = \int_0^s v(t)dt$ and finally the function $f(t) = y(t,1)$ on $(0,T)$ with the formula (4.41) and (4.42). The control u on $(T-2,T)$ is then given by (2.11). The direct approach permits not only to satisfy the condition $y(t,1) - \psi(t) \geq 0$ for all t but also to compute explicitly the control. In

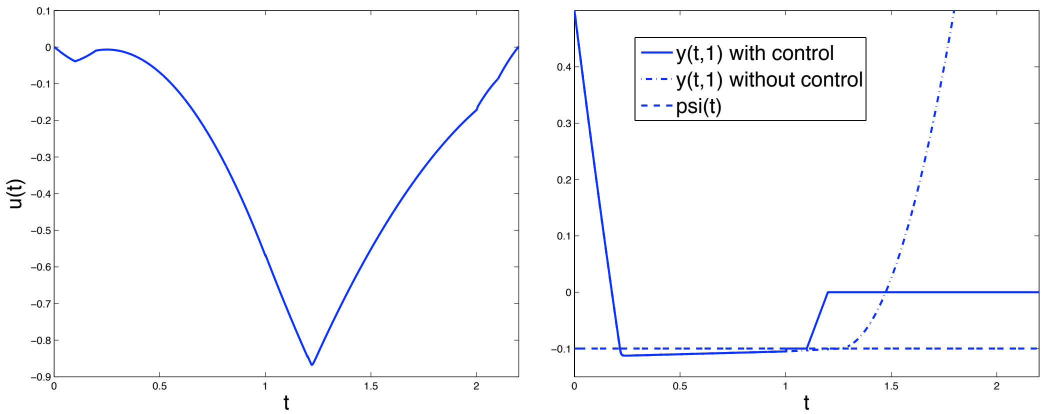

Figure 1: Penalty method - $\epsilon = 1/200$ - $\psi(t) = L = -1/10$ - Evolution of the control u_ϵ (**Left**) and corresponding displacement $y_\epsilon(\cdot, 1)$ (**Right**) vs $t \in [0, T]$ - $\|u_\epsilon\|_{L^2(0,T)} \approx 6.131 \times 10^{-1}$.

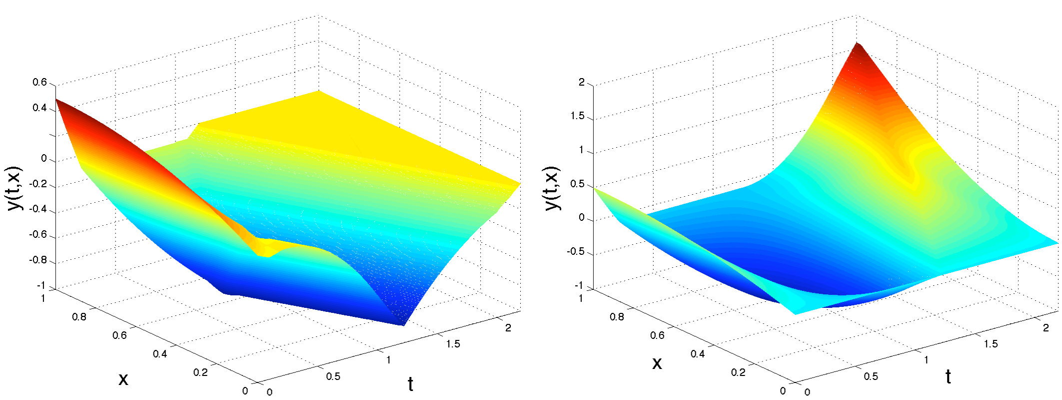

Figure 2: Penalty method - $\epsilon = 1/200$ - $\psi(t) = L = -1/10$ - Evolution of y_ϵ on Q_T in the controlled (**Left**) and uncontrolled case (**Right**).

	$\epsilon = 1/100$	$\epsilon = 1/200$	$\epsilon = 1/400$	$\epsilon = 1/800$
$\|u_\epsilon\|_{L^2(0,T)}$	6.175×10^{-1}	6.131×10^{-1}	6.108×10^{-1}	6.097×10^{-1}
$\|\epsilon^{-1}[y_\epsilon(\cdot,1) - \psi]^-\|_{L^2(0,T)}$	1.617	1.624	1.627	1.628
$\min_{t\in[0,T]}(y_\epsilon(t,1) - \psi(t))$	-2.47×10^{-2}	-1.25×10^{-2}	-6.34×10^{-3}	-3.19×10^{-3}

Table 1: Penalty approach - $\psi(t) = L = -1/10$.

	$\epsilon = 1/100$	$\epsilon = 1/200$	$\epsilon = 1/400$	$\epsilon = 1/800$
$\|u_\epsilon\|_{L^2(0,T)}$	5.586×10^{-1}	5.533×10^{-1}	5.506×10^{-1}	5.492×10^{-1}
$\|\epsilon^{-1}[y_\epsilon(\cdot,1) - \psi]^-\|_{L^2(0,T)}$	1.837	1.844	1.848	1.850
$\min_{t\in[0,T]}(y_\epsilon(t,1) - \psi(t))$	-3.09×10^{-2}	-1.57×10^{-2}	-7.97×10^{-3}	-4.01×10^{-3}

Table 2: Penalty approach - $\psi(t) = \sin(2\pi t/T)/5$.

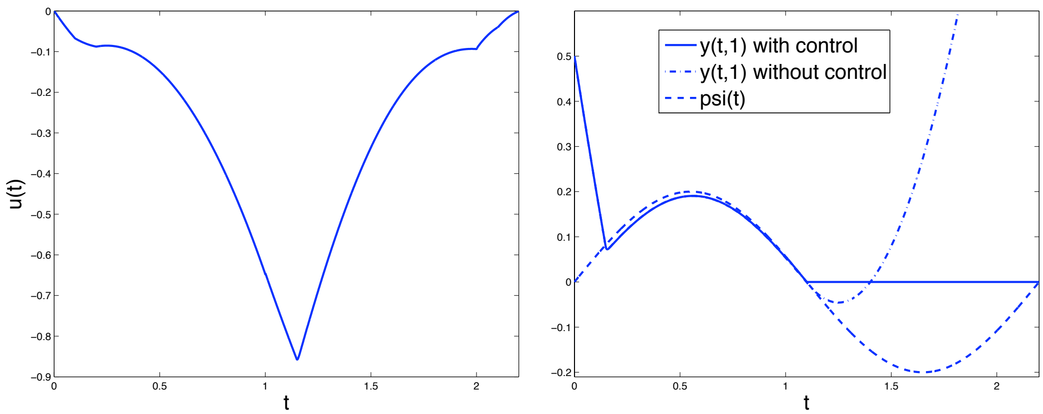

Figure 3: Penalty method - $\epsilon = 1/200$ - $\psi(t) = \sin(2\pi t/T)/5$ - Evolution of the control u_ϵ (**Left**) and corresponding displacement $y_\epsilon(\cdot, 1)$ (**Right**) vs $t \in [0, T]$ - $\|u_\epsilon\|_{L^2(0,T)} \approx 5.533 \times 10^{-1}$.

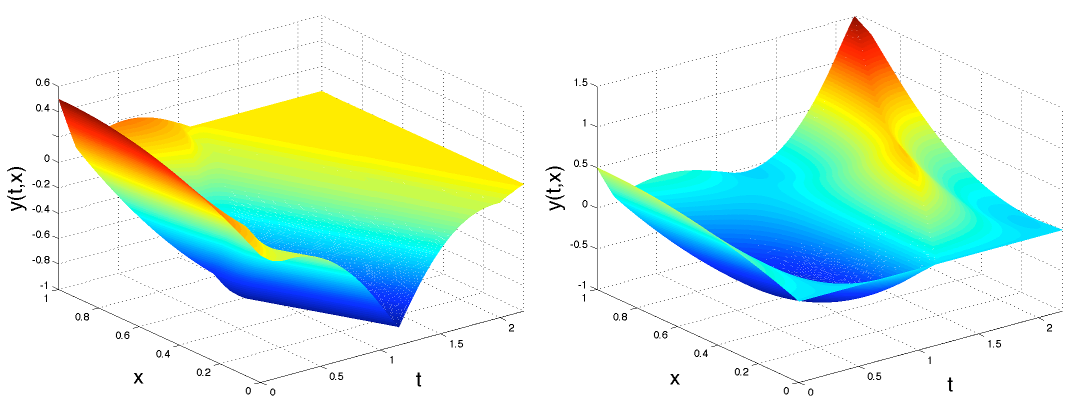

Figure 4: Penalty method - $\epsilon = 1/200$ - $\psi(t) = \sin(2\pi t/T)/5$ - Evolution of y_ϵ on Q_T in the controlled (**Left**) and uncontrolled case (**Right**).

the simple case $\psi(t) = L \in (-3/2, 0]$ in $[0, T]$, we obtain the following expressions. From (4.47), we deduce that

$$u(t) = -\frac{t}{2}\left(2t - 1 + \frac{L}{T-2}\right), \quad t \in (0, T-2) \tag{5.49}$$

leading to the function $V \in L^2(0, T-1)$ given by

$$V(t) = \begin{cases} t(-3+t) & 0 \leq t \leq 1 \\ \dfrac{4 - 2T - tL + L}{T-2} & 1 \leq t \leq T-1 \end{cases} \tag{5.50}$$

and to the function f given by

$$f(t) = \begin{cases} t(-3+t) + \dfrac{1}{2} & 0 \leq t \leq t_L \\ L & t_L \leq t \leq 1 \\ \dfrac{L(-t+T-1)}{T-2} & 1 \leq t \leq T-1 \\ 0 & T-1 \leq t \leq T \end{cases} \tag{5.51}$$

with $t_L = (3 - \sqrt{7+4L})/2 \in (0,1)$. From (2.11), the function f then provides the control u in $(T-2, T)$

$$\begin{cases} u(t) = -\dfrac{L}{2} + \dfrac{t}{2} - t^2 & T-2 < t < 1 \\ u(t) = \dfrac{3}{2} - \dfrac{L}{2} + \dfrac{t^2}{2} - \dfrac{5t}{2} & 1 < t < t_L + 1 \\ u(t) = -3 + \dfrac{L}{2} + \dfrac{5t}{2} - \dfrac{t^2}{2} & t_L + 1 < t < 2 \\ u(t) = -\dfrac{1}{2}\dfrac{L(t-T)}{T-2} & 2 < t < T \end{cases} \tag{5.52}$$

assuming that $L \in (-3/2, 0)$ and $T \in (2,3)$ are such that $t_L > T-2$. The knowledge of $(y(t,0), y(t,1))$ in $(0,t)$ then permits to compute the entire solution y on Q_T by using the formulae given in Section 2. In practice, it is simpler to approximate y by a numerical discretization of the wave equation (1.1). Figures 5 and 6 report the graph in the case $L = -1/10$. In particular, the set $\{t \in (0,T), f(t) = \psi(t) = L\}$ is reduced to one interval, corresponding to the contact period. Figure 6-left depicts the corresponding evolution of y on Q_T. The L^2-norm of the control is $\|u\|_{L^2(0,T)} \approx 4.84 \times 10^{-1}$. The others figures address the time dependent case $\psi(t) = \sin(n\pi t/T)/5$ for $n = 6, 11, 19$.

6 Comments and remarks

1. It is clear from the proof that instead of looking for controls such that the solution of (1.1) satisfies $y(T) = y'(T) = 0$ on $(0,1)$, we may look for a control such that, given $(z^0, z^1) \in H^1(0,1) \times L^2(0,1)$, the solution satisfies $y(T) = z^0$, $y'(T) = z^1$? It suffices to suitably change the expression of u' in Proposition 2.2. The exact result is then

   Theorem 6.1 *Let $T \in (2,3)$. For any $((y^0, y^1), (z^0, z^1)) \in [H^1(0,1) \times L^2(0,1)]^2$, $\psi \in H^1(0,T)$ with*

$$y^0(1) \geq \psi(0), \quad z^0(1) \geq \psi(T),$$

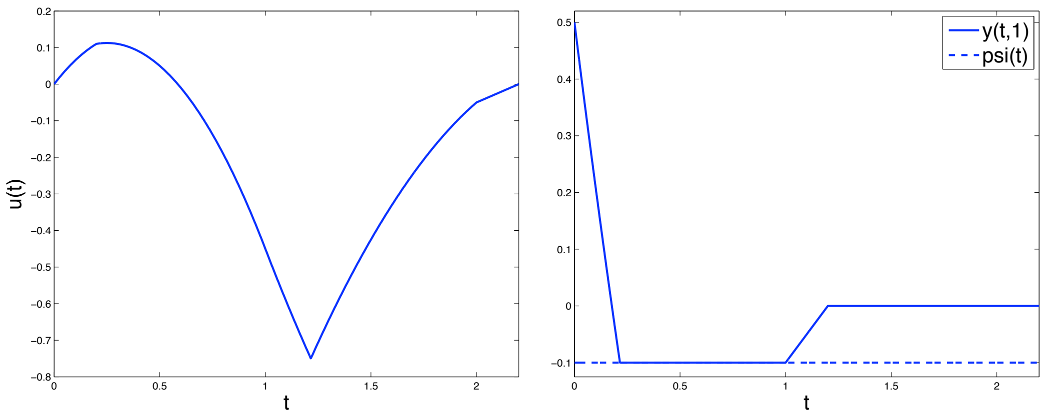

Figure 5: $\psi(t) = L = -1/10$ - Evolution of the control u (**Left**) and corresponding displacement $y(\cdot, 1)$ (**Right**) vs $t \in [0, T]$ - $\|u\|_{L^2(0,T)} \approx 4.84 \times 10^{-1}$.

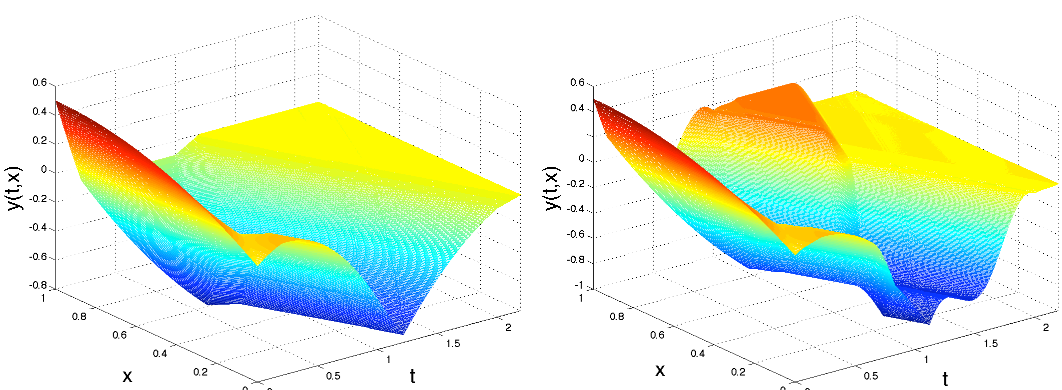

Figure 6: Evolution of the controlled solution y in Q_T coresponding to $\psi(t) = -1/10$ (**Left**) and $\psi(t) = \sin(6\pi t/T)/5$ (**Right**).

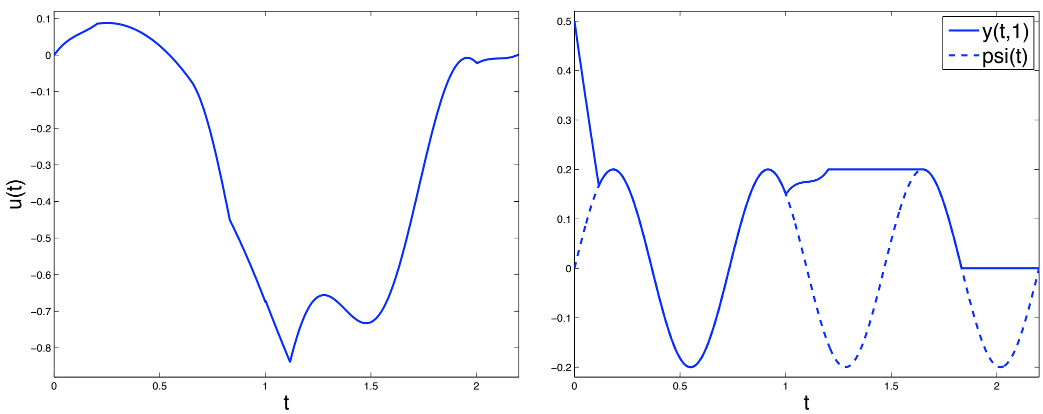

Figure 7: $\psi(t) = \sin(6\pi t/T)/5$− - Evolution of the control u (**Left**) and corresponding displacement $y(\cdot, 1)$ (**Right**) vs $t \in [0, T]$ - $\|u\|_{L^2(0,T)} \approx 6.44 \times 10^{-1}$.

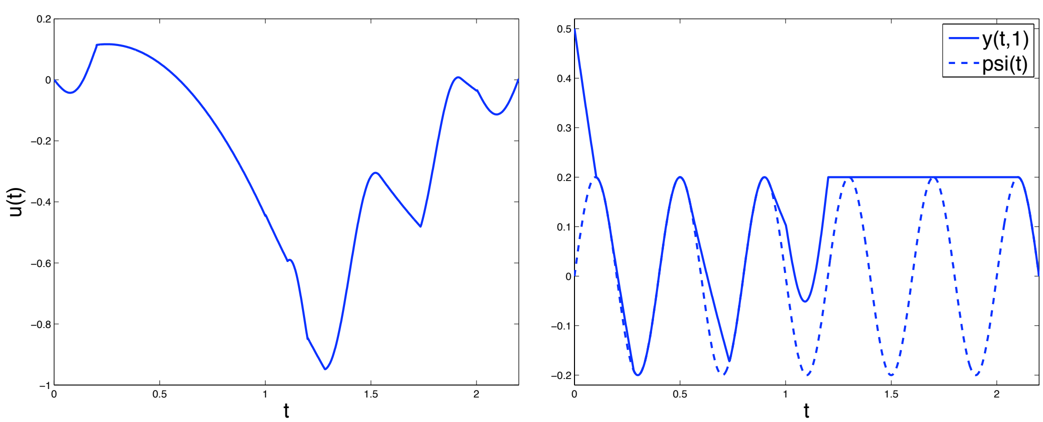

Figure 8: $\psi(t) = \sin(11\pi t/T)/5$− - Evolution of the control u (**Left**) and corresponding displacement $y(\cdot, 1)$ (**Right**) vs $t \in [0, T]$.

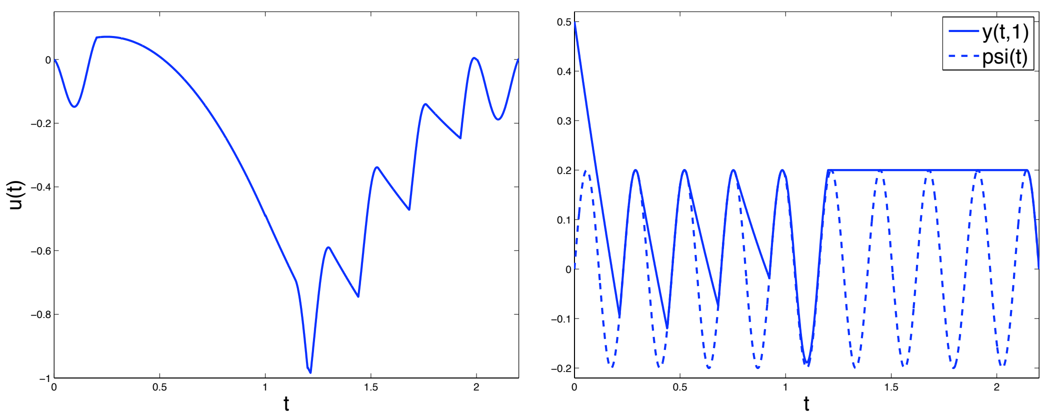

Figure 9: $\psi(t) = \sin(19\pi t/T)/5-$ - Evolution of the control u (**Left**) and corresponding displacement $y(\cdot, 1)$ (**Right**) vs $t \in [0, T]$.

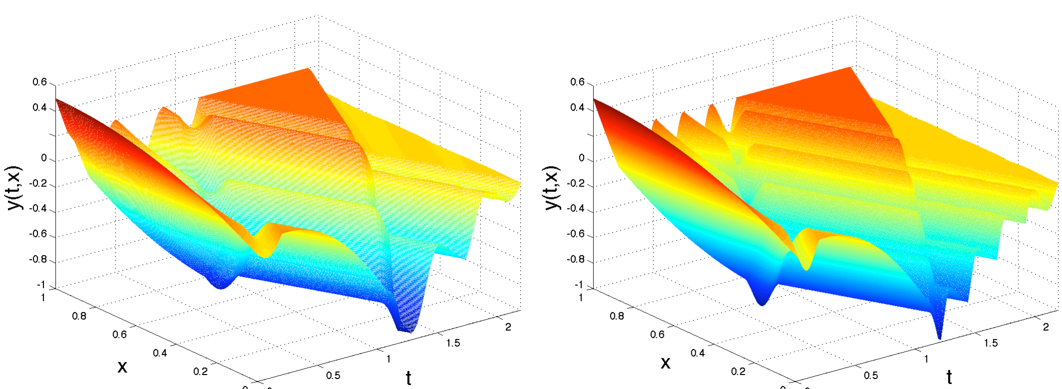

Figure 10: Evolution of the controlled solution y in Q_T coresponding to $\psi(t) = \sin(11\pi t/T)/5$ (**Left**) and $\psi(t) = \sin(19\pi t/T)/5$ (**Right**).

there exists $u \in H^1(0,T)$ *such that (1.1) admits a unique solution* y *such that* $y \in C\left([0,T], H^1(0,1)\right) \cap C^1\left([0,T], L^2(0,1)\right)$ *and satisfying* $y(T) = z^0,\ y'(T) = z^1$ *on* $(0,1)$.

2. **The case** $T = 2$. For simplicity with respect to the use of the caracteristic method, we have assumed that $T \in (2,3)$, but the controllability is *a fortiori* true for any $T > 2$. Concerning the limit case $T = 2$, if we use the penalty method, we first note that for the linear problem, in view of corollary 2.1, the condition $u(2) = f(2) = 0$ is possible for any initial data $\left(\phi^0, \phi^1\right) \in H^1(0,1) \times L^2(0,1)$ if and only if the following compatibility conditions hold

$$u(0) = \phi^0(0),\ f(0) = \phi^0(1).$$

The control is then given by

$$\begin{cases} u'(t) = f'(t+1) + \frac{1}{2}q^0(t) & \text{if } 0 < t < 1 \\ u'(t) = f'(t-1) - \frac{1}{2}p^0(2-t) & \text{if } 1 < t < 2 \end{cases}$$

and the control Dirichlet-to-Neumann map is given by

$$A_c(u, f, \phi^0, \phi^1)(t) = \begin{cases} f'(t) - p^0(1-t) & a.e.\, 0 < t < 1 \\ -f'(t) & a.e.\ 1 < t < 2 \end{cases}$$

and it does not depend on u anymore. Thus the differential equation corresponding to (3.18) is:

$$\begin{cases} f'(t) = \begin{cases} \frac{1}{\epsilon}[f(t) - \psi(t)]^- + p^0(1-t) & t \in (0,1) \\ \\ -\frac{1}{\epsilon}[f(t) - \psi(t)]^- & t \in (1,2) \end{cases} \\ f(0) = \phi^0(1),\ f(2) = 0 \end{cases} \tag{6.53}$$

For f to be a $H^1(0,2)$ function, we need the condition

$$f(1^-) = \lim_{t \to 1^-} f(t) = \lim_{t \to 1^+} f(t) = f(1^+). \tag{6.54}$$

If $v \in H^1(0,1)$ satisfies $v(0) = 0$ and $v(1) = f(1+) - \phi^0(1)$ then the couple $\left(f, p^0\right)$ defined on $(0,1)$ by

$$p^0(1-t) = v'(t) - \frac{\left[v(t) + \phi^0(1) - \psi(t)\right]^-}{\epsilon};\quad f(t) = v(t) + \phi^0(1).$$

With this choice, (6.53) and (6.54) are satisfied but the initial data depend on ϵ. To pass to the limit with respect to ϵ will impose supplementary conditions on the initial data.

The conclusion is that, in general, even the penalized problem is not controllable for any initial data. The same kind of problem occurs if one tries the direct method.

3. Using this approach, we may also address the case of a lower and upper obstacle $\psi_l, \psi_u \in H^1(0,T)$ so that $\psi_l(t) \leq y(t,1) \leq \psi_u(t)$, $t \in (0,T)$ with the condition $\psi_l(T) \leq 0 \leq \psi_u(T)$ (see [2]).

4. With the method used in section 3, we can consider the nonlinear control problem

$$\begin{cases} y'' - y_{xx} = 0 & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = u(t) & t \in (0,T), \\ y_x(t,1) = f(t,y) & t \in (0,T), \\ y(0,x) = y^0(x),\ y'(0,x) = y^1(x) & x \in (0,1). \end{cases}$$

and prove the controllability for $\left(y^0, y^1\right) \in H^1(0,1) \times L^2(0,1)$ at any time $T > 2$ whenever f is continuous with respect to t and Lipschitz with respect to y.

If f is superlinear in y, there will be a problem to act on the blow-up time of a solution of (3.18) in $(0,1)$ where the control u does not act. But once conditions on f ensures the existence of the solution of (3.18) on $(0,1)$, the same technique will provide global controllability.

5. The same problem for the wave equation in higher dimension is an open problem.

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