On the exact controllability of a cylindrical arch

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Abstract

We consider the exact controllability of a linear 2×2 hyperbolic system by one boundary force. The system - of the form $\boldsymbol{y''} + A_M \boldsymbol{y} = 0$ - models the membranal behavior of a cylindrical arch of radius of curvature r clamped on the extremities and submitted to initial condition $(\boldsymbol{y^0}, \boldsymbol{y^1})$. A spectral analysis first exhibits that the eigenvalue $\lambda = 0$ belongs to the essential spectrum of A_M implying the lack of controllability for some $(\boldsymbol{y^0}, \boldsymbol{y^1})$. An Ingham theorem then permits to show the observability of the system in the orthogonal of $KerA_M$ and finally to construct the HUM control for all $(\boldsymbol{y^0}, \boldsymbol{y^1})$ in this orthogonal. We also study the behavior of the control with respect to r and in particular precise the subspace of $KerA_M$ for which the convergence as r goes to infinity is uniform. *To cite this article: F. Ammar-Khodja, G. Geymonat and A. Münch, C. R. Acad. Sci. Paris, Ser. I (2007).*

Résumé

Sur la contrôlabilité exacte d'une arche cylindrique

On considère la contrôlabilité exacte d'un système hyperbolique linéaire 2×2 par une seule force. Le système du type - $\boldsymbol{y''} + A_M \boldsymbol{y} = 0$ - modélise le comportement membranaire d'une arche cylindrique de rayon de courbure r partiellement fixées et soumise au condition initiale $(\boldsymbol{y^0}, \boldsymbol{y^1})$. Une analyse spectrale montre que 0 est dans le spectre essentiel de l'opérateur A_M impliquant la non contrôlabilité uniforme. Un théorème d'Ingham permet alors de montrer l'observabilité du système dans l'orthogonal du noyau de A_M et ainsi de construire le contrôle HUM pour toute donnée dans cet orthogonal. Nous étudions également le comportement du contrôle par rapport à la courbure r^{-1} et précisons pour quel sous-espace de $KerA_M$ les données sont uniformément contrôlables lorsque r^{-1} tend vers zéro. *Pour citer cet article : F. Ammar-Khodja, G. Geymonat and A. Münch, C. R. Acad. Sci. Paris, Ser. I (2007).*

Version française abrégée

Cette note est une contribution à la contrôlabilité frontière à zéro de systèmes du type

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$$\boldsymbol{y_\epsilon''} + A_M \boldsymbol{y_\epsilon} + \epsilon^2 A_F \boldsymbol{y_\epsilon} = 0 \text{ dans } \omega \times (0,T) \quad + \text{Conditions initiales et frontières} \tag{1}$$

qui modélisent la vibration de coques élastiques $\omega \times]-\epsilon/2, \epsilon/2[$ d'épaisseur ϵ (voir [7]). Les opérateurs A_M et A_F respectivement d'ordre 2 et 4 en espace désignent les opérateurs de membrane et de flexion associés à la carte définissant la surface moyenne ω de la coque. Pour ϵ positif fixe, les propriétés de compacité de l'opérateur auto-adjoint $A_\epsilon \equiv A_M + \epsilon^2 A_F$ permettent d'obtenir des résultats de contrôlabilité uniforme (citons [6] dans le cas des coques peu profondes en utilisant la méthode HUM et la technique des mutliplicateurs). En revanche, l'influence du paramètre ϵ sur les propriétés de contrôlabilité reste encore un point ouvert malgré des résultats partiels dans le cas membranaire [2,3]. Lorsque ϵ tend vers zero, la coque exhibe principalement deux régimes : l'un deux est le régime membranaire qui apparait lorsque le noyau de l'opérateur A_M est réduit à zéro (la coque est dite bien-inhibée). Dans ce cas, le terme $\epsilon^2 A_F$ est un terme de perturbation régulière et la limite du champ $\boldsymbol{y_\epsilon}$ est solution de

$$\boldsymbol{y''} + A_M \boldsymbol{y} = 0 \text{ dans } \omega \times (0,T) \quad + \text{Conditions initiales et frontières.} \tag{2}$$

Dans le cas d'une hémi-sphère (coque bien inhibée), [2] exhibe la perte de contrôlabilité lorsque ϵ tend vers zéro. Le spectre de A_ϵ ne vérifie pas la propriété de gap uniforme par rapport à ϵ de sorte que le temps de contrôlabilité T ($T = O(1/\sqrt{\epsilon})$ dans ce cas) n'est pas uniformément borné. La transition coque-membrane implique ainsi une perte de compacité et l'apparition d'un spectre essentiel $\sigma_{ess}(A_M)$ de l'opérateur A_M. [3] étudie le cas général d'opérateur d'ordre mixte tel que A_M et fait le lien entre spectre essentiel et non exacte contrôlabilité.

L'objet de cette note est l'étude de la contrôlabilité frontière du système limite (2) afin d'identifier l'espace des données uniformément contrôlables par rapport à ϵ pour le système (1). L'étude est menée ici dans le cas simplifié mais suffisamment riche d'une arche cylindrique de courbure constante r^{-1} pour lequel $\omega = (0,1)$. Dans ce cas, on cherche un contrôle dirichlet $v \in L^2(0,T)$ agissant sur la composante tangentielle y_1 du déplacement $\boldsymbol{y} = (y_1, y_3)$ solution de (3) tel que (4) ait lieu. L'étude est ramenée à l'analyse du problème homogène adjoint (5), de donnée initiale dans $\boldsymbol{V} \times \boldsymbol{H}$, avec $\boldsymbol{V} = H_0^1(\omega) \times L^2(\omega)$, $\boldsymbol{H} = L^2(\omega) \times L^2(\omega)$ et dont l'opérateur a pour spectre $\{0, r^{-2}, r^{-2} + k^2\pi^2\}, k > 0$. La valeur propre 0 associée à (8) est de multiplicité infinie de sorte que $0 \in \sigma_{ess}(A_M)$ tandis que les valeurs r^{-2} et $r^{-2} + k^2\pi^2$ sont de multiplicité 1 associés à V_0 et V_k respectivement donnés par (9). L'utilisation d'un résultat d'Ingham (voir Théorème 2.1) permet alors d'obtenir l'observabilité de (5) pour toute donnée de $\boldsymbol{V} \times \boldsymbol{H}$ dans $(KerA_M)^\perp$ défini par (14) (Proposition 2.1). L'observabilité uniforme par rapport à r^{-1} est discutée dans la Proposition 2.2. On en conclut, selon la méthode HUM, la contrôlabilité de (3) pour toute donnée dans le dual de $(KerA_M)^\perp$ (voir Théorème 3.1). La perte de contrôlabilité de (2) dans $KerA_M$ est sans conséquence puisque (2) apparait comme limite de (1) en ϵ si et seulement si précisément $KerA_M = \{\boldsymbol{0}\}$! De ce point de vue, le cas bi-dimensionnel d'une coque cylindrique est plus subtile dans la mesure où l'opérateur A_M est de noyau réduit à zéro mais possède un spectre essentiel non vide sous la forme d'un interval de $\mathbb{R}$ associ aux modes de résonance. On renvoit à [1] pour l'analyse théorique et numérique dans ce cas.

1. Statement of the problem

Let $T, r \in \mathbb{R}^+$ and $\omega = (0,1)$. For some initial condition $(\boldsymbol{y^0}, \boldsymbol{y^1})$ in a suitable space, we address in this note the null controllability of the following linear hyperbolic system of order two

$$\begin{cases} \boldsymbol{y''} + A_M \boldsymbol{y} = \boldsymbol{0}, & \text{in} \quad \omega \times (0,T), \\ y_1(0,t) = 0, \quad y_1(1,t) = v(t), & t \in (0,T), \\ (\boldsymbol{y}(\cdot,0), \boldsymbol{y'}(\cdot,0)) = (\boldsymbol{y^0}, \boldsymbol{y^1}), & \text{in} \quad \omega, \end{cases} \qquad A_M \boldsymbol{y} = \begin{pmatrix} -(y_{1,11} + r^{-1} y_{3,1}) \\ r^{-1}(y_{1,1} + r^{-1} y_3) \end{pmatrix}, \tag{3}$$

where the symbol $'$ denotes the differentiation with respect to time. The derivatives with respect to $\xi \in \omega$ is denoted by $y_{1,1}(\xi,t) = \partial y_1(\xi,t)/\partial\xi$. v is a control in $L^2(0,T)$ which drives the variable $\boldsymbol{y} = (y_1, y_3)$ to rest at time of controllability T so that :

$$\boldsymbol{y}(\cdot,T) = \boldsymbol{y'}(\cdot,T) = 0, \quad \text{in} \quad \omega. \tag{4}$$

y_3 is free on the boundary so that there is only one control for the two components of $\boldsymbol{y}$. System (3) models the membranal behavior of an elastic homogeneous arch submitted to the initial position $\boldsymbol{y^0} = (y_1^0, y_3^0)$ and velocity $\boldsymbol{y^1} = (y_1^1, y_3^1)$ at time 0. For all $\xi \in \omega$ and $t \in (0,T)$ $y_1(\xi,t)$ and $y_3(\xi,t)$ denotes the tangential and normal displacement of the arch at the point of curvilinear abscissa ξ and at time t. Finally, r designates the constant radius of curvature of the arch (the curvature is then $C = 1/r$).

2. Adjoint homogeneous system : Spectral property and Observability

2.1. *Spectral property and decomposition*

We introduce the Hilbert spaces $\boldsymbol{V} = H_0^1(\omega) \times L^2(\omega)$, $\boldsymbol{H} = L^2(\omega) \times L^2(\omega)$ and then consider for any $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \boldsymbol{V} \times \boldsymbol{H}$, the unique weak solution $\boldsymbol{\phi} = (\phi_1, \phi_3) \in C(0,T;\boldsymbol{V}) \cap C^1(0,T;\boldsymbol{H})$ of

$$\begin{cases} \boldsymbol{\phi''} + A_M \boldsymbol{\phi} = \boldsymbol{0} \ \text{ in } \ \omega \times (0,T), \\ \phi_1(0,\cdot) = \phi_1(1,\cdot) = 0 \ \text{ in } \ (0,T), \quad (\boldsymbol{\phi}(\cdot,0), \boldsymbol{\phi'}(\cdot,0)) = (\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \ \text{ in } \ \omega. \end{cases} \tag{5}$$

Introducing the bilinear symmetric form $b_M(\boldsymbol{\phi}, \boldsymbol{v}) = (\phi_{1,1} + r^{-1}\phi_3)(v_{1,1} + r^{-1}v_3)$, the weak formulation is

$$\int_\omega A_M \boldsymbol{\phi} \cdot \boldsymbol{v}\, d\xi = \int_\omega b_M(\boldsymbol{\phi}, \boldsymbol{v})\, d\xi - \int_{\partial\omega} (\phi_{1,1} + r^{-1}\phi_3) v_1 \nu \, d\sigma, \quad \forall \boldsymbol{v} \in H^1(\omega) \times L^2(\omega) \tag{6}$$

where ν designates the outward normal to ω. Finally, we denote by E the "natural" energy of the arch

$$E(t,\boldsymbol{\phi}) = \frac{1}{2}\int_\omega (|\boldsymbol{\phi'}|^2 + b_M(\boldsymbol{\phi},\boldsymbol{\phi}))\, d\xi = \frac{1}{2}\int_\omega (|\phi_1'|^2 + |\phi_3'|^2 + (\phi_{1,1} + r^{-1}\phi_3)^2)\, d\xi, \quad \forall t \in (0,T) \tag{7}$$

which is constant along all the trajectories : $E(t,\boldsymbol{\phi}) = E(0,\boldsymbol{\phi}) = \frac{1}{2}\int_\omega (|\boldsymbol{\phi^1}|^2 + b_M(\boldsymbol{\phi^0},\boldsymbol{\phi^0}))d\xi$, for all $t > 0$. Then, we consider the spectral problem ($A_M \boldsymbol{\psi} = \lambda \boldsymbol{\psi}$ in ω, $\psi_1 = 0$ on $\partial\omega$) and obtain the solution $\lambda = 0$ and $\lambda = r^{-2} + k^2\pi^2$ for all $k \geq 0$. The kernel of A_M is of infinite dimension :

$$Ker A_M \equiv \widetilde{V}_0 = \left\{ V_\zeta = (-r^{-1}\xi, \xi_{,1}), \quad \xi \in H_0^1(\omega) \right\}. \tag{8}$$

$\lambda = 0$ then belongs to the essential spectrum $\sigma_{ess}(A_M)$ of A_M (see [4,7]). The eigenfunction associated with $\lambda_0 = r^{-2}$ and $\lambda_k = r^{-2} + k^2\pi^2, k > 0$ of multiplicity one are respectively

$$V_0 = (0,1), \quad V_k = \left(\sin(k\pi\xi), \frac{r^{-1}}{k\pi}\cos(k\pi\xi) \right). \tag{9}$$

Introducing $\mu_0 = r^{-1}$ and $\mu_k = \sqrt{r^{-2} + k^2\pi^2}$, $W_k = (-r^{-1}\sin(k\pi\xi), k\pi\cos(k\pi\xi)) \in Ker A_M$ for all $k \geq 1$, we may then expand the solution of the dynamical system (5) in term of a Fourier serie as follows

$$\boldsymbol{\phi}(\xi,t) = \sum_{k=1}^{\infty} (a_k + b_k t) W_k + (A_0 \cos(\mu_0 t) + B_0 \sin(\mu_0 t)) V_0 + \sum_{k=1}^{\infty} (A_k \cos(\mu_k t) + B_k \sin(\mu_k t)) V_k \tag{10}$$

for any function $\zeta_k \in L^2(\omega)$ such that $\int_\omega \zeta_k(\xi) d\xi = 0$. At time $t = 0$, we have

$$\boldsymbol{\phi^0}(\xi) = \sum_{k=1}^{\infty} a_k W_k + A_0 V_0 + \sum_{k=1}^{\infty} A_k V_k, \quad \boldsymbol{\phi^1}(\xi) = \sum_{k=1}^{\infty} b_k W_k + \mu_0 B_0 V_0 + \sum_{k=1}^{\infty} \mu_k B_k V_k \tag{11}$$

assumed in $\boldsymbol{V}$ and $\boldsymbol{H}$ respectively, i.e. such that for all $r > 0$,

$$A_0^2 < \infty, \quad \sum_{k=1}^{\infty}(k\pi)^2(r^{-2}a_k^2 + A_k^2) < \infty, \quad \sum_{k=1}^{\infty}((k\pi)^2 a_k^2 + r^{-2}A_k^2) < \infty, \tag{12}$$

and

$$\sum_{k=1}^{\infty}(r^{-2}b_k^2 + \mu_k^2 B_k^2) \leq \infty, \quad (\mu_0 B_0)^2 + \sum_{k=1}^{\infty}\left((k\pi)^2 b_k^2 + (\mu_k B_k)^2 \frac{r^{-2}}{(k\pi)^2}\right) < \infty. \tag{13}$$

Observe that if $\boldsymbol{\phi^0}, \boldsymbol{\phi^1} \in KerA_M$ then $\boldsymbol{\phi}(\cdot,t) \in KerA_M$ for all t. Similarly, if $\boldsymbol{\phi^0} \in KerA_M$ and $\boldsymbol{\phi^1} = (0,0)$, then $\boldsymbol{\phi}(\cdot,t) = \boldsymbol{\phi^0}$ for all $t > 0$. We now introduce the orthogonal of the subspace $\widetilde{V}_0$ for the L^2-norm: $\widetilde{V}_0^{\perp} = \{\boldsymbol{\psi} = (\psi_1,\psi_3) \in \boldsymbol{V}, \int_\omega (\psi_1\phi_1 + \psi_3\phi_3)d\xi = 0,\ \forall(\phi_1,\phi_3) \in \widetilde{V}_0\}$. From the definition of $\widetilde{V}_0$, we obtain that

$$\widetilde{V}_0^{\perp} = \{(\psi_1,\psi_3) \in \boldsymbol{V},\ r^{-1}\psi_1 + \psi_{3,1} = 0 \text{ in } H^{-1}(\omega)\}. \tag{14}$$

We check that $V_0 \subset \widetilde{V}_0^{\perp}$ and $V_k \subset \widetilde{V}_0^{\perp}$, for all $k > 0$ and obtain easily that $\widetilde{V}_0^{\perp}$ is generated by V_0 and V_k for all $k > 0$, assuming (12) and (13).

2.2. *Observability inequality*

The control property of the system (3) is related to the existence of two positive constants C_1 and C_2 such that, for all $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \boldsymbol{V} \times \boldsymbol{H}$ and $T > 0$ large enough

$$C_1 \|(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})\|^2_{\boldsymbol{V}\times\boldsymbol{H}} \leq \int_0^T b_M(\boldsymbol{\phi},\boldsymbol{\phi})(1,t)dt \leq C_2 \|(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})\|^2_{\boldsymbol{V}\times\boldsymbol{H}}. \tag{15}$$

Since $\lambda = 0 \in \sigma(A_M)$, the left inequality (called the observability inequality) can not hold for all $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \boldsymbol{V} \times \boldsymbol{H}$. It suffices to take $\boldsymbol{\phi^0}, \boldsymbol{\phi^1} \in KerA_M$ so that $b_M(\boldsymbol{\phi},\boldsymbol{\phi}) = 0$. Introducing the spaces $\widetilde{V}_{0,V}^{\perp} = \boldsymbol{V} \cap \widetilde{V}_0^{\perp}$ and $\widetilde{V}_{0,H}^{\perp} = \boldsymbol{H} \cap \widetilde{V}_0^{\perp}$, we have the following result :

Proposition 2.1 *Let $r > 0$ and $\gamma^\star(r) = \min(2r^{-1}, \sqrt{r^{-2}+\pi^2} - r^{-1})$. For all time $T > T^\star(r) \equiv 2\pi/\gamma^\star(r)$, there exist two strictly positive constants $C_1(r)$ and $C_2(r)$ such that (15) holds for all $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \widetilde{V}_{0,V}^{\perp} \times \widetilde{V}_{0,H}^{\perp}$.* ■

Since the energy $E(0,\boldsymbol{\phi})$ defines a norm over $\widetilde{V}_{0,V}^{\perp} \times \widetilde{V}_{0,H}^{\perp}$ (as a consequence of the Korn's inequality, see [7]) and since the control is only active at $\xi = 1$, (15) is equivalent to the existence of two positive constants C_1 and C_2 such that

$$C_1(r)E(0,\boldsymbol{\phi}) \leq \int_0^T (\phi_{1,1}(1,t) + r^{-1}\phi_3(1,t))^2 dt \leq C_2(r)E(0,\boldsymbol{\phi}), \quad \forall(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \widetilde{V}_{0,V}^{\perp} \times \widetilde{V}_{0,H}^{\perp}. \tag{16}$$

These inequalities may be obtained from a direct application of Ingham's theorem on Nonharmonic series (see [5] page 59) :

Theorem 2.1 (Ingham) *Let $K \in \mathbb{Z}$ and $(w_k)_{k\in K}$ be a family of real numbers satisfying the uniform gap condition $\gamma = \inf_{k\neq n}|w_k - w_n| > 0$. If I is a bounded interval of length $|I| > 2\pi/\gamma$, then there exist two positives constants C_1 and C_2 such that $C_1\sum_{k\in K}|x_k|^2 \leq \int_I |x(t)|^2 dt \leq C_2 \sum_{k\in K}|x_k|^2$ for all functions given by the sum $x(t) = \sum_{k\in K} x_k e^{iw_k t}$ with square-summable complex coefficients x_k.* ■

Proof of proposition 2.1. On one hand, we compute that

$$E(0,\boldsymbol{\phi}) = \frac{\mu_0^2}{2}(A_0^2 + B_0^2) + \frac{1}{4}\sum_{k=1}^{\infty}\frac{\mu_k^4}{k^2\pi^2}(A_k^2 + B_k^2). \tag{17}$$

On the other hand, we have

$$\begin{aligned}\phi_{1,1}(1,t)+r^{-1}\phi_3(1,t) &= r^{-1}(A_0\cos(\mu_0 t)+B_0\sin(\mu_0 t))+\sum_{k=1}^{\infty}\frac{(-1)^k\mu_k^2}{k\pi}(A_k\cos(\mu_k t)+B_k\sin(\mu_k t))\\ &= \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k\mu_k^2}{k\pi}(A_k+iB_k)e^{-i\mu_k t}+\frac{r^{-1}}{2}(A_0+iB_0)e^{-i\mu_0 t}\\ &\quad +\frac{r^{-1}}{2}(A_0-iB_0)e^{i\mu_0 t}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k\mu_k^2}{k\pi}(A_k-iB_k)e^{i\mu_k t}.\end{aligned} \tag{18}$$

We then apply Theorem 2.1 with $I=(0,T)$ and the sequence $w=(\cdots,-\mu_2,-\mu_1,-\mu_0,\mu_0,\mu_1,\mu_2,\cdots)$ to obtain that the existence of two positives constants $C_1(r)$ and $C_2(r)$ such that (16) holds for all $r>0$ under the condition $T>2\pi/\gamma$ with $\gamma=\min(\mu_0-(-\mu_0),\inf_{k\in\mathbb{N}}|\mu_k-\mu_{k-1}|)$. From the concavity of the square root function, we deduce that $|\mu_1-\mu_0|\le|\mu_{k+1}-\mu_k|$ for all $k\ge 0$ and then that $\gamma=\gamma^\star(r)$. □

The lower bound value $T^\star$ of observability may be precised as follows

$$T^\star(r)=\frac{\pi}{r^{-1}}\mathcal{X}_{(r^{-1}\le\pi^2/8)}+\frac{2\pi}{\sqrt{r^{-2}+\pi^2}-r^{-1}}\mathcal{X}_{(r^{-1}>\pi^2/8)} \tag{19}$$

and reaches its minimum for $r^{-1}=\pi^2/8$ for which $T^\star=8/\pi$. We observe that $T^\star$ goes to infinity as r^{-1} goes to infinity (in practice, r^{-1} is not greater than 2π which corresponds to the circle). The time of controllability also blows up as the curvature r^{-1}. This is due to the eigenvalue λ_0 which vanishes as r^{-1} goes to zero. Precisely, if we restrict the initial condition $(\boldsymbol{\phi^0},\boldsymbol{\phi^1})$ to be in V_0 so that $\phi_{1,1}(1,t)+r^{-1}\phi_3(1,t)=r^{-1}(A_0\cos(\mu_0 t)+B_0\sin(\mu_0 t))$ then we obtain explicitly that the constant $C_1(r)=2\min(T,T^3r^{-2}/3)$ goes to zero as r^{-1} goes to zero unless $T=O(r)$. Consequently, the observability inequality is not uniform with respect to r^{-1} for an arbitrarily shallow arch. The observability is uniform if $B_0=0$, i.e. in this case if $\phi_3^1=0$. If we denote V_K the space generated by V_k for all $k>0$, and $V_{K,H}=V_K\cap H$, $V_{K,V}=V_K\cap V$, we have the following result :

Proposition 2.2 *[Uniform observability w.r.t. r^{-1}] Let $r>0$ and $\gamma^{\star\star}(r)=\sqrt{r^{-2}+4\pi^2}-\sqrt{r^{-2}+\pi^2}$. For all $T>T^{\star\star}(r)\equiv 2\pi/\gamma^{\star\star}(r)$, there exist two positive constants C_1 and C_2 independent of r such that (15) holds for all $(\boldsymbol{\phi^0},\boldsymbol{\phi^1})\in V_{K,V}\times V_{K,H}$.* ■

In this case, the observability is uniform with respect to r. The lower bound $T^{\star\star}$ is now a monotonous increasing function of r^{-1} such that $\lim_{r^{-1}\to 0}T^{\star\star}(r)=2$, lower bound for the wave equation controlled at one extremity. We also remark that $T^{\star\star}(r)<T^\star(r)$ for all r.

3. Exact Controllability

We now apply the Hilbert Uniqueness Method and then assume that $(\boldsymbol{y^0},\boldsymbol{y^1})\in\boldsymbol{H}\times\boldsymbol{V'}$ where $\boldsymbol{V'}=H^{-1}(\omega)\times L^2(\omega)$. Formal integrations by part provide that v is a control for the system (3) if and only if

$$\int_0^T(\phi_{1,1}+r^{-1}\phi_3)(1,t)v(t)dt=<(\boldsymbol{\phi^0},\boldsymbol{\phi^1}),(\boldsymbol{y^1},-\boldsymbol{y^0})>_{\boldsymbol{V}\times\boldsymbol{H},\boldsymbol{V'}\times\boldsymbol{H}}. \tag{20}$$

We then introduce the continuous and convex functional $\mathcal{J}:\boldsymbol{V}\times\boldsymbol{H}\to\mathbb{R}$ defined by

$$\mathcal{J}(\boldsymbol{\phi^0},\boldsymbol{\phi^1})=\frac{1}{2}\int_0^T(\phi_{1,1}+r^{-1}\phi_3)^2(1,t)dt-<(\boldsymbol{\phi^0},\boldsymbol{\phi^1}),(\boldsymbol{y^1},-\boldsymbol{y^0})>_{\boldsymbol{V}\times\boldsymbol{H},\boldsymbol{V'}\times\boldsymbol{H}}. \tag{21}$$

If $\mathcal{J}$ is coercive, then $\mathcal{J}$ admits a unique minimum and the HUM control of minimal L^2-norm is given by $v=(\phi_{1,1}+r^{-1}\phi_3)(1,\cdot)$. Since $\mathcal{J}$ is only coercive on the orthogonal of $KerA_M$ (provided T be large enough), we first observe that the minimization of $\mathcal{J}$ is over $\widetilde{V}_{0,V}^{\perp} \times \widetilde{V}_{0,H}^{\perp}$. Furthermore, if $\boldsymbol{y^0}, \boldsymbol{y^0}$ belongs to $KerA_M$, then $< (\boldsymbol{\phi^0}, \boldsymbol{\phi^1}), (\boldsymbol{y^1}, -\boldsymbol{y^0}) >_{\boldsymbol{V}\times\boldsymbol{H},\boldsymbol{V'}\times\boldsymbol{H}} = 0$ for all $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1}) \in \widetilde{V}_{0,\boldsymbol{V}}^{\perp} \times \widetilde{V}_{0,\boldsymbol{H}}^{\perp}$ and from (20), the control is zero; in this case, the solution $\boldsymbol{y}$ remains in $KerA_M$ for all $t > 0$ but is not controlled ! Consequently, we have to enforce that $\boldsymbol{y^0}$ and $\boldsymbol{y^1}$ be in the dual of $\widetilde{V}_{0,\boldsymbol{H}}^{\perp}$ and $\widetilde{V}_{0,\boldsymbol{V}}^{\perp}$ respectively. Summarizing, we have the following result:

Theorem 3.1 *Let $r > 0$. For any $T > T^{\star}(r)$ and any initial data $(\boldsymbol{y^0}, \boldsymbol{y^1}) \in (\widetilde{V}_{0,\boldsymbol{H}}^{\perp})' \times (\widetilde{V}_{0,\boldsymbol{V}}^{\perp})'$, there exists a control function $v \in L^2(0,T)$ which drives to rest at time T the solution $\boldsymbol{y}$ of (3) associated with $(\boldsymbol{y^0}, \boldsymbol{y^1})$. Moreover, the control of minimal L^2-norm is given by $v = (\phi_{1,1} + r^{-1}\phi_3)(1, \cdot)$ where $\boldsymbol{\phi}$ is solution of (5) and associated with $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})$ minimum of $\mathcal{J}$ defined by (21) over $\widetilde{V}_{0,\boldsymbol{V}}^{\perp} \times \widetilde{V}_{0,\boldsymbol{H}}^{\perp}$.* ■

Similarly, from Proposition 2.2, we obtain directly the following uniform controllability result (we refer to [1] for the details) :

Theorem 3.2 (Uniform controllability w.r.t. r^{-1}) *Let $r > 0$. For any $T > T^{\star\star}(r)$ and any initial data $(\boldsymbol{y^0}, \boldsymbol{y^1}) \in (V_{K,\boldsymbol{H}})' \times (V_{K,\boldsymbol{V}})'$, there exists a control function $v \in L^2(0,T)$ which drives to rest at time T the solution $\boldsymbol{y}$ of (3) associated with $(\boldsymbol{y^0}, \boldsymbol{y^1})$. Moreover, the control of minimal L^2-norm is given by $v = (\phi_{1,1} + r^{-1}\phi_3)(1, \cdot)$ where $\boldsymbol{\phi}$ is solution of (5) and associated with $(\boldsymbol{\phi^0}, \boldsymbol{\phi^1})$ minimum of $\mathcal{J}$ defined by (21) over $(V_{K,\boldsymbol{V}}^{\perp}) \times (V_{K,\boldsymbol{H}}^{\perp})$. Finally, this control converges weakly in $L^2(0,T)$ as r^{-1} goes to zero toward the control of minimal L^2-norm which drives to rest the solution of the wave equation associated with the weak limit of (y_0^1, y_1^1) as r^{-1} goes to zero.* ■

4. Concluding remarks and extension

From a mathematical viewpoint, the system (3) is thus null controllable by one Dirichlet force only for the data in the orthogonal of $KerA_M$. In $(KerA_M)^{\perp}$ the unknown y_1 and y_3 are connected by the relation (14) which explains actually why only one control is sufficient. A similar phenomenon appears if we consider a Neumann control by acting on the longitudinal strain $y_{1,1} + r^{-1}y_3$. The analysis also reveals that the time of controllability increases linearly with the curvature. On a mechanical point of view, this is not restrictive at all since the system (3) appears through the limit process in ϵ in (1) if and only if the kernel is precisely reduced to zero. Consequently, any (membrane dominated) arch is asymptotically null controllable (with respect to ϵ). From this perspective, the 2D situation of a cylindrical shell for which the kernel of A_M is reduced to zero is more subtle: in this case, the essential spectrum is a non empty interval containing zero revealing (using [3]) that any (membrane dominated) cylindrical shell is not asymptotically (w.r.t. ϵ) uniformly null controllable. We refer to [1] for the theoretical and numerical analysis.

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