2d density-dependent Leray's problem for inhomogeneous fluids

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Dedicated to the memory of Prof. Hebe de Azevedo Biagioni

Abstract

We formulate the Leray's problem for inhomogeneous fluids in a two-dimensional domain and prove the existence of a solution. The given density is assumed to be continuous and the obtained solution attains its value in the supremum norm.

Key words. stationary Navier-Stokes equations, incompressible flow, inhomogeneous fluid.
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1 Introduction

Throughout Ω will denote an *admissible domain* of the plane in the sense of Amick [1], i.e. a domain of the plane with two straight unbounded channels. More precisely, Ω is an open and simply connected set of $\mathbb{R}^2$ with a smooth boundary Γ and such that $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ where Ω_0 is a bounded set and, in possibly different coordinate systems, $\Omega_1 = \{(x, y) \in \mathbb{R}^2 : x < 0, -d_1 < y < d_1\}$ and $\Omega_2 = \{(x, y) \in \mathbb{R}^2 : x > 0, -d_2 < y < d_2\}$ for given constants $d_1, d_2 > 0$. Cf. [1, Definition 1.1]. Also in [1] the reader can see a typical draw of Ω; [1, Figure 1].

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We consider a stationary inhomogeneous incompressible planar fluid in Ω, where 'inhomogeneous' stands for variable density. The mass density, velocity, pressure and the given constant viscosity of the fluid are denoted, respectively, by ρ, $\mathbf{v} = (v_1, v_2)$, p, and ν. The stationary Navier-Stokes equations describing such a fluid are the following:

$$\left\{ \begin{array}{l} \nu\Delta\mathbf{v} = \rho(\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p \\ \nabla\cdot\mathbf{v} = 0, \quad \nabla\cdot(\rho\mathbf{v}) = 0\,. \end{array} \right. \tag{1}$$

The first equation represents the conservation of momentum and the second and third equations represent the incompressibility of the fluid and the conservation of mass, respectively.

Inhomogeneous fluids are important to be investigated in both mathematical and physical aspects. They can model, for instance, stratified fluids, see e.g. [7]. From the mathematical point of view, some challenging questions are pertinent to domains with unbounded channels even for the case of constant density. For instance, the solvability of the nonlinear Leray's problem under no restriction on the size of the fluid fluxes through the cross sections of the channels is still an open problem. It consists of finding a solution of (1) such that the fluid flows are Poiseuille flows (i.e. parallel flows) at large distances. This problem seems to have been proposed, in the 1950s, by Jean Leray to Olga A. Ladyzhenskaya, cf. [1, p. 476]. Despite the effort made by brilliant mathematicians, see e.g. [5], up to now its solution is known only in the case of Poiseuille flows with small fluxes, a result due to Charles J. Amick [1, Theorem 3.8]. Not surprisingly, the main difficulty in solving the problem is to deal with the nonlinear term in the Navier-Stokes equations. This difficulty is overcome by seeking a solution with the velocity field $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{u} + \mathbf{a}$, for a new unknown $\mathbf{u}$, where $\mathbf{a}$ is a suitable extension of the given Poiseuille flows. It turns out that the nonlinear term can be estimated by the fluxes of the Poiseuille flows [1], thus the result comes off under the restriction that these fluxes are small, in comparison with the viscosity of the fluid. In the case of inhomogeneous fluids, besides the given values for the fluid velocity at the ends of the channels, we give values for the density in the 'incoming channel' (i.e. in the channel where the fluid is incoming).

We extend Amick's theorem [1, Theorem 3.8], in the two dimensional case, to inhomogeneous fluids with a continuous density. Our solution is based on the streamline formulation, an approach strictly inherent to the two dimensional case and which was used first by N.N. Frolov [2] to solve the boundary value problem for inhomogeneous fluid in a bounded domain. Indeed, the density ρ we obtain is of the form $\rho = \omega(\psi)$ where ψ is a streamline function, i.e. a scalar function such that $\nabla^\perp\psi = \mathbf{v}$ ($\nabla^\perp\psi := (-\partial_2\psi, \partial_1\psi)$), and ω is some scalar function connected to the given values for the density and velocity at the end of the incoming channel; see (26).

We use the Sobolev embedding $W^{2,2}(\Omega') \subset C(\overline{\Omega'})$, where Ω' is any smooth bounded domain contained in Ω, to get the decay of the solution to the given Poiseiulle flow $\mathbf{v}_i$ at infinity in the supremum norm (see (13)) and, as a consequence, the density decays to the given density ρ_1 at infinity in the supremum norm as well; see (10). Our solution, which is given in Theorem 1, has some extra properties. We compute explicitly the flux β_i of the momentum $\rho\mathbf{v}$ on the channel Ω_i, namely, $\beta_i = (-1)^i \int_{\psi(x,\, d_1)}^{\psi(x,-d_1)} \rho_1\left(\psi_1^{-1}(s)\right) ds$, where ψ_1 is a streamline function associated with $\mathbf{v}_1$, i.e. $\nabla^\perp\psi_1 = \mathbf{v}_1$; see (11) and Remark 1. Moreover, the density satisfies the 'maximum principle' $\sup|\rho| \le \sup|\rho_1|$ and we compute for the velocity an analogue estimate one has when the density is constant, yielding the explicit dependence on the given density; see (12).

Besides this Introduction, this paper contains the next Section in which we formulate the *density-dependent Leray problem* and prove the existence of a weak solution in the case of given Poiseuille flows with small fluxes.

2 Density-dependent Leray's problem

Together with equations (1) we take the following boundary conditions. First we assume that the fluid is non slippery on the boundary of Ω, i.e.

$$\mathbf{v} = 0 \quad \text{on} \ \ \Gamma\,. \tag{2}$$

Second, the fluid flow is a Poiseuille flow at the end of the channel Ω_i, $i = 1, 2$, i.e.

$$\lim_{x\to-\infty} \mathbf{v} = \mathbf{v}_1 \quad \text{and} \quad \lim_{x\to\infty} \mathbf{v} = \mathbf{v}_2 \tag{3}$$

where $\lim_{\,x\to(-1)^i\infty} \mathbf{v}$, in the coordinate systems of Ω_i, stands for $\lim_{x\to(-1)^i\infty} \mathbf{v}(x, y)$ with $(x, y) \in \Omega_i$ and $\mathbf{v}_i = \mathbf{v}_i(y)$, $-d_i < y < d_i$, is a given Poiseuille flow in Ω_i; see (6) below. We suppose that the fluid is incoming in Ω_1 and outgoing in Ω_2, i.e. $\mathbf{v}_1$ and $-\mathbf{v}_2$ are pointing toward Ω_0. Since the conservation of mass equation, $\nabla\cdot(\rho\mathbf{v}) = 0$, for smooth solutions is a transport equation with transport vector given by $\mathbf{v}$, it is natural to give the density only at the end of the channel Ω_1 where the fluid is incoming. Then we set

$$\lim_{x\to-\infty} \rho = \rho_1, \tag{4}$$

where ρ_1 is a given function in $C_b(\Sigma_1)$, $\Sigma_i := (-d_i, d_i)$, $i = 1, 2$. Here and throughout, if X is a topological space, $C_b(X)$ will denote the space of bounded and continuous functions defined on X, endowed with the supremum norm $||f||_{C_b(X)} := \sup_{x\in X} |f(x)|$. We call the problem (1)-(4) *density-dependent Leray's problem.*

Before state our main result, Theorem 1 below, we need some more notations. First, let

$$\alpha_i = \int_{\Sigma_i(x)} \mathbf{v}_i \cdot \mathbf{n}_i, \qquad i = 1, 2 \tag{5}$$

(the flux of the Poiseuille flow $\mathbf{v}_i$ in Ω_i) where $\mathbf{n}_i$ is the unit normal to $\Sigma_i(x) := \{x\} \times (-d_i, d_i)$ (a cross section of Ω_i) pointing toward $|x| = \infty$, i.e. pointing to the exterior of Ω_0; see Remark 1 below. In the coordinates systems of Ω_i, we have $\mathbf{n}_i = (\pm 1, 0)$ and

$$\mathbf{v}_i \equiv \mathbf{v}_i(y) = (\theta_i(y), 0) \quad \text{for} \quad \theta_i(y) = \pm \frac{3}{4d_i^3}\alpha_i(d_i^2 - y^2), \quad y \in (-d_i, d_i) \tag{6}$$

where the sign $\pm$ is $-$ if $i = 1$ and $+$ if $i = 2$; cf. [1, p.485]. Because the incompressibility equation $\nabla \cdot \mathbf{v} = 0$, condition (2) and Divergence Theorem, we assume the compatibility condition $\alpha_1 + \alpha_2 = 0$, i.e. $\alpha_2 = -\alpha_1$. Since the fluid is incoming in Ω_1 (and outgoing in Ω_2) we have $\alpha_1 < 0$ (and so $\alpha_2 > 0$) which is an accordance with the direction of $\mathbf{n}_i$, i.e. the incoming given velocity $\mathbf{v}_1$ in Ω_1 is pointing to the opposite direction of $\mathbf{n}_1$ and $\mathbf{v}_2$ in Ω_2 is pointing to the same direction of $\mathbf{n}_2$.

Let $\mathbf{H}_{k,loc}(\overline{\Omega})$ be the space of vector fields $\mathbf{v}$ in Ω such that $\mathbf{v}$ belongs to the Sobolev space $W^{k,2}(\Omega')$, for any open bounded subset Ω' of Ω, $\mathbf{v}$ is divergent free, i.e. $\nabla \cdot \mathbf{v} = 0$, and whose derivatives up to order $k-1$ have null trace on Γ. Let also $\mathcal{V}$ be the space of the vector fields $\mathbf{\Phi}$ in $C_0^\infty(\Omega)$ (the underscript '0' stands for compact support, i.e. the support set of $\mathbf{\Phi}$ is a compact set contained in Ω) and $\mathbf{\Phi}$ is divergent free.

Our main result is the following Theorem.

Theorem 1 *Assume that $\rho_1 \in C_b(\Sigma_1)$ and let $l := ||\rho_1||_{C_b(\Sigma_1)}$. Then there is a constant $c = c(\Omega) > 0$ such that for $c\alpha_2 l < \nu$, the problem (1)-(4) has a weak solution $(\rho, \mathbf{v}) \in C_b(\Omega) \times \mathbf{H}_{1,loc}(\overline{\Omega})$, in the following sense:*

i.

$$\nu \int_\Omega \nabla \mathbf{v} \cdot \nabla \mathbf{\Phi} dx = \int_\Omega \rho(\mathbf{v} \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{v} dx, \tag{7}$$

for all $\mathbf{\Phi} = (\Phi_1, \Phi_2)$ in $\mathcal{V}$, where $\nabla \mathbf{v} \cdot \nabla \mathbf{\Phi} := \nabla v_1 \cdot \nabla \Phi_1 + \nabla v_2 \cdot \nabla \Phi_2$ and $\mathbf{v} \cdot \nabla \mathbf{\Phi} := (\mathbf{v} \cdot \nabla \Phi_1, \mathbf{v} \cdot \nabla \Phi_2)$,

ii.

$$\int_\Omega \rho \mathbf{v} \cdot \nabla \varphi dx = 0 \qquad \textit{for all} \quad \varphi \quad \textit{in} \quad C_0^\infty(\Omega), \tag{8}$$

iii.

$$\mathbf{v} - \mathbf{v}_i \in W^{2,2}(\Omega_t^c), \quad i = 1, 2, \qquad \textit{for some } t > 0, \tag{9}$$

where $\Omega_t^c := \cup_{i=1}^2 \Omega_{i,t}^c, \quad \Omega_{i,t}^c := \{(x, y) \in \Omega_i \,;\, |x| > t\}$; *and*

iv.

$$\lim_{x \to -\infty} ||\rho(x, \cdot) - \rho_1||_{C_b(\Sigma_1)} = 0. \tag{10}$$

Furthermore, the flux of $\mathbf{v}$ in Ω_i is equal to α_i and the flux β_i of the momentum $\rho\mathbf{v}$ in Ω_i (see Remark 1 below) can be written as

$$\beta_i = \pm \int_{\alpha_1}^0 \rho_1 \left(\psi_1^{-1}(s)\right) ds, \tag{11}$$

where ψ_1 *is a stream function associated with* $\mathbf{v}_1$*, i.e.* ${\psi_1}' = -\theta_1$*; more precisely, we take* $\psi_1(y) = -\int_{-d_i}^{y} \theta_1(y')\,dy'$*,* $y \in \Sigma_i$*. Finally, we have* $||\rho||_{C_b(\Omega)} \le l$ *and*

$$||\nabla(\mathbf{v}-\mathbf{v}_i)||_{L^2(\Omega_i)} + ||\nabla \mathbf{v}||_{L^2(\Omega_0)} \le C|\alpha_i| \left(1 + \frac{\nu + |\alpha_i| l}{\nu - c|\alpha_i| l}\right) \tag{12}$$

for some other constant $C = C(\Omega)$*.*

Equations (7) and (8) are just the weak formulations (in the sense of distributions) of the conservation of momentum and mass equations, respectively, i.e. just multiply these equations by the indicated (in (7) and (8)) test functions and formally integrate them by parts. In equation (7) the pressure p is canceled out because the (vector valued) test functions $\mathbf{\Phi}$ are divergent free. It is classical that we can recover the pressure from (7); see e.g. [9, Propositions I.1.1 and I.1.2, p. 14] or [3, Corollary III.5.2]. The incompressibility equation, $\nabla \cdot \mathbf{v} = 0$, is inserted in the space $\mathbf{H}_{1,loc}(\overline{\Omega})$. Condition (9) implies that $\mathbf{v} \in C_b(\Omega_t^c)$ and

$$\lim_{|x|\to\infty} ||\mathbf{v}(x,\cdot) - \mathbf{v}_i||_{C_b(\Sigma_i)} = 0, \qquad i = 1, 2. \tag{13}$$

Indeed, since $\Omega_{i,t}^c$ is bounded in one direction, from the Sobolev Imbedding Theorem, we have $\mathbf{v} - \mathbf{v}_i \in C_b(\Omega_{i,t}^c)$ and there is a constant k, independent of $|x| > t+1$, such that

$$||\mathbf{v}(x,\cdot) - \mathbf{v}_i||_{C_b(\Sigma_i)} \le ||\mathbf{v} - \mathbf{v}_i||_{C_b(\Omega_{i,|x|-1}^c)} \le k ||\mathbf{v} - \mathbf{v}_i||_{W^{2,2}(\Omega_{i,|x|-1}^c)};$$

thus

$$\lim_{|x|\to\infty} ||\mathbf{v}(x,\cdot) - \mathbf{v}_i||_{C_b(\Sigma_i)} \le k \lim_{|x|\to\infty} ||\mathbf{v} - \mathbf{v}_i||_{W^{2,2}(\Omega_{i,|x|-1}^c)} = 0\,. \tag{14}$$

Before proving Theorem 1 we give some important remarks.

Remark 1 Since $\mathbf{v} \in W^{1,2}_{loc}(\overline{\Omega})$, it is classical that $\mathbf{v}$ has a trace on the cross section $\Sigma_i(x)$, so the flux $\int_{\Sigma_i(x)} \mathbf{v} \cdot \mathbf{n}_i$ of $\mathbf{v}$ through the cross section $\Sigma_i(x)$ is well defined. The same conclusion holds true for the flux of the momentum $\beta_i := \int_{\Sigma_i(x)} \rho \mathbf{v} \cdot \mathbf{n}_i$, but here we use that $\rho \mathbf{v}$ belongs to $L^2(\Omega')$ for each bounded subset Ω' of Ω and $\nabla \cdot (\rho \mathbf{v}) = 0$ in the sense of distributions (see (8)), then $\rho \mathbf{v}$ has a normal trace on each cross section $\Sigma_i(x)$, see e.g. [9, Theorem I.1.2]. Moreover, since $\nabla \cdot \mathbf{v} = \nabla \cdot (\rho \mathbf{v}) = 0$ in Ω, the fluxes $\int_{\Sigma_i(x)} \mathbf{v} \cdot \mathbf{n}_i$ and β_i are constant with respect to x ($-\infty < x < 0$ if $i = 1$ and $0 < x < \infty$ if $i = 2$). We also note that in the local coordinates of Ω_i we have $\int_{\Sigma_i(x)} \mathbf{v} \cdot \mathbf{n}_i = \pm \int_{-d_i}^{d_i} v_1(x,y)\,dy$ and $\beta_i = \pm \int_{-d_i}^{d_i} (\rho v_1)(x,y)\,dy$.

Remark 2 If $\mathbf{v} = \nabla^\perp \psi$ and $\rho = \omega(\psi)$ for some $\omega \in C_b(\mathbb{R})$, the equation $\nabla \cdot (\rho \mathbf{v}) = 0$ is automatically satisfied in the weak sense. More precisely, we have (8) if $\rho = \omega(\psi)$ with $\omega \in C_b(\mathbb{R})$ and $\psi \in W^{2,2}_{loc}(\overline{\Omega})$. Indeed, for a smooth ω it is straightforward to obtain $\nabla \cdot (\omega(\psi) \nabla^\perp \psi) = 0$ in the classical sense, and for $\omega \in C_b(\mathbb{R})$, we can obtain the result by passing to the limit in (8) with $\rho = \omega^\epsilon(\psi)$ and $\mathbf{v} = \nabla^\perp\psi$, with ϵ tending to zero, where ω^ϵ is a sequence of standard mollifications of ω. In this passage to the limit we use the compact embedding of $W^{2,2}(\Omega')$ into $C_b(\Omega')$ for a smooth bounded domain Ω' in Ω containing the support of the test function $\varphi \in C_0^\infty(\Omega)$. Indeed, let $\varphi \in C_0^\infty(\Omega)$ and $\Omega' \supset \mathrm{spt.}\varphi$, where spt.$\varphi$ stands for the support set of φ. Then $\nabla\cdot(\omega^\epsilon(\psi)\mathbf{v}) = (\omega^\epsilon)'(\psi)\nabla\psi\cdot\nabla^\perp\psi = 0$ and

$$\begin{aligned} & |\int_\Omega \rho\mathbf{v}\cdot\nabla\varphi\,dx| \\ = \; & |\int_{\Omega'} \omega(\psi)\mathbf{v}\cdot\nabla\varphi\,dx - \int_{\Omega'}\omega^\epsilon(\psi)\mathbf{v}\cdot\nabla\varphi\,dx| \\ \leq \; & \left(\sup_{x\in\overline{\Omega'}}|\omega(\psi(x)) - \omega^\epsilon(\psi(x))|\right)|\Omega'|^{1/2}||\mathbf{v}||_{W^{1,2}(\Omega')}||\varphi||_{L^\infty(\Omega')} \end{aligned}$$

tends to zero as $\epsilon \to 0$, because ω^ϵ tends to ω as $\epsilon \to 0$ uniformly on compact sets and $W^{2,2}(\Omega') \subset C_b(\overline{\Omega'})$, so $\psi(x)$ lies in a compact set for $x \in \overline{\Omega'}$.

Remark 3 If $\mathbf{v} = \nabla^\perp\psi$ then $\psi|\Gamma$ is constant on each component of Γ, because $\mathbf{v}|\Gamma \equiv 0$; see (2). In particular, $\psi(x, \pm d_i)$ is independent of x. Then we may fix arbitrarily the constant value $\psi(x, -d_i)$, since $\mathbf{v}$ does not change by modifying ψ by a constant. We set $\psi(x, -d_i) \equiv 0$. Then from $\alpha_i = \int_{\Sigma_i(x)}\mathbf{v}\cdot\mathbf{n}_i = \pm\int_{-d_i}^{d_i} v_1(x,y)\,dy = \pm\int_{-d_i}^{d_i}(-\psi_y(x,y))\,dy$ we have $\pm\psi(x,d_i) = -\alpha_i$.

To prove Theorem 1, let $\Omega_t = \cup_{i=2}^2\{(x,y)\in\Omega_i\,;\,|x|<t\}\cup\Omega_0$ $(t>0)$ and $\mathbf{a}$ be a smooth vector field in $\mathbf{H}_{1,loc}(\overline{\Omega})$ such that it coincides with the Poiseuille flow $\mathbf{v}_i$ in $\Omega_{i,t}^c$ for some $t > 0$ and

$$||\nabla\mathbf{a}||_{L^2(\Omega_t)} \leq c_t\alpha_2 \tag{15}$$

for some constant c_t depending only on t and Ω. For a construction of $\mathbf{a}$, see [1, §3.1/Theorem 3.3(b)] and [4, Lemma XI.3.1]. First we look for a weak solution of (1)-(3). In view of Remark 2, we reformulate this problem in the following way: Given $\omega \in C_b(\mathbb{R})$, find $\mathbf{u} = \nabla^\perp\psi - \mathbf{a}$, such that

$$\nu\int_\Omega \nabla(\mathbf{u}+\mathbf{a})\cdot\nabla\mathbf{\Phi} = \int_\Omega \omega(\psi)((\mathbf{u}+\mathbf{a})\cdot\nabla\mathbf{\Phi})\cdot(\mathbf{u}+\mathbf{a}) \tag{16}$$

for all $\mathbf{\Phi} \in \mathcal{V}$. With this reformulation, equations (7) and (8) (the generalized form of equations (1)) are automatically satisfied with $\mathbf{v} = \mathbf{u}+\mathbf{a}$ and $\rho = \omega(\psi)$. Afterwards we will chose ω appropriately (see (26)) such that all the other statements in Theorem 1 are satisfied. We shall seek a solution $\mathbf{u}$ of (16) in the closure of $\mathcal{V}$ with respect the Dirichlet norm $||\nabla\mathbf{u}||_{L^2(\Omega)}$. We denote this space by $\mathbf{V}$.

Theorem 2 *Let $l_\omega := ||\omega||_{C_b(\mathbb{R})}$. Then there is a constant $c = c(\Omega) > 0$ such that for $c\alpha_2 l_\omega < \nu$, the equation (16) has a solution $\mathbf{u} \in \mathbf{V}$.*

Proof: Given an orthonormal basis $\{\mathbf{\Phi}_k\} \subset \mathcal{V}$ of $\mathbf{V}$, we consider the approximated problem

$$\begin{cases} \mathbf{u}_m = \sum_{k=1}^m \xi_{km}\mathbf{\Phi}_k, \qquad \mathbf{u}_m + \mathbf{a} = \nabla^\perp \psi_m \\ \nu \displaystyle\int_\Omega \nabla(\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{\Phi}_k = \int_\Omega \omega(\psi_m)((\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{\Phi}_k) \cdot (\mathbf{u}_m + \mathbf{a}), \\ k = 1, 2, \cdots, m. \end{cases} \tag{17}$$

We remark that the existence of a scalar function ψ_m such that $\mathbf{u}_m + \mathbf{a} = \nabla^\perp \psi_m$, a stream function associated with the vector field $\mathbf{v}_m := \mathbf{u}_m + \mathbf{a}$, is assured because $\mathbf{v}_m$ is a smooth vector field with null divergent and Ω is an open simply connected set of $\mathbb{R}^2$. Besides, in view of Remark 3, we may assume $\psi_m(x, -d_i) \equiv 0$, $\psi_m(x, d_i) \equiv \alpha_1$ for $i = 1, 2$.

For each $m \in \mathbb{R}^m$, (17) is a system of nonlinear algebraic equations for the unknown $\xi = (\xi_{1m}, \cdots, \xi_{1m}) \in \mathbb{R}^m$. In fact, setting

$$F_k(\xi) := \nu \int_\Omega \nabla(\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{\Phi}_k - \int_\Omega \omega(\psi_m)((\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{\Phi}_k) \cdot (\mathbf{u}_m + \mathbf{a})$$

equation (17) becomes the problem of finding a singular point of the vector field $\mathbf{F} := (F_1, \cdots, F_m)$ in $\mathbb{R}^m$, i.e. a point $\xi \in \mathbb{R}^m$ such that $\mathbf{F}(\xi) = \mathbf{0}$. An enough condition to this holds is that $\mathbf{F}(\xi)$ points towards the exterior of some ball $B_r(0)$ (an open ball in $\mathbb{R}^m$ of radius r centered at the origin) at every point of its border (the sphere in $\mathbb{R}^m$ of radius r centered at the origin) i.e. $\mathbf{F}(\xi) \cdot \xi > 0$ for every $|\xi| \equiv |\xi|_{\mathbb{R}^m} = r$ for some $r > 0$. (This fact can be inferred by contradiction from Brouwer's fixed point theorem; see e.g. [4, Lemma VIII.3.1].) Thus we estimate $\mathbf{F}(\xi) \cdot \xi$: First we note that

$$\mathbf{F}(\xi) \cdot \xi = \nu|\xi|^2 - \nu \int_\Omega \Delta \mathbf{a} \cdot \mathbf{u}_m - \int_\Omega \omega(\psi_m)((\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{u}_m) \cdot \mathbf{a}\,, \tag{18}$$

which can be easily verified from (17) and using that

$$\int_\Omega \omega(\psi_m)\,((\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{u}_m) \cdot \mathbf{u}_m \;=\; 0\,. \tag{19}$$

This last identity is obviously true in the case of a smooth ω. Indeed,

$$\mathbf{u}_m + \mathbf{a} = \nabla^\perp \psi_m \quad \text{so} \quad \nabla \cdot (\omega(\psi_m)(\mathbf{u}_m + \mathbf{a})) = 0\,;$$

besides,

$$\omega(\psi_m)\,((\mathbf{u}_m + \mathbf{a}) \cdot \nabla \mathbf{u}_m) \cdot \mathbf{u}_m = \frac{1}{2}\,(\omega(\psi_m)(\mathbf{u}_m + \mathbf{a})) \cdot \nabla |\mathbf{u}_m|^2,$$

so (19) can be derived from integration by parts. For the case of a non smooth $\omega \in C_b(\mathbb{R})$, the identity (19) can be proved by using a sequence of mollifiers ω^ϵ ($\epsilon \to 0$) approximating ω. Indeed, let $\Omega' \subset \Omega$ be a smooth bounded domain containing spt.$\mathbf{u}_m$. Then

$$\begin{aligned}
&|\int_\Omega \omega(\psi_m)\left((\mathbf{u}_m+\mathbf{a})\cdot\nabla\mathbf{u}_m\right)\cdot\mathbf{u}_m| \\
= \;& |\int_{\Omega'} \omega(\psi_m)(\mathbf{v}_m\cdot\nabla\mathbf{u}_m)\cdot\mathbf{u}_m - \int_{\Omega'} \omega^\epsilon(\psi_m)(\mathbf{v}_m\cdot\nabla\mathbf{u}_m)\cdot\mathbf{u}_m| \\
\le \;& \left(\sup_{x\in\overline{\Omega'}} |\omega(\psi_m(x)) - \omega^\epsilon(\psi(\psi_m(x)))|\right) ||(\mathbf{v}_m\cdot\nabla\mathbf{u}_m)\cdot\mathbf{u}_m||_{L^\infty(\Omega')}
\end{aligned}$$

tends to zero as $\epsilon\to 0$; cf. Remark 2. Now we estimate each term in (18). The strategy is to split the integrals $\int_\Omega$ over the domains Ω_t and $\Omega_t^c = \Omega_{1,t}^c \cup \Omega_{2,t}^c$ and use (15) and $\mathbf{a}=\mathbf{v}_i$ on $\Omega_{i,t}^c$, with the help of Hölder inequalities and Sobolev embedding type estimates. Using (6), Fubini's theorem and that $\mathbf{u}_m$ has null flux on each cross section $\Sigma_i(x)$ of Ω_i (recall that $\mathbf{u}_m|\Gamma = 0$ and $\mathbf{u}_m\in\mathcal{V}$, so $\nabla\cdot\mathbf{u}_m = 0$), we have

$$\begin{aligned}
\int_{\Omega_{i,t}^c} \Delta\mathbf{a}\cdot\mathbf{u}_m &= \int_{\Omega_{i,t}^c} \theta''(y)\mathbf{n}_i\cdot\mathbf{u}_m\,dxdy = -(\pm\tfrac{3}{2d_i^3}\alpha_i)\int_{\Omega_{i,t}^c} \mathbf{u}_m\cdot\mathbf{n}_i\,dxdy \\
&= -\tfrac{3}{2d_i^3}\alpha_i \int_{\pm t}^{\pm\infty}\left(\int_{\Sigma_i(x)} \mathbf{u}_m\cdot\mathbf{n}_i\,dy\right)dy = 0\,,
\end{aligned}$$

so

$$\begin{aligned}
&|\int_\Omega \Delta\mathbf{a}\cdot\mathbf{u}_m| = |\int_{\Omega_{t+1}} \Delta\mathbf{a}\cdot\mathbf{u}_m| \\
= \;& |-\int_{\Omega_{t+1}} \nabla\mathbf{a}\cdot\nabla\mathbf{u}_m + \int_{\Sigma_2(t+1)} \mathbf{u}_m\cdot\tfrac{\partial\mathbf{a}}{\partial\mathbf{n}_2} + \int_{\Sigma_1(-t-1)} \mathbf{u}_m\cdot\tfrac{\partial\mathbf{a}}{\partial\mathbf{n}_1}| \\
= \;& |-\int_{\Omega_t} \nabla\mathbf{a}\cdot\nabla\mathbf{u}_m - \textstyle\sum_{i=1}^2 \displaystyle\int_{\Omega_{t+1}\cap\Omega_{i,t}^c} \nabla\mathbf{v}_i\cdot\nabla\mathbf{u}_m \\
& + \int_{\Sigma_2(t+1)} \mathbf{u}_m\cdot\tfrac{\partial\mathbf{v}_2}{\partial\mathbf{n}_2} + \int_{\Sigma_1(-t-1)} \mathbf{u}_m\cdot\tfrac{\partial\mathbf{v}_1}{\partial\mathbf{n}_1}| \\
\le \;& ||\nabla\mathbf{a}||_{L^2(\Omega_t)}||\nabla\mathbf{u}_m||_{L^2(\Omega_t)} + c\alpha_2\,||\nabla\mathbf{u}_m||_{L^2(\Omega_{t+1}\cap\Omega_{i,t}^c)} + c\alpha_2\,||\mathbf{u}_m||_{L^2(\partial\Omega_{t+1})} \\
\le \;& c\alpha_2\,||\nabla\mathbf{u}_m||_{L^2(\Omega)} + c\alpha_2\,||\nabla\mathbf{u}_m||_{L^2(\Omega)} + c\alpha_2\,||\nabla\mathbf{u}_m||_{L^2(\Omega_{t+1})} \\
\le \;& c\alpha_2\,||\nabla\mathbf{u}_m||_{L^2(\Omega)} = c\alpha_2\,|\xi|\,,
\end{aligned}$$

where above and from now on c stands for some constant depending only on Ω;

cf. [4, XI.(3.8)] and [1, (3.15)]. For the second integral, we start with

$$\begin{aligned}
&|\int_{\Omega^c_{i,t}} \omega(\psi_m)(\mathbf{u}_m\cdot\nabla\mathbf{u}_m)\cdot\mathbf{a}| \\
\le\;& l_\omega \left(\pm\int_{\pm t}^{\pm\infty}\int_{-d_i}^{d_i} |\theta_i(y)\mathbf{u}_m|^2 dy dx\right)^{1/2} ||\nabla\mathbf{u}_m||_{L^2(\Omega^c_{i,t})} \\
\le\;& l_\omega \left(\pm\int_{\pm t}^{\pm\infty} (\int_{-d_i}^{d_i} |\theta_i(y)|^4 dy)^{1/2} (\int_{-d_i}^{d_i} |\mathbf{u}_m|^4 dy)^{1/2} dx\right)^{1/2} ||\nabla\mathbf{u}_m||_{L^2(\Omega^c_{i,t})} \\
\le\;& c\, l_\omega |\alpha_i| \left(\pm\int_{\pm t}^{\pm\infty} ||\mathbf{u}_m||^2_{L^4((-d_i,d_i))} dx\right)^{1/2} ||\nabla\mathbf{u}_m||_{L^2(\Omega^c_{i,t})} \\
\le\;& c\, l_\omega |\alpha_i| \left(\pm\int_{\pm t}^{\pm\infty} ||\nabla\mathbf{u}_m||^2_{L^2((-d_i,d_i))} dx\right)^{1/2} ||\nabla\mathbf{u}_m||_{L^2(\Omega^c_{i,t})} \\
=\;& c\, l_\omega |\alpha_i|\, ||\nabla\mathbf{u}_m||^2_{L^2(\Omega^c_{i,t})} \le c\, l_\omega|\alpha_i|\, ||\nabla\mathbf{u}_m||^2_{L^2(\Omega)} = c\, l_\omega|\alpha_i|\, |\xi|^2.
\end{aligned}$$

Analogously, we have

$$|\int_{\Omega^c_{i,t}} \omega(\psi_m)(\mathbf{a}\cdot\nabla\mathbf{u}_m)\cdot\mathbf{a}| \le c\, l_\omega |\alpha_i|^2\, |\xi|\,.$$

Next, using in particular (15), we obtain

$$\begin{aligned}
|\int_{\Omega_t} \omega(\psi_m)(\mathbf{u}_m\cdot\nabla\mathbf{u}_m)\cdot\mathbf{a}| \;&\le l_\omega\, ||\; |\mathbf{u}_m|\, |\mathbf{a}|\; ||_{L^2(\Omega_t)}\; ||\nabla\mathbf{u}_m||_{L^2(\Omega_t)} \\
&\le l_\omega\, ||\mathbf{u}_m||_{L^4(\Omega_t)} ||\mathbf{a}||_{L^4(\Omega_t)} ||\nabla\mathbf{u}_m||_{L^2(\Omega_t)} \\
&\le c\, l_\omega\, ||\nabla\mathbf{u}_m||_{L^2(\Omega_t)} ||\nabla\mathbf{a}||_{L^2(\Omega_t)} ||\nabla\mathbf{u}_m||_{L^2(\Omega_t)} \\
&\le c\, l_\omega\, ||\nabla\mathbf{u}_m||_{L^2(\Omega_t)} ||\nabla\mathbf{a}||_{L^2(\Omega_t)} ||\nabla\mathbf{u}_m||_{L^2(\Omega_t)} \\
&\le c\, l_\omega\, |\alpha_i|\, ||\nabla\mathbf{u}_m||^2_{L^2(\Omega_t)} = \; c\, l_\omega\, |\alpha_i|\, |\xi|^2.
\end{aligned}$$

Analogously,

$$|\int_{\Omega_t} \omega(\psi_m)(\mathbf{a}\cdot\nabla\mathbf{u}_m)\cdot\mathbf{a}| \le c\, l_\omega\, |\alpha_i|^2\, |\xi|\,.$$

From the above estimates and (18) we arrive at

$$\mathbf{F}(\xi)\cdot\xi \ge (\nu - c\, l_\omega|\alpha_i|)|\xi|^2 - (\nu + |\alpha_i| l_\omega) c|\alpha_i|\, |\xi|\,.$$

Then $\mathbf{F}(\xi)\cdot\xi > 0$ if $c\, l_\omega|\alpha_i| < \nu$ and $|\xi| = r$ for any $r > c|\alpha_i|\frac{\nu+|\alpha_i|l_\omega}{\nu - c\, l_\omega|\alpha_i|}$. Therefore the system (17) has a solution $\mathbf{u}_m = \sum_{k=1}^m \xi_{km}\mathbf{\Phi}_k$ and $||\nabla\mathbf{u}_m||_{L^2(\Omega)} = |\xi|$ satisfies the following estimate

$$||\nabla\mathbf{u}_m||_{L^2(\Omega)} \le c|\alpha_i|\frac{\nu + |\alpha_i| l_\omega}{\nu - c|\alpha_i| l_\omega} \tag{20}$$

for some constant $c = c(\Omega) > 0$. Then, by Banach-Alaoglu's theorem, there exists a subsequence (u_{m_k}) that converges weakly to some $\mathbf{u}$ in $\mathbf{V}$. Since it holds Poincaré's inequality

$$||\mathbf{z}||_{L^2(\Omega)} \le c||\nabla\mathbf{z}||_{L^2(\Omega)}, \quad \forall\, \mathbf{z}\in\mathbf{V}\,, \tag{21}$$

(where c is a constant depending only on Ω; notice that (21) holds separately in each Ω_i, $i = 0, 1, 2$, because Ω_i is bounded in some direction) we may assume that $(\mathbf{u}_{m_k})$ also converges weakly to $\mathbf{u}$ in $W^{1,2}(\Omega)$ and, by Rellich-Kondrachov's theorem, strongly in $L^p_{loc}(\Omega)$ for any $p \in [1, \infty)$. As a consequence (possibly taking another subsequence of (m_k)), (ψ_{m_k}) converges weakly to some function ψ in $W^{2,2}_{loc}(\Omega)$, such that $\nabla^\perp \psi = \mathbf{u} + \mathbf{a}$. We recall that ψ_m is defined in (17) by the equation $\mathbf{u}_m + \mathbf{a} = \nabla^\perp \psi_m$. Noting that $W^{2,2}(\Omega')$ is compactly embedded in $C_b(\overline{\Omega'})$ for any smooth bounded open subset Ω' of Ω, we deduce that $\omega(\psi_{m_k})$ converges to $\omega(\psi)$ strongly in $C_b(\Omega')$. Then we can verify that $\mathbf{u} = \nabla^\perp \psi - \mathbf{a}$ satisfies (16) for all $\mathbf{\Phi}$ in $\mathcal{V}$. Indeed, let $\Omega' \subset \Omega$ be a smooth bounded domain containing spt.$\mathbf{\Phi}$. From the weak convergence of $(\mathbf{u}_{m_k})$ to $\mathbf{u}$ in $\mathbf{V}$, it is obvious that the left hand side of the second equation in (17), $\int_\Omega \nabla(\mathbf{u}_{m_k} + \mathbf{a}) \cdot \nabla \mathbf{\Phi} = \int_\Omega \nabla \mathbf{u}_{m_k} \cdot \nabla \mathbf{\Phi} + \int_\Omega \nabla \mathbf{a} \cdot \nabla \mathbf{\Phi}$, converges to $\int_\Omega \nabla(\mathbf{u}+\mathbf{a}) \cdot \nabla \mathbf{\Phi}$. Regarding the right hand side of the second equation in (17), let let $\mathbf{v} := \mathbf{u} + \mathbf{a}$ and $\mathbf{v}_m := \mathbf{u}_m + \mathbf{a}$ (as we defined right after (17)). Then we can write

$$
\begin{aligned}
& \int_\Omega \omega(\psi_{m_k})((\mathbf{u}_{m_k} + \mathbf{a}) \cdot \nabla \mathbf{\Phi}) \cdot (\mathbf{u}_{m_k} + \mathbf{a}) = \int_{\Omega'} \omega(\psi_{m_k})(\mathbf{v}_{m_k} \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{v}_{m_k} \\
= & \int_{\Omega'} [\omega(\psi_{m_k}) - \omega(\psi)](\mathbf{v}_{m_k} \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{v}_{m_k} + \int_{\Omega'} \omega(\psi)(\mathbf{v}_{m_k} - \mathbf{v}) \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{v}_{m_k} \\
& + \int_{\Omega'} \omega(\psi)(\mathbf{v} \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{v}_{m_k} \\
\equiv & \ I + II + III\,.
\end{aligned}
$$

The first term I converges to zero (as $m_k \to \infty$) since

$$
|I| \leq (\sup_{x \in \overline{\Omega'}} |\omega(\psi_{m_k}(x)) - \omega(\psi(x))|) ||\mathbf{v}_{m_k}||_{L^4(\Omega')} ||\mathbf{\Phi}||_{L^\infty(\Omega')},
$$

(ψ_{m_k}) converges to ψ in $C_b(\overline{\Omega'})$ (cf. Remark 2) and $(\mathbf{v}_{m_k})$ is bounded in $L^4(\Omega')$ (it converges strongly in $L^p(\Omega')$ for any $p \in [1, \infty)$. For the second term II, we have

$$
|II| \leq l_\omega ||\nabla \mathbf{\Phi}||_{L^\infty(\Omega')} ||\mathbf{u}_{m_k} - \mathbf{u}||_{L^2(\Omega')} ||\mathbf{v}_{m_k}||_{L^2(\Omega')}
$$

so it converges also to zero, since $(\mathbf{u}_{m_k})$ converges strongly to $\mathbf{u}$ in $L^2(\Omega')$ and $(\mathbf{v}_{m_k})$ is bounded in $L^2(\Omega')$. Finally, the third term III converges to $\displaystyle\int_{\Omega'} \omega(\psi)(\mathbf{v} \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{v} = \int_\Omega \omega(\psi)(\mathbf{v} \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{v}$ from previous arguments or because it equals to $\displaystyle\int_{\Omega'} \omega(\psi)(\mathbf{v} \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{u}_{m_k} + \int_{\Omega'} \omega(\psi)(\mathbf{v} \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{a}$ and the functional $\displaystyle\mathbf{z} \mapsto \int_{\Omega'} \omega(\psi)(\mathbf{v} \cdot \nabla \mathbf{\Phi}) \cdot \mathbf{z}$ is a bounded linear functional on $\mathbf{V}$ and $(\mathbf{u}_{m_k})$ converges weakly to $\mathbf{u}$ in $\mathbf{V}$. Therefore, by passing to the limit when $m \equiv m_k \to \infty$, from (17) we obtain (16). ■

Proof of Theorem 1: From Theorem 2, there exists a constant $c = c(\Omega)$ such that for any $\omega \in C_b(\omega)$ satisfying $c\alpha_2 l_\omega < \nu$, equation (16) has a solution $\mathbf{u} \in \mathbf{V}$. As we noticed right before the statement of Theorem 2, the pair of functions

$\mathbf{v} = \mathbf{u} + \mathbf{a} = \nabla^\perp \psi$ ($\psi \in W^{2,2}_{loc}(\Omega)$) and $\rho = \omega(\psi)$ satisfies equations (7) and (8). From $\mathbf{u} \in \mathbf{V}$, $\omega \in C_b(\mathbb{R})$ and the Sobolev embedding $W^{2,2}(\Omega') \subset C(\overline{\Omega'})$ for any bounded open subset of Ω, it is clear that $(\rho, \mathbf{v}) \in C_b(\Omega) \times \mathbf{H}_{1,loc}(\overline{\Omega})$. Then we proceed to prove the other statements (9)-(12).

From (16) we have that $\mathbf{u} = \mathbf{v} - \mathbf{v}_i$, along with some pressure function $\tau \in L^2_{loc}(\Omega_i)$, is a weak solution of the Stokes equation

$$\nu \Delta \mathbf{u} = \nabla \tau + \mathbf{f}, \tag{22}$$

in the domain $\Omega^c_{i,t}$, where

$$\mathbf{f} := \rho(\mathbf{v} \cdot \nabla)\mathbf{v} = \rho(\mathbf{u} \cdot \nabla)\mathbf{u} + \rho(\mathbf{a} \cdot \nabla)\mathbf{u} + \rho(\mathbf{u} \cdot \nabla)\mathbf{a},$$

with $\rho = \omega(\psi)$ and $\nabla^\perp \psi = \mathbf{v}$. Here we used that $\mathbf{a}$ coincides with the Poiseuille flow $\mathbf{v}_i$ in $\Omega^c_{i,t}$ (in particular, $\nu\Delta\mathbf{a} = \nu\Delta\mathbf{v}_i = \nabla\tilde{p}$ for some function $\tilde{p} \in L^2_{loc}(\Omega^c_{i,t})$ and $\mathbf{a} \cdot \nabla \mathbf{a} = \mathbf{v}_i \cdot \nabla \mathbf{v}_i = 0$). For $0 \le x_1 < x_2$, let $\Omega_{i,x_1,x_2} := \{(x,y) \in \Omega_i : x_1 < |x| < x_2\}$. For any $j = 0, 1, 2, \cdots$, using Hölder inequality and the Sobolev embedding $W^{1,2}(\Omega_{i,x_1,x_2}) \subset L^6(\Omega_{i,x_1,x_2})$, we have

$$\begin{aligned} ||\rho \mathbf{u} \cdot \nabla \mathbf{u}||_{L^{3/2}(\Omega_{i,t+j,t+j+1})} &\le l_\omega ||\mathbf{u}||_{L^6(\Omega_{i,t+j,t+j+1})} ||\nabla \mathbf{u}||_{L^2(\Omega_{i,t+j,t+j+1})} \\ &\le c\, l_\omega ||\nabla \mathbf{u}||^2_{L^2(\Omega_{i,t+j,t+j+1})} \end{aligned}$$

where from now on c is a constant depending only on Ω.Summing over j, we obtain

$$||\rho \mathbf{u} \cdot \nabla \mathbf{u}||_{L^{3/2}(\Omega^c_t)} \le c\, l_\omega ||\nabla \mathbf{u}||^2_{L^2(\Omega)}. \tag{23}$$

We also have,

$$||\rho \mathbf{a} \cdot \nabla \mathbf{u}||_{L^{3/2}(\Omega_{i,t+j,t+j+1})} \le l_\omega\, ||\mathbf{a}||_{L^6(\Omega_{i,t+j,t+j+1})} ||\nabla \mathbf{u}||_{L^2(\Omega_{i,t+j,t+j+1})};$$

notice that

$$||\mathbf{a}||^2_{L^6(\Omega_{i,t+j,t+j+1})} = \int_{t+j}^{t+j+1} \int_{-d_i}^{d_i} |\mathbf{v}_i|^6\, dy\, dx = \int_{-d_i}^{d_i} |\mathbf{v}_i|^6\, dy\,,$$

since $\mathbf{a} = \mathbf{v}_i$ in Ω^c_t and $\mathbf{v}_i$ does not depend on x; then

$$||\rho \mathbf{a} \cdot \nabla \mathbf{u}||_{L^{3/2}(\Omega^c_t)} \le l_\omega\, ||\mathbf{v}_i||_{L^6((-d_i,d_i))} ||\nabla \mathbf{u}||_{L^2(\Omega)}. \tag{24}$$

Analogously to (23) and (24), we obtain

$$||\rho(\mathbf{u} \cdot \nabla)\mathbf{a}||_{L^{3/2}(\Omega^c_t)} \le c\, l_\omega\, ||\nabla \mathbf{v}_i||_{L^2((-d_i,d_i))} ||\nabla \mathbf{u}||_{L^2(\Omega)}. \tag{25}$$

From (23)-(25) and [3, Lemma VI.1.2] together with its footnote [3, p.314], we obtain

$$||\mathbf{u}||_{W^{1,3/2}(\Omega^c_{t+1})} \le c\, l_\omega (||\nabla \mathbf{u}||_{L^2(\Omega)} + 1) ||\nabla \mathbf{u}||_{L^2(\Omega)}\,.$$

Since $||\mathbf{u}||_{L^\infty(\Omega^c_{t+1})} \le c ||\mathbf{u}||_{W^{1,3/2}(\Omega^c_{t+1})}$, it follows the estimate

$$\begin{aligned} ||\mathbf{f}||_{L^2(\Omega^c_{t+1})} \le \quad & l_\omega \Big(||\mathbf{u}||_{L^\infty(\Omega^c_{t+1})} ||\nabla \mathbf{u}||_{L^2(\Omega^c_{t+1})} \\ & + ||\mathbf{v}_i||_{L^\infty(\Omega)} ||\nabla \mathbf{u}||_{L^2(\Omega^c_{t+1})} + ||\nabla \mathbf{a}||_{L^\infty(\Omega)} ||\mathbf{u}||_{L^2(\Omega^c_{t+1})} \Big) \\ \le \quad & c\, l_\omega\, \big(l_\omega (||\nabla \mathbf{u}||_{L^2(\Omega)} + 1) ||\nabla \mathbf{u}||_{L^2(\Omega)} + 1 \big)\, ||\nabla \mathbf{u}||_{L^2(\Omega^c_{t+1})}. \end{aligned}$$

Therefore, by employing [3, Lemma VI.1.2] again, we arrive at (9), with $t+1$ in place of t, and we have the estimate $||\mathbf{u}||_{W^{2,2}(\Omega^c_{t+1})} \leq c(||\mathbf{f}||_{L^2(\Omega^c_t)} + ||\nabla\mathbf{u}||_{L^2(\Omega^c_t)})$, i.e.

$$||\mathbf{u}||_{W^{2,2}(\Omega^c_{t+1})} \leq c\left(l^2_\omega(||\nabla\mathbf{u}||_{L^2(\Omega)}+1)||\nabla\mathbf{u}||_{L^2(\Omega)} + l + 1\right)||\nabla\mathbf{u}||_{L^2(\Omega^c_t)}.$$

Next, as we shall see, to have condition (10) satisfied it is enough to choose ω in $C_b(\mathbb{R})$ such that

$$\omega(\psi_1(y)) = \rho_1(y)\,. \tag{26}$$

We recall that we defined ψ_1 in Theorem 1 as $\psi_1(y) = -\int_{-d_1}^{y}\theta_1(y')\,dy'$. We also note that ψ_1 is a monotonic function for $y \in (-d_1, d_1)$, thus there exists a function ω in $C_b(\mathbb{R})$ satisfying (26), i.e. $\omega(s) = \rho_1(\psi_1^{-1}(s)),\ \ \forall\ s \in \mathrm{Im}\psi_1$ and outside the interval $\mathrm{Im}\psi_1$ (the image set of ψ_1, which is defined in $\Sigma_1 = (-d_1, d_1)$) ω is arbitrary but continuous and bounded in $\mathbb{R}$. Now, since $\mathbf{u} = \mathbf{v} - \mathbf{v}_1 \in W^{1,2}_0(\Omega)$, from Remark 3 and Poincaré's inequality, we have that $\psi - \psi_1 \in W^{2,2}(\Omega_1)$. (Note that Ω_1 is bounded in one direction and $\psi - \psi_1$ vanishes on the 'horizontal boundary' of Ω_1, so we can apply Poincaré's inequality.) Thus, reasoning as in (14), we obtain $\lim_{x\to-\infty} ||\psi(x,\cdot) - \psi_1||_{C_b(\Sigma_1)} = 0$. Then, ψ is bounded in Ω_1 and given any $\epsilon > 0$ there exists a $s > 0$ such that $|x| \geq s$ implies

$$|\rho(x,y) - \rho_1(y)| = |\omega(\psi(x,y)) - \omega(\psi_1(y))| < \epsilon,$$

for all $y \in \Sigma_1 = (-d_1, d_1)$, since ω is locally uniformly continuous. Thus we have condition (10) satisfied.

With the above choice for ω, we set $W(s) \stackrel{\text{def}}{=} \displaystyle\int_0^s \omega(r)\,dr$ and using Remark 3 and Remark 1, we compute the flux of the momemtum on Ω_i to verify (11):

$$\begin{aligned}
\beta_i \quad &:= \int_{\Sigma_i(x)} \rho\mathbf{v}\cdot\mathbf{n}_i = \pm\int_{-d_i}^{d_i} \rho v_1(x,y)\,dy = \pm\int_{-d_i}^{d_i} \omega(\psi)(-\psi_y(x,y))\,dy \\
&= \pm\int_{-d_i}^{d_i} (-\partial_y W(\psi(x,y)))\,dy = \pm\left(W(\psi(x,-d_i)) - W(\psi(x,d_i)\right) \\
&= \pm\int_{\psi(x,d_i)}^{\psi(x,-d_i)} \omega(s)\,ds = \int_{\alpha_1}^{0} \omega(s)\,ds = \int_{\alpha_1}^{0} \rho_1(\psi_1^{-1}(s))\,ds\,.
\end{aligned}$$

Imposing on ω, besides (26), the condition $||\omega||_{C_b(\mathbb{R})} \leq l$, we get $\rho = \omega(\psi)$ satisfying also $||\rho||_{C_b(\Omega)} \leq l$.

Finally, we have (12) by the following steps:

$$\begin{aligned}
||\nabla(\mathbf{v}-\mathbf{v}_i)||_{L^2(\Omega_i)} \quad &= ||\nabla(\mathbf{a}-\mathbf{v}_i)||_{L^2(\Omega_i)} + ||\nabla\mathbf{u}||_{L^2(\Omega_i)} \\
&= ||\nabla(\mathbf{a}-\mathbf{v}_i)||_{L^2(\Omega_i\cap\Omega_t)} + ||\nabla\mathbf{u}||_{L^2(\Omega)} \\
&\leq c|\alpha_i| + ||\nabla\mathbf{u}||_{L^2(\Omega)} \leq c|\alpha_i| + c|\alpha_i|\tfrac{\nu+|\alpha_i|l}{\nu-c|\alpha_i|l},
\end{aligned}$$

where we used that $\mathbf{u}$ satisfies the same estimate as (20) and $l_\omega = l$;

$$\begin{aligned}
||\nabla\mathbf{v}||_{L^2(\Omega_0)} \quad &= ||\nabla(\mathbf{u}+\mathbf{a})||_{L^2(\Omega_0)} \leq ||\nabla\mathbf{a}||_{L^2(\Omega_0)} + ||\nabla\mathbf{u}||_{L^2(\Omega_0)} \\
&\leq c|\alpha_i| + ||\nabla\mathbf{u}||_{L^2(\Omega)} \\
&\leq c|\alpha_i| + c|\alpha_i|\tfrac{\nu+\alpha l}{\nu-c\alpha l}\,. \blacksquare
\end{aligned}$$

Remark 4 (on uniqueness). *The uniqueness of solution of Problem (1)-(4) is not clear for us, since following the usual procedure of taking the difference* $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$, $\rho = \rho_1 - \rho_2$ *of two solutions* $(\rho_1, \mathbf{v}_i)$, $i = 1, 2$, *we get stuck with the term* $\int_\Omega \rho(\mathbf{v}_i \nabla \mathbf{v}) \cdot \mathbf{v}_i$. *We conjecture that uniqueness of the velocity field is true under an assumption of smallness on the density, i.e. if we assume that* $||\rho||_{C_b(\Omega)}$ *is sufficiently small, but some new ingredient is necessary to improve the usual prove (or to find a new one). Regarding the uniqueness of the density, it is necessary to find new criteria to select the physically relevant solution (cf. [6, p.34]), unless the velocity vector field has not undesirable singularities, in a way that its stream lines foliate* Ω. *In this case, if the velocity* $\mathbf{v}$ *is unique then the density* ρ *is also unique, because the equations* $\nabla \cdot (\rho \mathbf{v}) = 0$ *and* $\nabla \cdot \mathbf{v} = 0$ *imply that* ρ *is constant along the stream lines of* $\mathbf{v}$.

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