Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 327 (2007) 525–538
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Stabilization of the nonuniform Timoshenko beam

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Received 6 October 2004
Available online 5 June 2006
Submitted by M.C. Nucci

Abstract

In this paper we consider a Timoshenko beam with variable physical parameters, we prove that the model can be stabilize by one control force for both internal and boundary cases.

Keywords: Asymptotic stability; Multiplier method; Lyapunov function; Timoshenko beam; Uniform stabilization

1. Introduction

The equations of motion of a Timoshenko beam are

$$\begin{cases} \alpha w_{tt} = (\beta(\varphi + w_x))_x, \\ \gamma \varphi_{tt} = (\delta \varphi_x)_x - \beta(\varphi + w_x) \end{cases} \quad \text{on } (0,1) \times \mathbb{R}^+. \tag{1}$$

Here, t is the time variable and x the space coordinate along the beam. The function w is the transverse displacement of the beam and φ is the rotation angle of a filament of the beam. The coefficients α, β, γ and δ are the mass per unit length, the polar moment of inertia of a cross

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.04.016

section, Young's modulus of elasticity, the moment of inertia of a cross section and the shear modulus respectively. The natural energy of the beam is

$$\mathcal{E}(t) = \frac{1}{2}\int_0^1 \left\{\alpha|w_t|^2 + \gamma|\varphi_t|^2 + \beta|\varphi + w_x|^2 + \delta|\varphi_x|^2\right\} dx. \tag{2}$$

The aim of this paper is to study the internal and the boundary stabilization of this beam. For the internal stabilization we will assume that

$$\alpha, \beta, \gamma \text{ and } \delta \text{ are positive } \mathbf{C}^1 \text{ functions of } x. \tag{3}$$

We will first prove that it is possible to stabilize uniformly this nonuniform beam, by using a unique internal feedback acting only on the rotation angle, namely:

$$\begin{cases} \alpha w_{tt} = (\beta(\varphi + w_x))_x, \\ \gamma\varphi_{tt} = (\delta\varphi_x)_x - \beta(\varphi + w_x) - a(x)\varphi_t \end{cases} \quad \text{on } (0,1) \times \mathbb{R}^+, \tag{4}$$

$$w(x,0) = w_0, \quad w_t(x,0) = w_1, \quad \varphi(x,0) = \varphi_0 \quad \text{and} \quad \varphi_t(x,0) = \varphi_1,$$

and we consider two boundary cases

$$w(0,t) = w(1,t) = 0; \qquad \varphi(0,t) = \varphi(1,t) = 0, \tag{5}$$

$$w_x(1,t) = 0; \qquad w(0,t) = \varphi(0,t) = \varphi(1,t) = 0, \tag{6}$$

where $a \geqslant 0$ *is a continuous function of the space variable*.

Second we will prove that it is possible to stabilize uniformly this nonuniform beam by using a one boundary feedback, more precisely we consider system (1) under the following boundary conditions:

$$\begin{cases} w(0,t) = w(1,t) = 0, \\ \varphi_x(0,t) = c\varphi_t(0,t), \\ \varphi_x(1,t) = -d\varphi_t(1,t), \end{cases} \tag{7}$$

where c and d are positive real numbers.

Let us mention some known results related to the stabilization of the Timoshenko beam. Kim and Renardy [6] proved the exponential stability of the Timoshenko beam under two boundary controls. Soufyane [10] showed the exponential stability of the uniform Timoshenko beam by using one distributed feedback. Shi et al. [3] considered the case of the uniform Timoshenko beam under two locally distributed feedbacks. Ammar-Khodja et al. [2] studied the stabilization of the uniform Timoshenko beam of memory type. Soufyane and Wehbe [11] proved the uniform stabilization of the Timoshenko beam under one locally distributed feedback. Xu and Yung [4] proved an exponential stability of the uniform Timoshenko beam by two pointwise controls. The first analysis for a Timoshenko beam with variable physical parameters seems to be the one of Taylor [12]. He studied the boundary control of system (1) under two feedbacks. Yan et al. [5] studied the case of the nonuniform Timoshenko beam under two locally distributed feedbacks.

The main result of this paper is that the energy of the nonuniform beam (4), (5), or (4), (6), or (1), (7) decays exponentially if the wave speeds $\frac{\delta}{\gamma}$ and $\frac{\beta}{\alpha}$ are the same on the whole interval. Also we prove that if the wave speeds are different on the whole interval, we prove the asymptotic stability and the nonuniform stability.

The plan of the paper is the following. In Section 2, the uniform stabilization of (4), (5), or (4), (6) is proved under the condition $\frac{\delta}{\gamma} = \frac{\beta}{\alpha}$ on the whole interval, using multipliers techniques and Neves et al. [8] results. Section 3 is devoted to the proof of the uniform stability of (1), (7), using eigenvalue system and Ammar-Khodja and Bader [1] results.

2. Preliminaries

We need to recall some definitions and results in view of the proof of some of our results.

First, for a continuous linear operator T from a Banach space into itself, we define its essential spectral radius $r_e(T)$ to be

$$r_e(T) = \inf\{R > 0\colon\ \mu \in \sigma(T), |\mu| > R \text{ implies } \mu \text{ is an isolated eigenvalue of finite multiplicity}\}, \tag{8}$$

where $\sigma(T)$ denotes the spectrum of T. It is well known (see, for instance, [7]) that if $r(T)$ is the spectral radius of T, then

$$r_e(T) \leqslant r(T); \qquad r_e(T+K) = r_e(T) \quad \forall K \in L(X),\ K \text{ compact}.$$

Now, if e^{Lt} is a C_0-semigroup generated by L, let us recall here that:

- e^{Lt} is asymptotically stable if, for any $Y_0 \in H$
$$\lim_{t\to\infty} e^{Lt} Y_0 = 0.$$
- e^{Lt} is uniformly (or exponentially) stable if there exist $\omega < 0$ and $M > 0$ such that:
$$\|e^{Lt}\| \leqslant M e^{\omega t}, \quad t \in \mathbb{R}^+.$$

Actually, there exist two real numbers $\omega = \omega(L)$ and $\omega_e = \omega_e(L)$ such that

$$r(e^{Lt}) = e^{\omega t}, \tag{9}$$
$$r_e(e^{Lt}) = e^{\omega_e t} \quad \forall t \in \mathbb{R}^+, \tag{10}$$

ω is often called the *type* and ω_e the *essential type* of the semigroup. A third real number which plays an essential role in stability theory is the *spectral abscissa* $s(L)$ of L defined by

$$s(L) = \sup\{\operatorname{Re}\lambda,\ \lambda \in \sigma(L)\}.$$

One has the following relation between these three real numbers:

$$\omega(L) = \max(\omega_e(L), s(L)).$$

Clearly, the uniform stability of e^{Lt} is equivalent to $\omega(L) < 0$ and, if L has a compact resolvent, the asymptotic stability is equivalent to $s(L) < 0$.

On the other hand, we introduce the Riemann invariants

$$u_1 = \frac{1}{2}\big(\sqrt{\alpha} w_t - \sqrt{\beta}(w_x + \varphi)\big), \qquad u_2 = \frac{1}{2}\big(\sqrt{\gamma}\varphi_t - \sqrt{\delta}\varphi_x\big),$$
$$v_1 = \frac{1}{2}\big(\sqrt{\alpha} w_t + \sqrt{\beta}(w_x + \varphi)\big), \qquad v_2 = \frac{1}{2}\big(\sqrt{\gamma}\varphi_t + \sqrt{\delta}\varphi_x\big).$$

Then system (4), (5) transforms into

$$Y_t = MY_x + CY \quad \text{in } (0,1) \times \mathbb{R}^+, \tag{11}$$

where M is the diagonal 4×4 matrix given by

$$M = \operatorname{diag}\left(-\sqrt{\frac{\beta}{\alpha}}, -\sqrt{\frac{\delta}{\gamma}}, \sqrt{\frac{\beta}{\alpha}}, \sqrt{\frac{\delta}{\gamma}}\right), \tag{12}$$

and $C = C_0 + C_1$ with

$$C_0 = \operatorname{diag}\left(-\left(\sqrt{\frac{\beta}{\alpha}}\right)', -\left(\sqrt{\frac{\delta}{\gamma}}\right)' - \frac{a}{\gamma}, \left(\sqrt{\frac{\beta}{\alpha}}\right)', \left(\sqrt{\frac{\delta}{\gamma}}\right)' - \frac{a}{\gamma}\right), \tag{13}$$

and C_1 is the skew-symmetric matrix whose entries are

$$\begin{aligned}
&c_{12} = -c_{14} = c_{23} = -c_{34} = -\frac{1}{2}\sqrt{\frac{\beta}{\gamma}},\\
&c_{13} = \frac{1}{2}\left(\left(\beta^{1/2}\right)'\alpha^{-1/2} - \beta^{1/2}\left(\alpha^{-1/2}\right)'\right),\\
&c_{24} = \frac{1}{2}\left(\left(\delta^{1/2}\right)'\gamma^{-1/2} - \delta^{1/2}\left(\gamma^{-1/2}\right)' - \frac{a}{\gamma}\right).
\end{aligned} \tag{14}$$

The boundary conditions (5) become:

$$\begin{cases} u_i(0,t) = -v_i(0,t), \\ u_i(1,t) = -v_i(1,t), \end{cases} \quad i = 1, 2, \tag{15}$$

the boundary conditions (4), (6) transforms into

$$\begin{cases} u_1(0,t) = -v_1(0,t) \quad \text{and} \quad u_1(1,t) = v_1(1,t), \\ u_2(0,t) = -v_2(0,t) \quad \text{and} \quad u_2(1,t) = -v_2(1,t), \end{cases} \tag{16}$$

and last, the boundary conditions (7) transforms into

$$\begin{cases} u_1(0,t) + v_1(0,t) = 0, \\ u_1(1,t) + v_1(1,t) = 0, \\ \left(c\sqrt{\frac{\delta}{\gamma}(0)} + 1\right)u_2(0,t) + \left(c\sqrt{\frac{\delta}{\gamma}(0)} - 1\right)v_2(0,t) = 0, \\ \left(-d\sqrt{\frac{\delta}{\gamma}(1)} + 1\right)u_2(1,t) + \left(d\sqrt{\frac{\delta}{\gamma}(1)} + 1\right)v_2(1,t) = 0. \end{cases} \tag{17}$$

Let us define, on the new energy space $G = (L^2(0,1))^4$, the operators

$$\begin{aligned}
&A = M\frac{\partial}{\partial x} + C,\\
&D(A) = \left\{Y = (u_1, u_2, v_1, v_2) \in \left(H^1(0,1)\right)^4,\ \text{boundary conditions}\right\},
\end{aligned} \tag{18}$$

where "boundary conditions" refers to one the group of boundary conditions (15), (16) or (17). It is easy to check that solutions of (11) in G correspond to solutions of (1) or (19) in H and the converse holds true.

3. Uniform stability of (4), (5) or (4), (6)

In the sequel, we assume that α, β, γ and δ *are positive C^1 functions of the space variable*. The energy space H associated to system (4), (5) will be

$$H = \left(H(0,1) \times L^2(0,1)\right)^2,$$

where $H(0,1)$ is $H_0^1(0,1)$ if we deal with (5) and $H_-^1(0,1) \times L^2(0,1) \times H_0^1(0,1) \times L^2(0,1)$ with $H_-^1(0,1) = \{w \in H^1(0,1)\colon\ w(0) = w_x(1) = 0\}$ if we deal with (6). In each case, H is equipped with the inner product:

$$\langle Y, \widetilde{Y}\rangle = \int_0^1 \left\{\alpha w_2 \widetilde{w}_2 + \gamma \varphi_2 \tilde{\varphi}_2 + \beta(\varphi_1 + \partial_x w_1)(\tilde{\varphi}_1 + \partial_x \widetilde{w}_1) + \delta \varphi_1 \tilde{\varphi}_1\right\} dx,$$

where $Y = (w_1, w_2, \varphi_1, \varphi_2)$ and $\widetilde{Y} = (\widetilde{w}_1, \widetilde{w}_2, \tilde{\varphi}_1, \tilde{\varphi}_2) \in H$.

The system (4), (5) or (4), (6) can be put in the abstract form

$$\frac{dY}{dt} = LY, \tag{19}$$

where

$$Y = (w\ w_t\ \varphi\ \varphi_t),$$

and

$$L = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{1}{\alpha}\partial_x(\beta\partial_x) & 0 & \frac{1}{\alpha}\partial_x(\beta.) & 0 \\ 0 & 0 & 0 & I \\ -\frac{\beta}{\gamma}\partial_x & 0 & \frac{1}{\gamma}(\partial_x(\delta\partial_x) - \beta) & -\frac{a}{\gamma} \end{pmatrix},$$

with domain

$$D(L) = \{Y \in H,\ LY \in H\}.$$

With our assumptions in hand, L is maximal dissipative and so, by the Lumer–Philips theorem, it is the infinitesimal generator of a C_0-semigroup (e^{Lt}).

We are now ready to state our first result.

Theorem 1. *Under assumption* (3) *and assume that* $a \in C([0,1])$ *with*

$$a \geqslant a_0 > 0 \quad \text{on } (0,1) \tag{20}$$

then:

1. *If* $\frac{\beta}{\alpha} = \frac{\delta}{\gamma}$ *on* $(0,1)$ *then* e^{Lt} *is uniformly stable.*
2. *If* $\frac{\beta}{\alpha} \neq \frac{\delta}{\gamma}$ *on* $(0,1)$ *then* e^{Lt} *is not uniformly stable.*

Proof. We prove the asymptotic stability in the last section. We deal now with the proof of the second claim.

(1) *The stability is non-uniform if* $\frac{\beta}{\alpha} \neq \frac{\delta}{\gamma}$ on $(0,1)$. We work with the transformed system (11). Noting that the eigenvalues of M (see (12) for its definition) are distinct in this case, a result of Neves, Ribeiro and Lopes [8, Theorems A and B, p. 324] asserts that

$$r_e(e^{At}) = r_e(e^{A_0 t}) = e^{\alpha_0 t},$$

where

$$A_0 = M\frac{\partial}{\partial x} + C_0, \qquad D(A_0) = D(A),$$

and $\alpha_0 = s(A_0)$ (see the definition of this number in the previous subsection). If we set $U = (u_1, u_2)$ and $V = (v_1, v_2)$ and if

$$A_0\begin{pmatrix} U \\ V \end{pmatrix} = \lambda \begin{pmatrix} U \\ V \end{pmatrix}, \qquad \begin{pmatrix} U \\ V \end{pmatrix} \in D(A_0),$$

then a straightforward computation (it is a diagonal differential system) leads us to the following equations for the eigenvalues of A_0:

$$e^{2\lambda \int_0^1 \sqrt{\frac{\alpha}{\beta}}\,dx} - 1 = 0 \quad \text{or} \quad e^{2\lambda \int_0^1 \sqrt{\frac{\gamma}{\delta}}\,dx + 2\int_0^1 \sqrt{\frac{\gamma}{\delta}}\frac{a}{\gamma}\,dx} - 1 = 0.$$

Therefore, the eigenvalues of A_0 are, if we assume (5):

$$\lambda_k^1 = i\frac{k\pi}{\int_0^1 \sqrt{\frac{\alpha}{\beta}}\,dx}; \qquad \lambda_k^2 = -\frac{\int_0^1 \sqrt{\frac{\gamma}{\delta}}\frac{a}{\gamma}\,dx}{\int_0^1 \sqrt{\frac{\gamma}{\delta}}\,dx} + i\frac{k\pi}{\int_0^1 \sqrt{\frac{\gamma}{\delta}}\,dx}, \quad k \in \mathbb{N},$$

and, if we assume (6):

$$\lambda_k^1 = i\frac{(2k+1)\pi}{\int_0^1 \sqrt{\frac{\alpha}{\beta}}\,dx}; \qquad \lambda_k^2 = -\frac{\int_0^1 \sqrt{\frac{\gamma}{\delta}}\frac{a}{\gamma}\,dx}{\int_0^1 \sqrt{\frac{\gamma}{\delta}}\,dx} + i\frac{k\pi}{\int_0^1 \sqrt{\frac{\gamma}{\delta}}\,dx}, \quad k \in \mathbb{N},$$

and it follows that

$$s(A_0) = 0,$$

which implies that $\omega(L) = \omega(A) = 0$. Thus e^{Lt} is not uniformly stable. This ends the proof of the first assertion.

(2). *The stability is uniform if* $\frac{\beta}{\alpha} \equiv \frac{\delta}{\gamma}$ on (0, 1) and a verifies (20).

We follow here [2,10]. The main idea is to construct a Lyapunov functional $\mathcal{L}_1$ that is a function which has the form

$$\mathcal{L}_1(t) = V\big(Y(t)\big), \quad t \in \mathbb{R}^+,$$

where $Y = (w, w_t, \varphi, \varphi_t)$ is a solution of (4), (5) or (4), (6) and V a functional from H into $\mathbb{R}^+$, such that for some positive constants c_1, c_2 and c_3:

$$c_1\|Y\|^2 \leqslant V(Y) \leqslant c_2\|Y\|^2 \quad \forall Y \in H \quad \text{and} \quad \frac{d}{dt}\mathcal{L}_1(t) \leqslant -c_3\big\|Y(t)\big\|^2,$$

for any solution of (4), (5) or (4), (6). A careful choice of multipliers and the sequence of estimates in the energy method will give the result.

Let

$$I_1 = \int_0^1 (-\gamma\varphi_t\varphi - \alpha w_t w)\,dx,$$

then

$$\frac{d}{dt}\int_0^1 -\gamma\varphi_t\varphi\,dx = \int_0^1 \delta(\varphi_x)^2\,dx - \int_0^1 \gamma(\varphi_t)^2\,dx + \int_0^1 \beta(\varphi + w_x)\varphi\,dx + \int_0^1 a\varphi_t\varphi\,dx,$$

$$\frac{d}{dt}\int_0^1 -\alpha w_t w\,dx = -\int_0^1 \alpha(w_t)^2\,dx + \int_0^1 \beta(\varphi + w_x)w_x\,dx.$$

Hence

$$\frac{d}{dt}I_1 = \int_0^1 \delta(\varphi_x)^2\,dx - \int_0^1 \alpha(w_t)^2\,dx - \int_0^1 \gamma(\varphi_t)^2\,dx + \int_0^1 \beta(\varphi + w_x)^2\,dx + \int_0^1 a\varphi_t\varphi\,dx.$$

With the Poincaré constant, we conclude that there is a constant $C > 0$ such that

$$\frac{d}{dt}I_1 \leqslant -\frac{1}{2}\int_0^1 \gamma(\varphi_t)^2\,dx - \int_0^1 \alpha(w_t)^2\,dx + C\int_0^1 \delta(\varphi_x)^2\,dx + \int_0^1 \beta(\varphi + w_x)^2\,dx.$$

Let us define the function f by

$$(\beta f_x)_x = -(\beta\varphi)_x, \quad f(0) = f(1) = 0,$$

if we deal with (5), and by

$$(\beta f_x)_x = -(\beta\varphi)_x, \quad f_x(0) = f(1) = 0,$$

if we deal with (6).

We treat here the case if we deal with (5) but the reader should see that minor changes in this proof lead to the same result.

We consider

$$I_2 = \int_0^1 (\gamma\varphi_t\varphi + \alpha w_t f)\,dx,$$

then

$$\frac{d}{dt}I_2 = -\int_0^1 \delta(\varphi_x)^2\,dx - \int_0^1 \beta\varphi^2\,dx - \int_0^1 a\varphi_t\varphi\,dx + \int_0^1 \alpha w_t f_t\,dx + \int_0^1 \beta f_x^2\,dx + \int_0^1 \gamma\varphi_t^2\,dx,$$

but, by using Poincaré constant

$$\int_0^1 \beta f_x^2\,dx \leqslant \int_0^1 \beta\varphi^2\,dx \leqslant c\int_0^1 \beta(\varphi_x)^2\,dx,$$

we have

$$\frac{d}{dt}I_2 \leqslant -\frac{1}{2}\int_0^1 \delta(\varphi_x)^2\,dx + \varepsilon_1\int_0^1 \alpha w_t^2\,dx + C_{\varepsilon_1}\int_0^1 \varphi_t^2\,dx.$$

Let

$$I_3 = \int_0^1 \big(\alpha\delta\varphi_t(\varphi + w_x) + \beta\gamma w_t\varphi_x\big)\,dx.$$

Then using $\frac{\beta}{\alpha} = \frac{\delta}{\gamma}$ we get

$$\frac{d}{dt}I_3 = -\int_0^1 \beta^2(\varphi + w_x)^2\,dx - \int_0^1 (\alpha\delta)_x w_t\varphi_t\,dx - \int_0^1 \beta a\varphi_t(\varphi + w_x)\,dx + \int_0^1 \beta\gamma\varphi_t\varphi\,dx + \int_0^1 \alpha\delta\varphi_t^2\,dx + [\beta\delta w_x\varphi_x]_{x=0}^{x=1},$$

which implies that for any $\varepsilon_2 > 0$ there exists a positive constant C_{ε_2} such that

$$\frac{d}{dt}I_3 \leqslant (-1+\varepsilon_2)\int_0^1 \beta^2(\varphi + w_x)^2\,dx + \varepsilon_2\int_0^1 \alpha w_t^2\,dx + C_{\varepsilon_2}\int_0^1 (\varphi_t)^2\,dx + [\beta\delta w_x\varphi_x]_{x=0}^{x=1}.$$

In order to deal with the boundary terms appearing in $\frac{d}{dt}I_3$, we consider the following multipliers, let

$$I_4 = N_1\int_0^1 \gamma\beta\varphi_t b(x)\varphi_x\,dx + N_2\int_0^1 \alpha\delta w_t b(x) w_x\,dx,$$

where b on $C^1[0,1]$ satisfy $b(0) > 0$, $b(1) < 0$ (for example, $b(x) = \frac{1-2x}{2}$), N_1 large number and N_2 sufficiently small. Then we obtain

$$\frac{d}{dt}I_4 \leqslant -\frac{N_2\beta\delta}{2}w_x^2(1) - \frac{N_2\beta\delta}{2}w_x^2(0) - \frac{N_1\beta\delta}{2}\varphi_x^2(1) - \frac{N_1\beta\delta}{2}\varphi_x^2(0) + N_1\left(C_1\int_0^1 \varphi_t^2\,dx + C_\varepsilon\int_0^1 \varphi_x^2\,dx + \varepsilon\int_0^1 w_x^2\,dx\right) + N_2\left(C_1\int_0^1 w_t^2\,dx + C_1\int_0^1 \varphi_x^2\,dx + C_1\int_0^1 w_x^2\,dx\right).$$

Now we consider

$$I_5 = I_3 + I_4,$$

and observing that

$$\frac{(-1+\varepsilon_2)}{2}\int_0^1 \beta^2(\varphi + w_x)^2\,dx \leqslant \frac{(-1+\varepsilon_2)}{4}\int_0^1 \beta^2 w_x^2\,dx + C_2\int_0^1 \beta^2\varphi^2\,dx,$$

for some positive constant C_2, ε, ε_i sufficiently small, we have for $0 \prec \mu \prec 1$ and some C_μ, C_3 positives, such that

$$\frac{d}{dt}I_5 \leqslant \frac{(-1+\varepsilon_2)}{2}\int_0^1 \beta^2(\varphi + w_x)^2\,dx + \mu C_3\int_0^1 \alpha w_t^2\,dx + C_\mu\left(\int_0^1 \varphi_t^2\,dx + \int_0^1 \varphi_x^2\,dx\right).$$

Finally, we consider

$$I_6 = I_5 + 2\mu C_3 \,.\, I_1,$$

choosing μ small, then

$$\frac{d}{dt} I_6 \leqslant \frac{(-1+\varepsilon_2)}{4} \int_0^1 \beta^2 (\varphi + w_x)^2 \, dx - \mu C_3 \int_0^1 \alpha w_t^2 \, dx + C_\mu \left(\int_0^1 \varphi_t^2 \, dx + \int_0^1 \varphi_x^2 \, dx \right).$$

The Lyapunov functional is now defined by

$$\mathcal{L}_1(t) = N\mathcal{E}(t) + I_2 + \mu_1 I_6$$

choosing ε_i $(i = 1, 2)$, μ_1 sufficiently small and N sufficiently large, then we get the uniform stability. □

4. Uniform stability of (1), (7)

In this section, we assume also that α, β, γ and δ *are positive* C^1 functions of the space variable.

Again, we denote by L the operator associated with system (1), (7) and it is easy to check that it is the infinitesimal generator of a C_0-semigroup e^{Lt} in the space H defined as in the previous section by taking into account (7).

We state and prove our second result.

Theorem 2. *If α, β, γ and δ are positive functions, then*:

1. *If $\frac{\beta}{\alpha} = \frac{\delta}{\gamma}$ on $(0, 1)$ then e^{Lt} is uniformly stable up to a finite-dimensional space of initial data.*
2. *If $\frac{\beta}{\alpha} \neq \frac{\delta}{\gamma}$ on $(0, 1)$ then e^{Lt} is not uniformly stable.*

Proof. We work with the transformed system (11), (17). In the present case, we just have to put $a = 0$ in the expression of C_0 and C_1.

(1) *The stability is non-uniform if* $\frac{\beta}{\alpha} \neq \frac{\delta}{\gamma}$. The eigenvalues of M (see (12) for its definition) are again distinct in this case and the result of Neves, Ribeiro and Lopes [8, Theorems A and B, p. 324] asserts that

$$r_e\big(e^{At}\big) = r_e\big(e^{A_0 t}\big) = e^{\alpha_0 t},$$

where $\alpha_0 = s(A_0)$. The eigenvalues of the system

$$A_0 \begin{pmatrix} U \\ V \end{pmatrix} = \lambda \begin{pmatrix} U \\ V \end{pmatrix}, \quad \begin{pmatrix} U \\ V \end{pmatrix} \in D(A_0),$$

where $U = (u_1, u_2)$ and $V = (v_1, v_2)$, are easily computed (recall that we work here with (17)):

$$\lambda_k^1 = i \frac{k\pi}{\int_0^1 \sqrt{\frac{\beta}{\alpha}(x)}\, dx};$$

$$\lambda_k^2 = \frac{1}{2 \int_0^1 \sqrt{\frac{\gamma}{\delta}(x)}\, dx} \ln \left| \frac{\big(c\sqrt{\frac{\delta}{\gamma}(1)} - 1\big)\big(d\sqrt{\frac{\delta}{\gamma}(1)} - 1\big)}{\big(c\sqrt{\frac{\delta}{\gamma}(1)} + 1\big)\big(d\sqrt{\frac{\delta}{\gamma}(1)} + 1\big)} \right| + i \frac{k\pi}{\int_0^1 \sqrt{\frac{\gamma}{\delta}(x)}\, dx}, \quad k \in \mathbb{Z}.$$

Therefore $\omega(L)=\omega(A)=0$ and the claim is proved.

(2) *The stability is uniform if* $\frac{\beta}{\alpha}=\frac{\delta}{\gamma}$. In this case, M has two double eigenvalues and we apply the result in [1] which asserts that

$$r_e\big(e^{At}\big)=r_e\big(e^{A_0t}\big),$$

with, this time, setting $2s:=\sqrt{\frac{\beta}{\alpha}}=\sqrt{\frac{\delta}{\gamma}}$

$$A_0=M\frac{\partial}{\partial x}+\widetilde{C},\quad \widetilde{C}=\begin{pmatrix}K & 0_2\\ 0_2 & -K\end{pmatrix},\ K=\begin{pmatrix}-2s' & -s\\ s & -2s'\end{pmatrix},$$

where 0_2 denotes the 2×2 null matrix. Here K and $-K$ are the 2×2 matrices extracted from C by taking $K=(c_{ij})_{1\leqslant i,j\leqslant 2}$ and $-K=(c_{ij})_{3\leqslant i,j\leqslant 4}$ as asserted in [1].

We first compute the eigenvalues of A_0. In order to do this, we introduce some reformulations and notations. If we set $U=(u_1,u_2)$, $V=(v_1,v_2)$ and

$$l=1+2cs(0),\qquad m=1+2ds(1),\qquad n=2cs(0)-1,\qquad p=1-2ds(1),$$
$$B_0=\begin{pmatrix}1 & 0\\ 0 & \frac{n}{l}\end{pmatrix},\qquad B_1=\begin{pmatrix}1 & 0\\ 0 & \frac{p}{m}\end{pmatrix},$$

then the boundary conditions (17) may be reformulated in the following way:

$$\begin{cases}U(0)=-B_0V(0),\\ V(1)=-B_1U(1).\end{cases}\tag{21}$$

We also introduce

$$P=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -i & i\end{pmatrix},\qquad P^{-1}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ 1 & -i\end{pmatrix},\quad i^2=-1,$$

then

$$K=PDP^{-1},\qquad D(x)=\begin{pmatrix}-2s'(x)+is(x) & 0\\ 0 & -2s'(x)-is(x)\end{pmatrix},\quad x\in(0,1).$$

The eigenvalue system of A_0,

$$A_0\begin{pmatrix}U\\ V\end{pmatrix}=\lambda\begin{pmatrix}U\\ V\end{pmatrix},$$

writes, with the notations we have introduced:

$$\begin{cases}-2s\frac{dU}{dx}+KU=\lambda U\quad \text{on } (0,1),\\ 2s\frac{dV}{dx}-KV=\lambda V\qquad \text{on } (0,1),\\ \quad\text{and } (21).\end{cases}\tag{22}$$

Introducing the new unknown functions $\widetilde{U}=P^{-1}U$ and $\widetilde{V}=P^{-1}V$, (22) writes

$$\begin{cases}\dfrac{d\widetilde{U}}{dx}=\dfrac{1}{2s}(-\lambda I+D)\widetilde{U}\quad \text{on } (0,1),\\ \dfrac{d\widetilde{V}}{dx}=\dfrac{1}{2s}(\lambda+D)\widetilde{V}\qquad \text{on } (0,1),\\ \widetilde{U}(0)=-P^{-1}B_0P\widetilde{V}(0),\qquad \widetilde{V}(1)=-P^{-1}B_1P\widetilde{U}(1).\end{cases}\tag{23}$$

From and since the two first equations in systems (23) are diagonal differential systems, we get the solutions:

$$\widetilde{U}(x) = e^{\int_0^x \frac{(-\lambda I+D)}{s}(\tau)\,d\tau}\,\widetilde{U}(0), \qquad \widetilde{V}(x) = e^{\int_0^x \frac{(\lambda I+D)}{s}(\tau)\,d\tau}\,\widetilde{V}(0)$$

and the boundary conditions lead to

$$\begin{aligned}\widetilde{V}(0) &= -e^{-\int_0^1 \frac{(\lambda I+D)}{s}(\tau)\,d\tau} P^{-1} B_1 P \widetilde{U}(1)\\ &= -e^{-\int_0^1 \frac{(\lambda I+D)}{s}(\tau)\,d\tau} P^{-1} B_1 P\, e^{\int_0^1 \frac{(-\lambda I+D)}{s}(\tau)\,d\tau}\,\widetilde{U}(0)\\ &= e^{-\int_0^1 \frac{(\lambda I+D)}{s}(\tau)\,d\tau} P^{-1} B_1 P e^{\int_0^1 \frac{(-\lambda I+D)}{s}(\tau)\,d\tau} P^{-1} B_0 P \widetilde{V}(0)\\ &= R(\lambda)\widetilde{V}(0).\end{aligned}$$

It follows that the eigenvalue equation is

$$\det\bigl(I - R(\lambda)\bigr) = 0, \tag{24}$$

which, after some computations, writes

$$h\bigl(e^{2\lambda\int_0^1 \frac{dr}{s(r)}}\bigr) := e^{4\lambda\int_0^1 \frac{dr}{s(r)}} - (z+\bar{z})e^{2\lambda\int_0^1 \frac{dr}{s(r)}} + |z|^2 - |\tau|^2 = 0, \tag{25}$$

with

$$\begin{aligned} z &= \frac{v_1 + v_2 e^{2i}}{(v_1+1)(1+v_2)}, \qquad \tau = \frac{1 + v_1 v_2 e^{2i}}{(v_1+1)(1+v_2)},\\ z+\bar{z} &= 2\frac{v_1 + v_2\cos(2)}{(v_1+1)(1+v_2)}, \qquad |z|^2 - |\tau|^2 = \frac{v_1^2 + v_2^2 - v_1^2 \,.\, v_2^2 - 1}{(v_1+1)^2(1+v_2)^2},\end{aligned}$$

where $v_1 = 2cs(0)$ and $v_2 = 2\,ds(1)$. The simplified discriminant of (25) is computed as

$$\Delta(v_1, v_2) = \frac{(v_1^2 - \sin^2(2))v_2^2 + 2(v_1\cos(2))v_2 + 1}{(v_1+1)^2(1+v_2)^2}.$$

Thus, there are two cases:

First case: $\Delta(v_1, v_2) \geqslant 0$. In this situation, we get the sequence of eigenvalues:

$$\lambda_k^{\pm} = \frac{1}{2\int_0^1 \frac{d\tau}{s(\tau)}} \ln\left|\frac{v_1 + v_2\cos(2)}{(v_1+1)(1+v_2)} \pm \sqrt{\Delta(v_1,v_2)}\right| + i\frac{k\pi}{\int_0^1 \frac{d\tau}{s(\tau)}}, \quad k \in \mathbb{Z}.$$

A simple computation shows that

$$h(\pm 1) > 0 \quad \Leftrightarrow \quad v_1 \neq 0 \quad \text{or} \quad v_2 \neq 0,$$

which implies, assuming this condition, that

$$\mathrm{Re}\bigl(\lambda_k^{\pm}\bigr) = \frac{1}{2\int_0^1 \frac{d\tau}{s(\tau)}} \ln\left|\frac{v_1 + v_2\cos(2)}{(v_1+1)(1+v_2)} \pm \sqrt{\Delta(v_1,v_2)}\right| < 0, \quad \forall k \in \mathbb{Z}.$$

Second case: $\Delta(v_1, v_2) < 0$. We get the sequence of eigenvalues:

$$\lambda_k^{\pm} = \frac{1}{4\int_0^1 \frac{d\tau}{s(\tau)}} \ln\left(\frac{(v_1 + v_2\cos(2))^2}{(v_1+1)^2(1+v_2)^2} - \Delta(v_1,v_2)\right) + i\frac{\frac{\theta}{2} \pm k\pi}{\int_0^1 \frac{d\tau}{s(\tau)}}, \quad k \in \mathbb{N},$$

where $\theta = \arg(\frac{v_1+v_2\cos(2)}{(v_1+1)(1+v_2)} + i\sqrt{-\Delta(v_1,v_2)})[2\pi]$. Again, it is not difficult to verify that

$$\mathrm{Re}\bigl(\lambda_k^{\pm}\bigr) = \frac{1}{4\int_0^1 \frac{d\tau}{s(\tau)}} \ln\left(\frac{(v_1 + v_2\cos(2))^2}{(v_1+1)^2(1+v_2)^2} - \Delta(v_1,v_2)\right) < 0, \quad \forall k \in \mathbb{Z}.$$

We conclude with the help of the following result.

Lemma 3. (M. Renardy [9, Theorem 1, p. 1300]) *Let H be a Hilbert space, and let $L = L_0 + B$ be the infinitesimal generator of a C_0-semigroup of operators in H. Assume that L_0 is normal and B is bounded. Assume that there exists a number $M > 0$ and an integer n such that the following hold*:

(a) *If $\lambda \in \sigma(L_0)$ and $|\lambda| > M - 1$, then λ is an isolated eigenvalue of finite multiplicity.*
(b) *If $|z| > M$, then the number of eigenvalues of L_0 in the unit disk centered at z (counted by multiplicity) does not exceed n.*

Then $\omega_e(L) \leqslant s(L)$.

Applying this lemma to A_0 and taking into account the previous computations of its eigenvalues and the definition of ω_e, we get that $s(A_0) = \omega_e(A_0)$ and uniform stability occurs if we prove the asymptotic stability of our initial system: this will be done in the next section. □

5. Asymptotic stability

In this section, we have:

Theorem 4. *Under the assumption* (3), *assume that $a \in C([0,1])$ and*

$$a \geqslant a_0 > 0 \quad \text{on } (0,1). \tag{26}$$

Then e^{Lt} is asymptotically stable.

Proof. Returning to system (4), (5) and differentiating the energy given by (2) one gets:

$$\mathcal{E}'(t) = -\int_0^1 a\varphi_t^2\,dx \leqslant 0.$$

From Lasalle's invariance principle, the asymptotic stability holds true if the unique solution of the system:

$$\begin{aligned} &\alpha w_{tt} = \big(\beta(\varphi + w_x)\big)_x, \qquad \gamma\varphi_{tt}(x,t) = (\delta\varphi_x)_x - \beta(\varphi + w_x),\\ &\varphi_t = 0 \quad \text{on } (0,1)\times\mathbb{R}^+, \end{aligned} \tag{27}$$

completed with the boundary conditions in (4), (5), is the trivial one. But, the third equation in (27) implies that φ is independent of t and, thus, from the second equation in (27) one deduces that w is also independent of t. We are lead to the system:

$$\begin{aligned} &\big(\beta(\varphi + w_x)\big)_x = 0, \qquad (\delta\varphi_x)_x - \beta(\varphi + w_x) = 0,\\ &w(0) = w(1) = 0, \qquad \varphi(0) = \varphi(1) = 0. \end{aligned}$$

From the first equation of the last system, there exists a constant $a_1 \in \mathbb{R}$ such that $\beta(\varphi + w_x) = a_1$. If we let

$$p(x) = 1/\delta, \qquad q(x) = xp(x), \qquad r(x) = 1/\beta, \tag{28}$$

and if we denote, $\bar{u} = \int_0^1 u(x)\,dx$ the mean value of any function u, then, taking into account the boundary conditions on φ the second equation allows the computation of φ and the first equation the computation of w:

$$\varphi(x) = a_1\left(\int_0^x q(\tau)\,d\tau - \frac{\bar{q}}{\bar{p}}\int_0^x p(\tau)\,d\tau\right),$$

$$w(x) = a_1\left(\int_0^x r(\tau)\,d\tau - \int_0^x\int_0^y q(\tau)\,d\tau\,dy + \frac{\bar{q}}{\bar{p}}\int_0^x\int_0^y p(\tau)\,d\tau\,dy\right).$$

Assume now that $a_1 \neq 0$. The condition $w(1) = 0$ is satisfied if and only if

$$\int_0^1 r(\tau)\,d\tau - \int_0^1\int_0^y q(\tau)\,d\tau\,dy + \frac{\bar{q}}{\bar{p}}\int_0^1\int_0^y p(\tau)\,d\tau\,dy = 0. \tag{29}$$

We will end the proof of the asymptotic stability by showing that this last identity is impossible. The following lemma allows to achieve this contradiction:

Lemma 5. *If p and q are given by* (28)*, then*

$$-\int_0^1\int_0^y q(\tau)\,d\tau\,dy + \frac{\bar{q}}{\bar{p}}\int_0^1\int_0^y p(\tau)\,d\tau\,dy \geqslant 0. \tag{30}$$

If we admit for a moment this lemma, then clearly (29) is impossible and, thus, $a_1 = 0$ and the desired result follows.

Proof of Lemma 5. From the Cauchy–Schwarz inequality, we get

$$\left(\int_0^1 yp(y)\,dy\right)^2 \leqslant \int_0^1 p(y)\,dy\int_0^1 y^2p(y)\,dy.$$

It follows that

$$\begin{aligned}\int_0^1 yp(y)\,dy&\left(\int_0^1 p(y)\,dy - \int_0^1 yp(y)\,dy\right)\\ &\geqslant \int_0^1 p(y)\,dy\left(\int_0^1 yp(y)\,dy - \int_0^1 y^2p(y)\,dy\right).\end{aligned} \tag{31}$$

But integrating by part, one can derive the following identities:

$$\int_0^1\int_0^y \tau p(\tau)\,d\tau\,dy = \int_0^1 yp(y)\,dy - \int_0^1 y^2p(y)\,dy,$$

$$\int_0^1\int_0^y p(\tau)\,d\tau\,dy = \int_0^1 p(y)\,dy - \int_0^1 yp(y)\,dy.$$

Inserting these identities in (31), we get

$$\int_0^1 yp(y)\,dy \int_0^1 \int_0^y p(\tau)\,d\tau\,dy \geqslant \int_0^1 p(y)\,dy \int_0^1 \int_0^y \tau p(\tau)\,d\tau\,dy, \quad \text{or}$$

$$\bar{q} \int_0^1 \int_0^y p(\tau)\,d\tau\,dy \geqslant \bar{p} \int_0^1 \int_0^y q(\tau)\,d\tau\,dy,$$

which is equivalent to the desired inequality (30). □

Acknowledgments

The authors thank the anonymous referees for their careful review and helpful suggestions and comments.

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