SUFFICIENT CONDITIONS FOR UNIFORM STABILIZATION OF SECOND ORDER EQUATIONS BY DYNAMICAL CONTROLLERS.

Farid AMMAR KHODJA, Assia BENABDALLAH

ammar@math.univ-fcomte.fr assia@math.univ-fcomte.fr

Université de Franche-Comté,
Laboratoire de Calcul Scientifique et CNRS- UMR 6623,
16, route de Gray, 25030 Besançon Cedex, France

in "Dynamics of Continuous Discrete and Impulsive Systems"
Volume 7, 2 (2000), 207-222

Abstract. We look for dynamical stabilizers for a given second order equation. Sufficient conditions on the stabilizers are obtained. A complete description of a one-parameter family of such stabilizers is given which includes a precise behaviour of the spectrum of the associated operators. These abstract results are applied to thermoelastic systems.

AMS (MOS) subject classification : 35B40, 35M10, 35L05, 35L70, 93C15,93C20.

1 Introduction, preliminaries, main results.

Let X and Y be two Hilbert spaces and A, B and C unbounded operators, with dense domains, acting on X, from Y to X and on Y respectively. We consider the system:

$$\left\{\begin{array}{rrl} u''(t)+Au(t)= & Bw(t) & t>0 \\ w'(t)+Cw(t)= & -B^*u'(t) & t>0 \\ u(0)=u_0, \quad u'(0)=v_0, & w(0)=w_0 & \end{array}\right. \tag{1}$$

where B^* is the adjoint operator of B. If A is a positive selfadjoint operator, our aim is to find sufficient conditions on B and C which insure the uniform stability of the preceding system in the space $H = D(A^{\frac{1}{2}}) \times X \times Y$. Usually, to stabilize (uniformly) the first equation of the system, one looks for static feedbacks as is done in [6], [7], for example (see also references therein). Our problem can be seen as the search of dynamical stabilizers for the first equation.

Throughout, we will assume two kinds of hypothesis. The first one will insure the semigroup property for system (1) (assumption (**H1)**). The second will insure the uniform (exponential) stability for the semigroup solution of (1) (assumption (**H2**)).

(H.1) (i) A (resp. C) is a self-adjoint operator on a Hilbert space X (resp. Y), strictly positive, with dense domain $D(A)$ (resp. $D(C)$)and compact resolvent $R(\lambda, A)$ (resp. $R(\lambda, C)$);

(ii) B is an operator from Y to X such that:

$$D(C) \subset D(B) \tag{2}$$

and is C-bounded.

(iii) There exists $a > 0$ such that

$$\| A^{-\frac{1}{2}} B C^{-\frac{1}{2}} w \|_X \leq \quad a ||w||_Y \qquad \forall w \in D(C^{\frac{1}{2}}); \tag{3}$$

(vi)

$$\overline{(D(A^{\frac{1}{2}}) \cap D(B^*))} = X$$

In the sequel, $H = D(A^{\frac{1}{2}}) \times X \times Y$ will be equipped with the inner product:

$$\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}, \begin{bmatrix} f \\ g \\ h \end{bmatrix} \right) = \left(A^{\frac{1}{2}} u, A^{\frac{1}{2}} f \right)_X + (v, g)_X + (w, h)_Y$$

and the induced norm will be denoted by $||.||$. System (1) can be equivalently written:

$$\begin{cases} Y'(t) = & LY(t) \quad t \in R^+ \\ Y(0) = & Y_0 \end{cases} \tag{4}$$

$$L = \begin{pmatrix} 0 & I & 0 \\ -A & 0 & B \\ 0 & -B^* & -C \end{pmatrix} \tag{5}$$

$$D(L) = D(A) \times (D(A^{\frac{1}{2}}) \cap D(B^*)) \times D(C)$$

and

$$Re(LY, Y) = -(Cw, w)_Y \leq 0 \tag{6}$$

for all $Y = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in D(L)$.

L is dissipative with dense domain in H thus L is closable (see [14], Theorem 4.5, p. 15). The previous assumptions allow to prove that range of $I - L$ is dense in H so, by the Lumer-Phillips theorem (see [14], Corollary 4.4,p. 15), $\bar{L}$, the closure of L, generates a strongly continuous semigroup of contractions $S_{\bar{L}}(t)$ on H. *In all what follows, $\bar{L}$ will be denoted by L.*

(H.2)

(i) B^* is invertible ($(B^*)^{-1} \in L(X,Y)$) and

$$\| (B^*)^{-1} C^{\frac{1}{2}} w \|_X \leq \quad b||w||_Y \qquad \forall w \in D(C^{\frac{1}{2}}); \tag{7}$$

a and b being positive real constants.

(ii) There exist three positive constants c_1, c_2, c_3 such that for all $Y \in D(L)$

$$|(Au, (B^*)^{-1} w)_X| \leq \frac{c_1}{2\alpha} ||C^{\frac{1}{2}} w||_Y^2 + \frac{\alpha}{2} \left(c_2 ||Y||^2 + (G(Y), LY) \right) \tag{8}$$

for all $\alpha > 0$, where G is a function from H to $L(H)$ which satisfies

$$||G(Y)|| \leq c_3 ||Y|| \qquad \forall Y \in H \tag{9}$$

In a previous paper (see [1,2]), we had already pointed out that the uniform stability is not achieved, in general, if B is a bounded operator. Thus, the assumption (**H2**) gives the "order" of the unboundedness of B which will insure uniform stability for the system (1).

The main result is then:

Theorem 1 *Under the assumption (**H**), $(S_L(t))$ is uniformly stable i.e. there exist $\omega > 0$ and $M \geq 1$ such that:*

$$\| S_L(t) \|_{L(H)} \leq M e^{-\omega t} \qquad \forall t \in R^+ \tag{10}$$

Moreover, ω can be estimated by (32).

A particular system of this kind is obtained if one takes $X = Y$, $A = C$ and $B = A^\alpha$ where $\alpha \in [0, 1]$. Denoting by

$$L_\alpha = \begin{pmatrix} 0 & I & 0 \\ -A & 0 & A^\alpha \\ 0 & -A^\alpha & -A \end{pmatrix}; \qquad D(L_\alpha) = D(A) \times \left(D(A^\alpha) \cap D(A^{\frac{1}{2}}) \right) \times D(A) \tag{11}$$

we show the following:

Theorem 2 *(i) The strongly continuous semigroup of contractions $(S_\alpha(t))$ generated by L_α is uniformly stable if and only if $\alpha \in [\frac{1}{2}, 1]$. In this case, $S_\alpha(t)$ satisfies (10) with $\omega = -\sup \mathrm{Re}\ \sigma(L_\alpha)$, $\sigma(L_\alpha)$ being the spectrum of L_α.*

Moreover

$$P_\alpha = \begin{pmatrix} \frac{1}{2}A^{2(\alpha-1)} + A^{1-2\alpha} + A^{-2\alpha} & \frac{1}{2}A^{-1} & -A^{-\alpha-1} + \frac{1}{2}A^{\alpha-2} \\ \frac{1}{2}I & A^{1-2\alpha} + A^{-2\alpha} + A^{-1} & A^{-\alpha} \\ -A^{-\alpha} + \frac{1}{2}A^{\alpha-1} & A^{-\alpha} & \frac{3}{2}A^{-1} \end{pmatrix}$$

is the unique positive, selfadjoint operator solution of

$$2\Re e\,(P_\alpha L_\alpha Y, Y) = - \parallel Y \parallel^2 \qquad \forall Y \in D(L_\alpha)$$

and it is bounded if and only if $\frac{1}{2} \le \alpha \le 1$ and

$$\omega \ge \frac{1}{2 \parallel P_\alpha \parallel_{L(H)}}$$

(ii) If $\alpha \in [0, \frac{1}{2}[$, $S_\alpha(t)$ is strongly stable.
(iii) The semigroup $S_\alpha(t)$ is analytic if and only if $\frac{3}{4} \le \alpha \le 1$.
(iv) The semigroup is compact if $\frac{1}{2} < \alpha \le 1$.

This result makes more precise our remark on the "order" of the unboundedness of B with respect to A and C.

Remark 1 *In fact, one can consider the little more complicated particular system corresponding to the operator*

$$L_{\alpha,\beta} = \begin{pmatrix} 0 & I & 0 \\ -A & 0 & A^\alpha \\ 0 & -A^\alpha & -A^\beta \end{pmatrix};$$

$$D(L_\alpha) = D(A) \times \Big(D(A^\alpha) \cap D(A^{\frac{1}{2}})\Big) \times D(A^\beta)$$

with $0 \le \alpha \le \beta$ and show that the associated semigroup $(S_{\alpha,\beta}(t))$ is uniformly stable on H if and only if

$$\max(1-\beta, \beta) \le 2\alpha \le 1 + \beta$$

Section 2 of this paper contains the proof of the first theorem and the method used is based on finding a Lyapunov function for system (1). In the third section, we prove THEOREM 2 by analyzing the spectrum of L_α and showing that it is a spectral operator. In the fourth section, we show how these last results provide direct proofs for the uniform stability of two models of thermoelastic plates and allow an estimate for the decay rate of the energy.

2 Proof of THEOREM 1.1

Let $Y_0 \in D(L)$ and $Y(t) = \begin{bmatrix} u(t) \\ v(t) \\ w(t) \end{bmatrix}$ the corresponding solution of system (4).

We introduce the real function:

$$\begin{aligned} \rho_\varepsilon(t) &= \| Y(t) \|^2 - \varepsilon \frac{\alpha}{2} F(Y(t)) \\ &\quad + \varepsilon \left((u(t), v(t))_X + 6 \left(w(t), B^{-1} v(t) \right)_Y + \frac{15}{4} \| C^{-\frac{1}{2}} w(t) \|_Y^2 \right) \end{aligned} \tag{12}$$

for $t \geq 0$ and F is a scalar differentiable function on H such that

$$F'(Y) = G(Y) \qquad Y \in H$$

In a first step, we show that for sufficiently small $\varepsilon > 0$, there exists a constant $D_\varepsilon > 0$ such that:

$$\frac{d}{dt} \rho_\varepsilon(t) \leq -D_\varepsilon \rho_\varepsilon(t) \qquad t \geq 0 \tag{13}$$

This inequality will imply that

$$\rho_\varepsilon(t) \leq e^{-D_\varepsilon t} \rho_\varepsilon(0) \qquad t \geq 0 \tag{14}$$

We end the proof by showing, in a second step, that for $\varepsilon > 0$ sufficiently small, there exist two positive constants M_ε and N_ε such that:

$$M_\varepsilon \| Y(t) \|^2 \leq \rho_\varepsilon(t) \leq N_\varepsilon \| Y(t) \|^2 \qquad t \geq 0 \tag{15}$$

from which we deduce the desired result by density.

Step 1: Differentiating (12), we obtain:

$$\begin{aligned} \rho'_\varepsilon &= 2Re\,(Y, Y') \\ &\quad + \varepsilon \left(\| v \|_X^2 + (u, v')_X + 6 \left(w', B^{-1} v \right)_Y \right) \\ &\quad + \varepsilon \left(6 \left(w, B^{-1} v' \right)_Y + \frac{15}{2} \Re e \left(C^{-\frac{1}{2}} w', C^{-\frac{1}{2}} w \right)_Y - \frac{\alpha}{2} \frac{d}{dt} F(Y(t)) \right) \end{aligned} \tag{16}$$

Using (6) and the two first equations of (1), we get:

$$\begin{aligned} \rho'_\varepsilon &= -2 \left(Cw, w \right)_Y \\ &\quad + \varepsilon \left(\| v \|_X^2 - \| A^{\frac{1}{2}} u \|_X^2 + (Bw, u)_X \right) \\ &\quad + \varepsilon \left(-6 \left(Cw, B^{-1} v \right)_Y - 6 \left(B^* v, B^{-1} v \right)_Y \right) \end{aligned}$$

$$+\varepsilon\left(-6\left((B^*)^{-1}w, Au\right)_X + 6\left((B^*)^{-1}w, Bw\right)_X\right)$$

$$+\varepsilon\left(-\frac{15}{2}\parallel w\parallel_Y^2 + \frac{15}{2}\left(B^*v, -C^{-1}w\right)_Y - \frac{\alpha}{2}\frac{d}{dt}F(Y(t))\right) \quad (17)$$

and after simplification

$$\begin{aligned}\rho'_\varepsilon &= -2\left(Cw, w\right)_Y + \varepsilon\left(-5\parallel v\parallel_X^2 - \parallel A^{\frac{1}{2}}u\parallel_X^2 - \frac{3}{2}\parallel w\parallel_Y^2\right)\\ &\quad +\varepsilon\left((Bw, u)_X - 6\left(Cw, B^{-1}v\right)\right)_Y\\ &\quad +\varepsilon\left(-6\left((B^*)^{-1}w, Au\right)_X - \frac{15}{2}\left(B^*v, C^{-1}w\right)_Y - \frac{\alpha}{2}\frac{d}{dt}F(Y(t))\right) \quad (18)\end{aligned}$$

Now, using inequality (3), one gets

$$\begin{aligned}|(Bw, u)_X| &= \left|\left(A^{-\frac{1}{2}}Bw, A^{\frac{1}{2}}u\right)_X\right|\\ &\leq a\parallel C^{\frac{1}{2}}w\parallel_Y\parallel A^{\frac{1}{2}}u\parallel_X \quad (19)\end{aligned}$$

In the same way, using inequality (7), one has

$$\begin{aligned}\left|\left(Cw, B^{-1}v\right)_Y\right| &= \left|\left((B^*)^{-1}Cw, v\right)_X\right|\\ &\leq c\parallel C^{\frac{1}{2}}w\parallel_Y\parallel v\parallel_X \quad (20)\end{aligned}$$

From the inequality (8), we deduce that

$$|(Au(t), (B^*)^{-1}w(t))| \leq \frac{c_1}{2\alpha}||C^{\frac{1}{2}}w(t)||_Y^2 + \frac{\alpha c_2}{2}||Y(t)||^2 + \frac{\alpha}{2}\frac{d}{dt}F(Y(t)) \quad (21)$$

Finally, using the C-boundedness of B ((**H1**)(ii))

$$\begin{aligned}\left|\left(B^*v, C^{-1}w\right)_Y\right| &= \left|\left(v, BC^{-1}w\right)_X\right|\\ &\leq d\parallel C^{\frac{1}{2}}w\parallel_Y\parallel v\parallel_X \quad (22)\end{aligned}$$

where $d = \mu_1^{-\frac{1}{2}}\parallel BC^{-1}\parallel_{L(Y,X)}$, μ_1 being the first eigenvalue of C.

The identity (18) and the estimations (19)-(22) give

$$\begin{aligned}\rho'_\varepsilon &\leq -2\left\|C^{\frac{1}{2}}w\right\|_Y^2 - \varepsilon\left(5\|v\|_X^2 + \left\|A^{\frac{1}{2}}u\right\|_X^2 + \frac{3}{2}\|w\|_Y^2\right)\\ &\quad +\varepsilon\left(a\left\|C^{\frac{1}{2}}w\right\|_Y\left\|A^{\frac{1}{2}}u\right\|_X + (6c + \frac{15}{2}d)\left\|C^{\frac{1}{2}}w\right\|_Y\|v\|_X\right)\\ &\quad +6\varepsilon\left(\frac{c_1}{2\alpha}\left\|C^{\frac{1}{2}}w\right\|_Y^2 + \frac{\alpha c_1}{2}\|Y(t)\|^2\right) \quad (23)\end{aligned}$$

Setting $a_1 = 6c + \frac{15}{2}d$ and using the following inequalities

$$\| \quad C^{\frac{1}{2}} w \|_Y \| A^{\frac{1}{2}} u \|_X \leq \frac{\beta}{2} \| A^{\frac{1}{2}} u \|_X^2 + \frac{1}{2\beta} \| C^{\frac{1}{2}} w \|_Y^2$$

$$\| \quad C^{\frac{1}{2}} w \|_Y \| v \|_X \leq \frac{\gamma}{2} \| v \|_X^2 + \frac{1}{2\gamma} \| C^{\frac{1}{2}} w \|_Y \tag{24}$$

β and γ being any positive real constants, we obtain from (23)

$$\begin{aligned} \rho'_\varepsilon \quad \leq \quad & \left(-2 + \varepsilon \left(\frac{a}{2\beta} + \frac{a_1}{2\gamma} + 3\frac{c_1}{\alpha} \right) \right) \| C^{\frac{1}{2}} w \|_Y^2 \\ & + \varepsilon \left(\left(-5 + \frac{a_1 \gamma}{2} \right) \| v \|_X^2 + \left(-1 + \frac{a\beta}{2} \right) \| A^{\frac{1}{2}} u \|_X^2 - \frac{3}{2} \| w \|_Y^2 \right) \\ & + 6\varepsilon \left(\frac{\alpha c_2}{2} ||Y(t)||^2 \right) \end{aligned} \tag{25}$$

Choosing $\alpha = \frac{1}{6c_2}$, $\beta = \frac{1}{2a}$, $\gamma = \frac{17}{2a_1}$, and

$$\varepsilon < \frac{34}{17a^2 + a_1^2 + 306 c_1 c_2}$$

inequality (25) becomes simply

$$\rho'_\varepsilon \leq -\frac{\varepsilon}{4} \| Y \|^2 \tag{26}$$

Step 2: We prove now (15). One has

$$\begin{aligned} |(u, v)_X| \quad &= \quad \left| \left(A^{\frac{1}{2}} u, A^{-\frac{1}{2}} v \right)_X \right| \\ &\leq \quad \frac{1}{2\delta_1^{\frac{1}{2}}} \left(\| A^{\frac{1}{2}} u \|_X^2 + \| v \|_X^2 \right) \end{aligned} \tag{27}$$

where $\delta_1 > 0$ is the first eigenvalue of A,

$$\left| \left(w, B^{-1} v \right)_Y \right| \quad \leq \frac{m}{2} \left(\| w \|_Y^2 + \| v \|_X^2 \right) \tag{28}$$

where m is the continuity constant of B^{-1}. From the definition of ρ_ε (see (12)) and (9), (27)-(28), it is clear that with

$$D = \frac{1}{2\delta_1^{\frac{1}{2}} + \frac{m}{2}} + \frac{c_3}{12 c_2}$$

one has

$$(1 - \varepsilon D) \| Y \|^2 \leq \rho_\varepsilon \leq (1 + \varepsilon D) \| Y \|^2 \tag{29}$$

and thus, (15) follows.

Using the second inequality of (15) in (26), we obtain

$$\rho'_\varepsilon \le -\frac{\varepsilon}{2N\varepsilon}\rho_\varepsilon \tag{30}$$

and

$$\rho_\varepsilon(t) \le e^{-\frac{\varepsilon}{2N_\varepsilon}t}\rho_\varepsilon(0)$$

The first inequality of (15) then gives

$$\parallel Y(t) \parallel^2 \le \frac{\rho_\varepsilon(0)}{M_\varepsilon} e^{-\frac{\varepsilon}{2N_\varepsilon}t} \tag{31}$$

which is the expected inequality with

$$\omega \ge \frac{\varepsilon}{2(1+\varepsilon D)} \tag{32}$$

and

$$1 - \varepsilon D > 0$$

3 Proof of THEOREM 1.2

Let $(\mu_n)_{n\ge1}$ denote the sequence of positive eigenvalues of A and $(\phi_n)_{n\ge1}$ the corresponding sequence of X-normalized eigenfunctions. Before proving THEOREM 2, we need the following lemma which is proved in the appendix.

Lemma 3 *$\sigma(L_\alpha)$, the spectrum of L_α, reduces to the eigenvalues of L_α which are the roots of the following algebraic equations*

$$\lambda^3 + \mu_n\lambda^2 + (\mu_n + \mu_n^{2\alpha})\lambda + \mu_n^2 = 0 \qquad n \ge 1 \tag{33}$$

and, if λ_n^1 denotes the smallest real root, then, for $0 \le \alpha < 1$,

$$\lambda_n^1 = \frac{1+\varepsilon_n}{\mu_n^{1-2\alpha}} - \mu_n \tag{34}$$

where

$$\lim_{n\to\infty} \varepsilon_n = 0$$

As a consequence of this lemma, we have

Corollary 4 *$(S_\alpha(t))$ is not uniformly stable if $\alpha \in [0, \frac{1}{2}[$.*

Proof: From the Hille-Yosida theorem, one has

$$\sup \Re e\, \sigma(L_\alpha) \le \omega$$

for all ω for which there is a positive constant M such that

$$\| S_\alpha(t) \|_{L(H)} \leq M e^{\omega t} \qquad \forall t \geq 0$$

But, if λ_n^2 and λ_n^3 are the two other roots of (33), then

$$\lambda_n^1 + \lambda_n^2 + \lambda_n^3 = -\mu_n \qquad n \geq 1$$

and using (34)

$$\lim_{n\to\infty} \left(\lambda_n^2 + \lambda_n^3\right) = -\lim_{n\to\infty} \frac{1+\varepsilon_n}{\mu_n^{1-2\alpha}} = 0$$

since $0 \leq \alpha < \frac{1}{2}$ and $\lim_{n\to\infty} \mu_n = +\infty$. The conclusion follows from the fact that if λ is a root of equation (33) then $\mathrm{Re}(\lambda) < 0$ (by applying the Routh theorem, see [17], Theorem 2.4, p.33).

Proof of THEOREM 2: Let $\alpha \in [\frac{1}{2}, 1[$. Assumption (**H.1**) is clearly satisfied. One has, for all $w \in D(A)$

$$\begin{aligned} \left\| A^{\alpha-\frac{1}{2}} w \right\|_X &= \left\| A^{\alpha-1} A^{\frac{1}{2}} w \right\|_X \\ &\leq \left\| A^{\alpha-\frac{1}{2}} \right\|_{L(X)} \left\| A^{\frac{1}{2}} w \right\|_X \end{aligned}$$

and

$$\begin{aligned} \left\| A^{\frac{1}{2}-\alpha} w \right\|_X &= \left\| A^{-\alpha} A^{\frac{1}{2}} w \right\|_X \\ &\leq \left\| A^{-\alpha} \right\|_{L(X)} \left\| A^{\frac{1}{2}} w \right\|_X \end{aligned}$$

Thus assumption (**H.2**) is satisfied and $(S_\alpha(t))$ is uniformly stable. The first part of the claim (i) is then proved. The second part is proved in the appendix. And lastly direct computations prove that P_α satisfies the so-called Lyapunov equation (see [15], p.33)

$$2\,\mathrm{Re}\,(P_\alpha L_\alpha Y, Y) = -\|Y\|^2 \quad \forall Y \in D(L_\alpha)$$

The claim (ii) is a consequence of the fact that L_α has a compact resolvent, generates a C_0−semogroup of contractions and $\sigma(L_\alpha) \cap \{\lambda \in C; \Re e(\lambda) = 0\} = \emptyset$ (see [4]).

(iii) From Theorem 5.1, (ii), in the Appendix, L_α has the representation

$$L_\alpha = \sum_{n\geq 1} \sum_{j=1}^{3} \lambda_n^j \left(., \Psi_{n,j}^*\right)_H \Psi_{n,j}$$

and $S_\alpha(t)$ is given by

$$S_\alpha(t) = \sum_{n\geq 1} \sum_{j=1}^{3} e^{\lambda_n^j t} \left(., \Psi_{n,j}^*\right)_H \Psi_{n,j}$$

where (λ_n^j) are the eigenvalues of L_α, $(\Psi_{n,j})$ are the corresponding eigenvectors and $(\Psi^*_{n,j})$ are the eigenvectors of L^*_α such that $\left(\Psi_{n,j}, \Psi^*_{m,l}\right)_H = \delta_{nm}\delta_{jl}$. The estimate (5.4) in Theorem 5.1 shows that

$$\left|\frac{\mathrm{Im}(\lambda_n^j)}{\mathrm{Re}(\lambda_n^j)}\right| \leq M \Longleftrightarrow \frac{3}{4} \leq \alpha \leq 1$$

and the conclusion follows from [14], Theorem 2.5, (d), p. 61.

(iv) Let

$$\eta_\varepsilon(t) = \| Y(t) \|^2 + \varepsilon \left(\left(A^{-\frac{1}{2}} w, v \right)_X + \left(A^\beta u, v \right)_X \right)$$

where $0 < \beta \leq \min(\alpha - \frac{1}{2}, 2 - 2\alpha)$. Then, one can prove that there exist positive constants such that C_1, C_2 and C_3

$$\eta'_\varepsilon(t) + C_1 \| A^{\frac{2\alpha-1}{4}} v(t) \|_X^2 + C_2 \| A^{\frac{\beta+1}{2}} u(t) \|_X^2 + C_3 \| A^{\frac{1}{2}} w(t) \|_X^2 \leq 0$$

As, for ε sufficiently small, there exist positive constants m and M such that

$$m \| Y(t) \|^2 \leq \eta_\varepsilon(t) \leq M \| Y(t) \|^2$$

one gets that, for a bounded sequence of initial data Y_0^n, $S_\alpha(.)Y_0^n$ is a bounded sequence in $L^2\left(R^+, D(A^{\frac{1+\beta}{2}}) \times D(A^{\frac{2\alpha-1}{4}}) \times D(A^{\frac{1}{2}})\right)$. But $D(A^{\frac{1+\beta}{2}}) \times D(A^{\frac{2\alpha-1}{4}}) \times D(A^{\frac{1}{2}})$ is compact in H for $\alpha > \frac{1}{2}$. We end the proof by using the Aubin compacity result (see [11],Theorem 5.1, p.58).

4 Application to Thermoelastic Plates

We consider in this section two models of thermoelastic plates (see [12]) which differ by the boundary conditions. Let Ω an open bounded set in R^n with smooth boundary Γ,

$$\begin{cases} u'' = -\Delta^2 u - \Delta w & R^+ \times \Omega \\ w' = \Delta w + \Delta u' & R^+ \times \Omega \\ u = \Delta u = 0 & R^+ \times \Gamma \\ w = 0 & R^+ \times \Gamma \\ u(0) = u_0, u'(0) = u_1, w(0) = w_0, & \Omega \end{cases} \tag{35}$$

The energy space will be the Hilbert space:

$$H = (H^2(\Omega) \cap H_0^1(\Omega)) \times L^2(\Omega) \times L^2(\Omega)$$

with the scalar product:

$$< \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \begin{bmatrix} f \\ g \\ h \end{bmatrix} >= \int_\Omega (\Delta u.\Delta f + vg + wh)\, dx$$

and we denote by $\| . \|$ the induced norm on H. $|.|$ will denote the L^2-norm.

Let:

$$L = \begin{pmatrix} \begin{pmatrix} 0 & I \\ -\Delta^2 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\Delta \end{pmatrix} \\ \begin{pmatrix} 0 & \Delta \end{pmatrix} & \Delta \end{pmatrix}$$

$$D(L) = \left\{ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in H;\ \Delta u \in H^2(\Omega) \cap H_0^1(\Omega), v \in H^2(\Omega) \cap H_0^1(\Omega), w \in H^2(\Omega) \cap H_0^1(\Omega) \right\}$$

This model has already been studied in [16] where uniform stability is proved. This system satisfies Assumption (**H**) with $A = \Delta^2$ and

$$D(A) = u \in H^2(\Omega) \cap H_0^1(\Omega); \Delta u \in H^2(\Omega) \cap H_0^1(\Omega)$$

$$C = B = A^{\frac{1}{2}} = -\Delta,\ D(C) = D(B) = H^2(\Omega) \cap H_0^1(\Omega)\ ,\ F = 0$$

So

Proposition 5 *The thermoelastic plates system (35) is uniformly stable. Moreover, the semigroup associated to L is compact and the decay rate of the energy is* $\sup \operatorname{Re} \sigma(L)$.

Proof : As Assumption (**H**) is satisfied, uniform stability is a direct consequence of Theorem 1.1. The compactness of the associated semigroup is obtained like in the proof of THEOREM 2, (iv) by setting

$$\eta_\varepsilon = \| Y \|^2 + \varepsilon \left(\left(w, (-\Delta)^{-\frac{1}{2}} v\right)_{L^2(\Omega)} + \left(v, (-\Delta)^{\frac{1}{2}} u\right)_{L^2(\Omega)} \right)$$

one gets that, for a bounded sequence of initial data Y_0^n, $S_L(.)Y_0^n$ is a bounded sequence in $L^2\left(R^+, D((-\Delta)^{\frac{5}{4}}) \times D((-\Delta)^{\frac{1}{4}}) \times D((-\Delta)^{\frac{1}{2}})\right)$. But $D((-\Delta)^{\frac{5}{4}}) \times D((-\Delta)^{\frac{1}{4}}) \times D((-\Delta)^{\frac{1}{2}}$ is compact in H and we conclude as before. The last claim is a consequence of the compactness of $S_L(t)$ (see [15], Proposition 3.2., p. 32).

The second model is the following

$$\begin{cases} u'' = -\Delta^2 u - \Delta w & R^+ \times \Omega \\ w' = \Delta w + \Delta u' & R^+ \times \Omega \\ u = \frac{\partial u}{\partial \nu} = w = 0 & R^+ \times \Gamma \\ u(0) = u_0, u'(0) = u_1, w(0) = w_0, & \Omega \end{cases} \quad (36)$$

The energy space will be the Hilbert space:

$$H = H_0^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$$

with the scalar product:

$$< \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \begin{bmatrix} f \\ g \\ h \end{bmatrix} >= \int_\Omega (\Delta u.\Delta f + vg + wh)\, dx$$

and we denote by $\| . \|$ the induced norm on H. $|.|$ will denote the L^2-norm.

Let:

$$L = \begin{pmatrix} \begin{pmatrix} 0 & I \\ -\Delta^2 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -\Delta \end{pmatrix} \\ \begin{pmatrix} 0 & \Delta \end{pmatrix} & \Delta \end{pmatrix}$$

$$D(L) = \left\{ \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in H;\ \Delta u \in H^2(\Omega), v \in H_0^2(\Omega), w \in H^2(\Omega) \cap H_0^1(\Omega) \right\}$$

This model has also been studied in [9] where it is proved that the system is uniformly stable using an indirect method. We obtain here the same result but with an estimate of the decay rate. Let's denote by $A = \Delta^2$ and $D(A) = H^4(\Omega) \cap H_0^2(\Omega)$, $C = B = -\Delta$ with $D(C) = D(B) = H^2(\Omega) \cap H_0^1(\Omega)$.

Proposition 6 *The thermoelastic plates system (36) is uniformly stable. Moreover, the decay rate of the energy is estimated by (32) with the given constants in (37).*

Proof. Relatively to the preceding system, we have only to find the function F such that the inequality (9) in the assumption (**H**) is satisfied (in this case $F = 0$ does not satisfy such a relation because of the occurence of nonzero boundary terms).

One has

$$\begin{aligned} \big(Au, (B^*)^{-1}w\big) &= \int_\Omega \Delta^2 u \Delta^{-1} w dx \\ &= \int_\Omega \Delta u w dx - \int_\Gamma \Delta u \tfrac{\partial \Delta^{-1} w}{\partial \nu} d\sigma \end{aligned}$$

The difficulty is to estimate the boundary integral:

$$\left| \int_\Gamma \Delta u \frac{\partial \Delta^{-1} w}{\partial \nu} d\sigma \right| \le |\Delta u|_{L^2(\Gamma)} \left| \frac{\partial \Delta^{-1} w}{\partial \nu} \right|_{L^2(\Gamma)}$$

From the continuity of the trace application, there exists a positive constant r such that

$$\left| \frac{\partial \Delta^{-1} w}{\partial \nu} \right|_{L^2(\Gamma)} \le r |w|_{L^2(\Omega)}$$

To estimate the other term, we proceed as in [10, p.244]. Let h be a vector field in $\left[C^2(\overline{\Omega})\right]^n$ which coincides with the outward unit normal vector on Γ. We multiply the first equation of system (36) by $h.\nabla u$ and obtain

$$\begin{aligned}\frac{1}{2}|\Delta u|^2_{L^2(\Gamma)} = \ & \frac{d}{dt}\int_\Omega v\ h.\nabla u dx \\ & +\frac{1}{2}\int_\Omega (\nabla.h)(v^2-(\Delta u)^2)dx \\ & +2\int_\Omega \sum_{i,j=1}^n \frac{\partial h_i}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\Delta u dx \\ & +\int_\Omega \Delta u(\Delta h.\nabla u)dx - \int_\Omega \nabla w.\nabla(h.\nabla u)dx\end{aligned}$$

We set

$$F(Y) = \int_\Omega v\ h.\nabla u dx$$

and

$$G(Y) = F'(Y) \qquad Y \in H$$

then (9) is satisfied with

$$\begin{aligned} c_1 &= (1+r)c_0 + \tfrac{1}{4}(|\nabla h|_{L^\infty(\Omega)} + |h|_{L^\infty(\Omega)}); \\ c_2 &= 1 + (1+s)|\nabla h|_{L^\infty(\Omega)} + |\Delta h|_{L^\infty(\Omega)} + 2c_0 \\ c_3 &= \tfrac{1}{2}|h|_{L^\infty(\Omega)}.Max(1, c_0)\end{aligned} \tag{37}$$

where c_0 is the Poincaré constant and $s = ||\Delta^{-1}||_{L\left(L^2(\Omega);H^2(\Omega)\cap H_0^1(\Omega)\right)}$.

5 Appendix

5.1 Proof of lemma 3.1

One can easily see that L has a compact resolvent; thus $\sigma(L)$ reduces to the point spectrum : equation (33) (by solving the equation $LY = \lambda Y$).

By setting $\lambda = \mu_n \delta$ in (33), we get the equation

$$p(\delta) := \delta^3 + \delta^2 + \left(\frac{1}{\mu_n} + \frac{1}{\mu_n^{2(1-\alpha)}}\right)\delta + \frac{1}{\mu_n} = 0 \tag{38}$$

If we denote by δ_n the smallest real root of (38), then $\delta_n < -\frac{2}{3}$ for n sufficiently large. This follows from the fact that the smallest root of

$$p'(\delta) = 3\delta^2 + 2\delta + \frac{1}{\mu_n} + \frac{1}{\mu_n^{2(1-\alpha)}} = 0$$

is

$$\tilde{\delta}_n = -\frac{1}{3} - \frac{1}{3}\left(1 - 3\left(\frac{1}{\mu_n} + \frac{1}{\mu_n^{2(1-\alpha)}}\right)\right)^{\frac{1}{2}}$$

and simple computations give for $0 \leq \alpha < 1$

$$\lim_{n\to\infty} \tilde{\delta}_n = -\frac{2}{3}$$

and

$$\lim_{n\to\infty} p(\tilde{\delta}_n) = \frac{4}{27} > 0$$

On the other hand, from (38)

$$\mu_n^{2(1-\alpha)}\,(1+\delta_n) = -\frac{\delta_n}{\delta_n^2 + \frac{1}{\mu_n}} \tag{39}$$

then necessarely $-1 < \delta_n < 0$. It follows that the right member in (39) is bounded and then, always from (39), $\lim_{n\to\infty} \delta_n = -1$. That means that there exists a sequence $\varepsilon_n \to 0$ as $n \to \infty$ such that

$$\mu_n^{2(1-\alpha)}\,(1+\delta_n) = 1 + \varepsilon_n \tag{40}$$

and this is exactly (34) since $\lambda_n^1 = \mu_n \delta_n$.

5.2 Additional properties of the operator L_α

We collect, in the following theorem, some spectral properties of the operator L_α.

Theorem 7 *(i)-If $\frac{3}{4} < \alpha < 1$, there is at most a finite number of complex eigenvalues.*

-If $0 \leq \alpha \leq \frac{3}{4}$, for every n sufficiently large, there are two conjugate complex eigenvalues λ_n^2 and λ_n^3 such that there exists a sequence $\varepsilon'_n \to 0$ as $n \to \infty$ and

$$\left[\frac{\mathrm{Im}(\lambda_n^2)}{\mathrm{Re}(\lambda_n^2)}\right]^2 = 4\,(1+\varepsilon'_n)\,\mu_n^{3-4\alpha} - 1 \tag{41}$$

(ii) The sequence of eigenfunctions $(\Psi_{n,j})$ corresponding to the sequence $\left(\lambda_n^j\right)$ of eigenvalues is a Riesz basis (that is, it's isomorphic to an orthonormal basis) of H.

Proof: (i) Setting $\lambda = \delta - \frac{1}{3}\mu_n$ in (33) reduces this equation to the canonical form

$$\delta^3 + p_n\delta^2 + q_n = 0 \tag{42}$$

with

$$p_n = -\frac{1}{3}\mu_n^2 + \mu_n + \mu_n^{2\alpha}; \qquad q_n = \frac{2}{27}\mu_n^3 + \frac{2}{3}\mu_n^2 - \frac{1}{3}\mu_n^{2\alpha+1} \tag{43}$$

If $\Delta_n = 4p_n^3 + 27q_n^2$, it is easy to see that for a sufficiently large n, $\Delta_n < 0$ when $\frac{3}{4} < \alpha < 1$ and $\Delta_n > 0$ if $0 \leq \alpha \leq \frac{3}{4}$. This proves the first claims in (i). As to the estimate (40), we note first that, for all n, (34) implies

$$\mathrm{Re}(\lambda_n^2) = -\frac{1+\varepsilon_n}{2\mu_n^{1-2\alpha}} \tag{44}$$

and

$$\left|\lambda_n^2\right|^2 = \left[\mathrm{Re}(\lambda_n^2)\right]^2 + \left[\mathrm{Im}(\lambda_n^2)\right]^2 = -\frac{\mu_n^2}{\lambda_n^1} \tag{45}$$

which are obtained from the well-known relations between the roots of a polynomial.

(ii) Direct computations give

$$\Psi_{n,j} = \begin{bmatrix} 1 \\ \\ \lambda_n^j \\ \\ \frac{\left(\lambda_n^j\right)^2+\mu_n}{\mu_n^\alpha} \end{bmatrix} \phi_n \qquad j = 1,2,3 \quad n \geq 1 \tag{46}$$

L_α^*, the adjoint operator of L_α, is

$$L_\alpha^* = \begin{pmatrix} 0 & -I & 0 \\ A & 0 & -A^\alpha \\ 0 & A^\alpha & -A \end{pmatrix};$$

and its eigenfunctions are

$$\Psi_{n,j}^* = \begin{bmatrix} 1 \\ \\ \overline{\lambda}_n^j \\ \\ \frac{\left(\overline{\lambda}_n^j\right)^2+\mu_n}{\mu_n^\alpha} \end{bmatrix} \phi_n \qquad j = 1,2,3 \quad n \geq 1 \tag{47}$$

In what follows, we assume that for every $n \geq 1$, the roots of (33) are simple. In fact, if it is not the case for a (necessarely) finite number of n, we complete the sequence of eigenfunctions with generalized eigenfunctions. Let

$$W = span\left\{\Psi_{n,j},\ n \geq 1,\ j = 1,2,3\right\}$$

then $\overline{W} = H$. To prove this, let $Z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in W^\perp$. For each $n \geq 1$

$$(Z, \Psi_{n,j}) = 0 \qquad j = 1,2,3$$

This system is equivalent to the following one

$$\begin{cases} \mu_n^{\alpha+1}(z_1,\phi_n)_X & +\mu_n^\alpha\lambda_n^1(z_2,\phi_n)_X & +\left(\lambda_n^1+\mu_n\right)(z_3,\phi_n)_X & = 0 \\ \mu_n^{\alpha+1}(z_1,\phi_n)_X & +\mu_n^\alpha\lambda_n^2(z_2,\phi_n)_X & +\left(\lambda_n^2+\mu_n\right)(z_3,\phi_n)_X & = 0 \\ \mu_n^{\alpha+1}(z_1,\phi_n)_X & +\mu_n^\alpha\lambda_n^3(z_2,\phi_n)_X & +\left(\lambda_n^3+\mu_n\right)(z_3,\phi_n)_X & = 0 \end{cases}$$

whose determinant det is

$$\det = \mu_n^{2\alpha+1}(\lambda_n^2-\lambda_n^1)(\lambda_n^3-\lambda_n^1)(\lambda_n^3-\lambda_n^2) \neq 0$$

It follows that $Z = 0$ and $W^{\perp} = \{0\}$.

Let $\left(\tilde{\Psi}_{n,j}\right)$ and $\left(\tilde{\Psi}^*_{n,j}\right)$ denote the normalized sequences of eigenfunctions of L_α and L^*_α respectively. We now prove that $\left(\frac{\tilde{\Psi}_{n,j}}{\left(\tilde{\Psi}_{n,j},\tilde{\Psi}^*_{n,j}\right)_H}\right)$ is isomorphic to an orthonormal basis. It is sufficient (and necessary) to prove that there exist two positive constants m and M such that

$$m\sum_{n=1}^{N}\sum_{j=1}^{3}|\alpha_n^j|^2 \leq \| \sum_{n=1}^{N}\sum_{j=1}^{3}\alpha_n^j\frac{\tilde{\Psi}_{n,j}}{\left(\tilde{\Psi}_{n,j},\tilde{\Psi}^*_{n,j}\right)_H} \|_H^2 \leq M\sum_{n=1}^{N}\sum_{j=1}^{3}|\alpha_n^j|^2 \tag{48}$$

for all $N \geq 1$ and any sequence $\left(\alpha_n^j\right)$ of complex numbers (see [8], Theorem 2.1, (3), p. 310). But

$$\begin{aligned} \| \sum_{n=1}^{N}\sum_{j=1}^{3}\alpha_n^j\frac{\tilde{\Psi}_{n,j}}{\left(\tilde{\Psi}_{n,j},\tilde{\Psi}^*_{n,j}\right)_H} \|_H^2 &= \| \sum_{n=1}^{N}\sum_{j=1}^{3}\alpha_n^j\frac{\tilde{\Psi}^*_{n,j}}{\left|\left(\tilde{\Psi}_{n,j},\tilde{\Psi}^*_{n,j}\right)_H\right|^2} \|_H^2 \\ &= \sum_{n=1}^{N}\sum_{j=1}^{3}\frac{|\alpha_n^j|^2}{\left|\left(\tilde{\Psi}_{n,j},\tilde{\Psi}^*_{n,j}\right)_H\right|^2} \end{aligned} \tag{49}$$

With the help of the estimates (34), (44) and (45), one can easily see that

$$\lim_{n\to\infty}\left|\left(\tilde{\Psi}_{n,j},\tilde{\Psi}^*_{n,j}\right)_H\right| = 1 \tag{50}$$

It follows that there exists $M > 0$ such that

$$\left|\left(\tilde{\Psi}_{n,j},\tilde{\Psi}^*_{n,j}\right)_H\right|^2 \geq \frac{1}{M} \qquad \forall n \geq 1 \; j = 1,2,3 \tag{51}$$

Using (49) and (50), one obtains (48) with $m = 1$.

6 References

[1] F.Ammar Khodja & A.Benabdallah , Stabilisation de l'équation des ondes par un contrôleur dynamique, *C. R. Acad. Sci Paris*, 321, I, p. 195-198, (1995)

[2] F.Ammar Khodja & A.Benabdallah, Stabilization of the wave equation by a dynamical controller, *Prébublication de l'équipe de Mathématiqyes de Besançon*, 95/22, (1995)

[3] F.Ammar Khodja , A.Benabdallah & D.Teniou, Stabilisation frontière et interne d'un système similaire à celui de la thermoélasticité, *C. R. Acad. Sci Paris*, 322, I, p. 551-556, (1996)

[4] **W.Arendt and C. J. K. Batty ,** Tauberian theorems and stability of one-parameter semigroups, *Trans. Amer. Math. Society*, 306, p. 837-852, (1988)

[6] **C.Bardos, G.Lebeau and J.Rauch,** Contrôle et stabilisation dans les problèmes hyperboliques, Appendix II in Vol 1 of [10], (1988)

[7] **S.Chen and R.Triggiani,** Proof of extensions of two conjectures on structural damping for elastic systems, *Pacific Journal of Mathematics*, 136, p. 15-55, (1989)

[8] **I.C.Gohberg and M.G.Krein,** Introduction to the Theory of Linear Nonselfadjoint Operators, 18, American Mathematical Society, (1969)

[9] **J.U.Kim,** On the energy decay of a linear thermoelasticitc bar and plate, *SIAM J.Math.Anal*, 23, 4, p. 889-899, (1992)

[10] **J.L.Lions,** Contrôlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués, Vol 1 et 2, RMA 8 et 9, Masson, 1988.

[11] **J.L.Lions,** Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod., 1969.

[12] **J.E. Lagnese and J.L.Lions,** Modelling Analysis and Control of Thin Plates, RMA 6, Masson, 1988.

[13] **Z.Liu & S.Zheng,** Exponential stability of the semigroup associated with a thermoelastic system, *Quarterly of Applied Mathematics*,II, 3, p. 535-545, (1993)

[14] **A.Pazy,** Semigroups of linear operators and applications to partial differential equations, Applied mathematical sciences ,Springer-Verlag , 1983.

[15] **A.J.Pritchard and J.Zabczyk,** Stability and stabilizability of infinite dimensional systems, *SIAM Rev.*, 23, 1, (1981)

[16] **R.Racke,** Existence and regularity of solutions to nonlinear thermoelastic systems, Proceedings of the Workshop on Qualitative aspects and applications of nonlinear evolution equations, Trieste May 3-14, p. 74-87, 1993.

[17] **J.Zabczyk,** Mathematical Control Theory: An introduction, Birkhäuser, 1992.

Additional references

The authors would like to thank the referee for indicating us the following paper which is related to ours in the spirite:

D. L.Russell , A general framework for the study of indirect damping mechanisms in elastic systems, *J. Math. Anal. Appl., 173, p. 339, (1993)*

By studying the smoothness effect of parabolic equation on hyperbolic equation, J.E. Munoz Rivera and R. Racke obtain similar conditions as ours in remark 1.

J. E.Munoz Rivera, R.Racke, Smoothing properties, decay and global existence of solutions to nonlinear coupled systems of thermoelastic type, *SIAM J. Math. Anal., 1547-1563, (1995)*