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Energy decay for Timoshenko systems of memory type

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Abstract

Linear systems of Timoshenko type equations for beams including a memory term are studied. The exponential decay is proved for exponential kernels, while polynomial kernels are shown to lead to a polynomial decay. The optimality of the results is also investigated.

MSC: 73 C 99; 35 Q 99

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1. Introduction

In this paper we consider linear systems of Timoshenko type with memory, which are written as

$$\rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0 \quad \text{in } (0,L)\times(0,\infty), \tag{1.1}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + g*\psi_{xx} + k(\varphi_x + \psi) = 0 \quad \text{in } (0,L)\times(0,\infty), \tag{1.2}$$

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where ρ_1, k, ρ_2, b and L are positive constants. The functions ϕ and ψ describe the transverse displacement of the beam and the rotation angle of a filament, respectively. The boundary conditions we consider here are given by

$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \quad t \geqslant 0. \tag{1.3}$$

The initial conditions are

$$\varphi(\cdot,0) = \varphi_0, \quad \varphi_t(\cdot,0) = \varphi_1, \quad \psi(\cdot,0) = \psi_0, \quad \psi_t(\cdot,0) = \psi_1 \quad \text{in } (0,L). \tag{1.4}$$

The usual convolution term

$$g * \psi_{xx}(x,t) = \int_0^t g(t-s)\psi_{xx}(x,s)\,ds$$

represents the memory effect with a real-valued C^2-function g.

Our main interest concerns the asymptotic behavior of the solution of the system above. That is, whether the dissipation given by the memory effect in Eq. (1.2) is strong enough to stabilize the whole system. Another natural question concerning the asymptotic behavior is about the rate of decay of the solution. That is, what type of rate of decay may we expect? (if there exist one). How can the damping mechanism given by the memory effect through the relaxation function g be effective to produce uniform stabilization?

Let us mention some known results about related viscoelastic systems. Dafermos [3] proved that the solutions to viscoelastic systems tend to zero as time tends to infinity, but without giving explicit rates of decay. Lagnese [10] considered a linear viscoelastic equation obtaining uniform rates of decay but introducing additional damping terms acting on the boundary. Greenberg [7] and Hrusa [8] proved an exponential rate of decay for the nonlinear viscoelastic equation when the relaxation function g is of the form $g(t) = e^{-\mu t}$. In this case using the fact that $g'(t) = -\mu g(t)$ the convolution term is eliminated by differentiation, therefore the resulting equation has no integral term, hence this method cannot be used for a more general class of relaxation functions even for those which are a linear combination of exponential terms with varying rates of decay. A similar result was obtained by Dassios and Zafiropoulos [4] for homogeneous and isotropic viscoelastic materials which occupy the whole three-dimensional space. They proved that the longitudinal and transverse waves decay to zero uniformly like $t^{-m-3/2}$, where m increases depending on the symmetry of the initial data, provided the relaxation is an exponential function like $t \mapsto \mu_0 e^{-\gamma t}$. The method the authors used is based on the study of the roots of the characteristic polynomial associated to the ordinary differential equation, which is obtained by taking Fourier transform of the system and then differentiating the resulting equation with respect to time. By using the fact that the kernel g is an exponential function, that is $g'(t) = -\gamma g(t)$, the convolution term is eliminated, so the authors work with the resulting purely ordinary differential equation. In [11,13,14] it was proved that the rate of decay of the solution depends on the rate of decay of the relaxation function, that is if the relaxation function decays exponentially then the solution decays exponentially, while if the relaxation function decays polynomially then the solution decays also polynomially with the same rate. For localized damping in viscoelasticity see Rivera and Peres [15] where it is shown that the first-order energy decays exponentially to zero provided the relaxation kernel also decays exponentially to zero. When the kernel decays polynomially, the problem is open.

Finally, we remark that the memory effect is a subtle damping mechanism, the effect of which depends on the rate of decay rather than its dissipative properties. In fact, if we consider the memory effect with a dissipative relaxation function together with other stronger dissipative effects, for example the frictional damping, then the resulting dissipation does not produce any rate of decay if the relaxation function does not decay uniformly, see the work of Frabrizio and Polidoro [6].

The main result of this paper is that the whole system decays uniformly if and only if the coefficients satisfy

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}. \tag{1.5}$$

Concerning the rate of decay, we will show that the solution decays exponentially to zero provided the kernel tends to zero also exponentially. When the kernel decays to zero polynomially, the solution also decays polynomially with the same rate. More precisely: If g is of exponential type, i.e. if the following assumption

$$\left.\begin{aligned} g>0, \quad \exists k_0, k_1, k_2>0:\ -k_0 g \leqslant g' \leqslant -k_1 g, \quad |g''| \leqslant k_2 g, \\ \lambda := b - \textstyle\int_0^\infty g(s)\, ds > 0 \end{aligned}\right\} \tag{1.6}$$

is satisfied, then the exponential decay of the energy

$$E(t) = \frac{1}{2}\int_0^L \rho_1 |\varphi_t|^2 + \rho_2 |\psi_t|^2 + \left(b - \int_0^t g\, d\tau\right)|\psi_x|^2 + k|\varphi_x + \psi|^2 + g\Box\psi_x\, dx$$

for a solution (ϕ, ψ) as time tends to infinity will be proved if and only if the coefficients satisfy (1.5). The symbol $\Box$ denotes the following convolution:

$$(g\Box f)(t) := \int_0^t g(t-s)|f(s) - f(t)|^2\, ds.$$

If g is of polynomial type, i.e. if it satisfies

$$\left.\begin{aligned} 0 < g(t) \leqslant b_0(1+t)^{-p}, \\ -b_1 g(t)^{\frac{p+1}{p}} \leqslant g'(t) \leqslant -b_2 g(t)^{\frac{p+1}{p}}, \\ -b_3 |g'(t)|^{\frac{p+2}{p+1}} \leqslant g''(t) \leqslant -b_4 |g'(t)|^{\frac{p+2}{p+1}}, \end{aligned}\right\} \tag{1.7}$$

with positive constants b_0, b_1, b_2, b_3, b_4 and $p>2$, then the polynomial decay of the energy will be proved. This result is also shown to be optimal in the sense, that there cannot occur an exponential decay. The typical example $\bar{g}$ satisfying (1.7) is of course

$$\bar{g}(t) = b_0(1+t)^{-p}.$$

In Sections 2 and 3 we consider exponential kernels showing the exponential decay result under assumption (1.5) and that there is no uniform decay if this assumption is not satisfied, respectively. In Sections 4 and 5 polynomial kernels are studied and the polynomial decay of the energy (under assumption (1.5)) is proved as well as the optimality, i.e. non exponential decay, respectively. The results in Sections 2, 4 and 5 are proved by energy methods, using suitably sophisticated estimates for multipliers, while Section 3 also uses sharp perturbation arguments for the spectral radius of a semigroup.

Remark. Timoshenko *plates* can be dealt with in a similar manner as the Timoshenko beams discussed here.

The uniform stabilization of Timoshenko beams with the memory term $g * \psi_{xx}$ in Eq. (1.2) replaced by some control function f was studied by Soufyane [19]. He showed the exponential decay of the associated energy for

$$f(x,t) = b(x)\psi_t(x,t)$$

and also if and only if assumption (1.5) is satisfied. Previous work of different authors considered two boundary control functions, like Kim and Renardy [9], or two forces, see [20]. In our paper the first results are presented for a memory type control term, both for exponential and for polynomial kernels.

In the sequel we shall always assume the unique existence of strong solutions to the initial–boundary value problem under consideration, cp. for example [18,19]. The problem is well posed for data $((\varphi_0, \varphi_1), (\psi_0, \psi_1))$ in the Sobolev space $[H^2((0,L)) \times H_0^1((0,L))]^2$. Weak solutions and the energy are well defined also in $[H_0^1((0,L)) \times L^2((0,L))]^2$.

2. Exponential decay

First, we consider exponential kernels of type (1.6) and we look for the exponential decay of the energy

$$E(t) := \frac{1}{2}\int_0^L \rho_1|\varphi_t|^2 + \rho_2|\psi_t|^2 + \left(b - \int_0^t g\,d\tau\right)|\psi_x|^2 + k|\varphi_x + \psi|^2 + g\Box\psi_x\,dx. \quad (2.1)$$

Using the following simple lemma (cf. Lemma 3.2 from [16]).

Lemma 2.1. *For* $f, h \in C^1([0, \infty), \mathbb{R})$ *we have*

$$2(f * h)(t)h_t(t) = (f' \Box h)(t) + \frac{d}{dt}\left\{ \int_0^t f(s)\, ds\, |h(t)|^2 - (f \Box h)(t) \right\} - f(t)|h(t)|^2.$$

We easily conclude that the energy decays:

$$\frac{d}{dt}E(t) = -\frac{1}{2}g(t)\int_0^L |\psi_x|^2\, dx + \frac{1}{2}\int_0^L g' \Box \psi_x\, dx \leqslant 0. \tag{2.2}$$

The main point to show the exponential decay is to construct a Lyapunov functional $\mathscr{L}$ satisfying

$$\beta_1 E(t) \leqslant \mathscr{L}(t) \leqslant \beta_2 E(t), \quad \frac{d}{dt}\mathscr{L}(t) \leqslant -\alpha \mathscr{L}(t)$$

for all $t \geqslant 0$ and some positive constants β_1, β_2, α. To achieve this we will use the multiplicative technique, and our starting point will be the multiplier $(g * \psi)_t$ to deal with the functional I given by

$$\begin{aligned} I(t) :=& \int_0^L \rho_2 \psi_t (g * \psi)_t\, dx + b\int_0^L \psi_x (g * \psi_x)\, dx + k\int_0^L \psi(g * \psi)\, dx \\ & - \frac{1}{2}\int_0^L |g * \psi_x|^2\, dx - \frac{b}{2}\left(\int_0^t g\, d\tau\right)\int_0^L |\psi_x|^2\, dx \\ & + \frac{b}{2}\int_0^L g \Box \psi_x\, dx + \frac{k}{2}\int_0^L g \Box \psi\, dx - \frac{k}{2}\left(\int_0^t g\, d\tau\right)\int_0^L |\psi|^2\, dx \end{aligned}$$

which will yield a negative term $-\int_0^L |\psi_t|^2\, dx$. To simplify notations let us introduce the symbol $\diamond$ by

$$(g \diamond h)(t) := \int_0^t g(t-s)\{h(t) - h(s)\}\, ds.$$

Then we have

Lemma 2.2. *There are* $c > 0$ *and for any* $\epsilon > 0$ *a positive constant* C_ϵ *such that for* $t \geqslant 0$

$$\begin{aligned} -\frac{d}{dt}I(t) \leqslant & -\frac{\rho_2}{2}g(0)\int_0^L |\psi_t|^2\, dx + C_\epsilon(|g'| + g)\int_0^L |\psi_x|^2\, dx \\ & + c\int_0^L |g'| \Box \psi_x\, dx + c\int_0^L g'' \Box \psi\, dx + \epsilon\int_0^L |\varphi_x|^2\, dx. \end{aligned} \tag{2.3}$$

Proof. Multiplying Eq. (1.2) by $(g*\psi)_t$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_0^L \rho_2\psi_t(g*\psi)_t\,dx\\
&\quad=\frac{d}{dt}\left\{-b\int_0^L \psi_x(g*\psi_x)\,dx+\frac{1}{2}\int_0^L |g*\psi_x|^2\,dx-k\int_0^L \psi(g*\psi)\,dx\right\}\\
&\qquad+\;b\int_0^L \psi_{xt}(g*\psi_x)\,dx+k\int_0^L \psi_t(g*\psi)\,dx\\
&\qquad+\;\rho_2 g(0)\int_0^L |\psi_t|^2\,dx+\rho_2\int_0^L g'\psi_t\psi\,dx-\rho_2\int_0^L \psi_t(g''\diamond\psi)\,dx\\
&\qquad-k\int_0^L \varphi_x\{g(0)\psi+g'*\psi\}\,dx.
\end{aligned}$$

Observing

$$\begin{aligned}
b\int_0^L \psi_{xt}(g*\psi_x)\,dx=&\frac{b}{2}\frac{d}{dt}\int_0^L\left(\int_0^t g\,ds\right)|\psi_x|^2\,dx-\frac{b}{2}\int_0^L g|\psi_x|^2\,dx\\
&-\frac{b}{2}\frac{d}{dt}\int_0^L g\Box\psi_x\,dx+\frac{b}{2}\int_0^L g'\Box\psi_x\,dx
\end{aligned}$$

and

$$\begin{aligned}
k\int_0^L \psi_t(g*\psi)\,dx=&\frac{k}{2}\frac{d}{dt}\int_0^L\left(\int_0^t g\,ds\right)|\psi|^2\,dx-\frac{k}{2}\int_0^L g|\psi|^2\,dx\\
&-\frac{k}{2}\frac{d}{dt}\int_0^L g\Box\psi\,dx+\frac{k}{2}\int_0^L g'\Box\psi\,dx,
\end{aligned}$$

we conclude

$$\begin{aligned}
\frac{d}{dt}I(t)=&\,\rho_2 g(0)\int_0^L |\psi_t|^2\,dx+\rho_2\int_0^L g'\psi_t\psi\,dx\\
&-\rho_2\int_0^L \psi_t(g''\diamond\psi)\,dx+k\int_0^L \varphi_x(g\psi-g'\diamond\psi)\,dx\\
&-\frac{b}{2}\int_0^L g|\psi_x|^2\,dx+\frac{b}{2}\int_0^L g'\Box\psi_x\,dx-\frac{k}{2}\int_0^L g|\psi|^2\,dx+\frac{k}{2}\int_0^L g'\Box\psi\,dx
\end{aligned}$$

which implies the assertion of the lemma. □

Now we introduce the multiplier w given by the solution of the Dirichlet problem

$$-w_{xx}=\psi_x,\quad w(0)=w(L)=0,$$

and we introduce the functional

$$J_1(t) := \rho_2 \int_0^L \psi_t \psi \, dx + \rho_1 \int_0^L \varphi_t w \, dx$$

in order to get a negative term $-\int_0^L |\psi_x|^2 \, dx$.

Lemma 2.3. *For any $\epsilon_1 > 0$ there exists a positive constant $C_{\epsilon_1} > 0$ such that for $t \geqslant 0$:*

$$\frac{d}{dt} J_1(t) \leqslant C_{\epsilon_1} \int_0^L |\psi_t|^2 \, dx - \frac{\lambda}{2} \int_0^L |\psi_x|^2 \, dx + C_{\epsilon_1} \int_0^L g \Box \psi_x \, dx + \epsilon_1 \int_0^L |\varphi_t|^2 \, dx. \quad (2.4)$$

Proof. Multiplying Eq. (1.2) by ψ we get

$$\frac{d}{dt} \int_0^L \rho_2 \psi_t \psi \, dx = \rho_2 \int_0^L |\psi_t|^2 \, dx - \left(b - \int_0^t g \, d\tau \right) \int_0^L |\psi_x|^2 \, dx - k \int_0^L |\psi|^2 \, dx$$
$$- k \int_0^L \varphi_x \psi \, dx - \int_0^L (g \diamond \psi_x) \psi_x \, dx. \quad (2.5)$$

Multiplying Eq. (1.1) by w we obtain

$$\frac{d}{dt} \int_0^L \rho_1 \varphi_t w \, dx = -k \int_0^L \varphi \psi_x \, dx + k \int_0^L |w_x|^2 \, dx + \rho_1 \int_0^L \varphi_t w_t \, dx. \quad (2.6)$$

Eqs. (2.5) and (2.6) lead to

$$\frac{d}{dt} J_1(t) = \rho_2 \int_0^L |\psi_t|^2 \, dx - \left(b - \int_0^t g \, d\tau \right) \int_0^L |\psi_x|^2 \, dx - k \int_0^L |\psi|^2 \, dx$$
$$+ k \int_0^L |w_x|^2 \, dx + \rho_1 \int_0^L \varphi_t w_t \, dx - \int_0^L (g \diamond \psi_x) \psi_x \, dx.$$

Observing that, for $\delta > 0$,

$$\left| \int_0^L (g \diamond \psi_x) \psi_x \, dx \right| \leqslant C_\delta \int_0^L g \Box \psi_x \, dx + \delta \int_0^L |\psi_x|^2 \, dx,$$

our conclusion follows. □

Let $\mathscr{E}_1(t)$ and $\mathscr{N}(t)$, respectively, denote the functionals

$$\mathscr{E}_1(t) := N_1 E(t) - N_2 I(t) + N_3 J_1(t),$$

$$\mathscr{N}(t) := \int_0^L |\psi_t|^2 + \lambda |\psi_x|^2 + g \Box \psi_x \, dx. \quad (2.7)$$

Using Lemmas 2.2, 2.3 and assumption (1.6) on g, it follows for any $\epsilon_2 > 0$ and for sufficiently large $N_1^{\epsilon_2} > N_2^{\epsilon_2}, N_3^{\epsilon_2}$ that $\mathscr{E}_1(t)$ satisfies

$$\frac{d}{dt}\mathscr{E}_1(t) \leqslant -\frac{N_2^{\epsilon_2}}{2}\mathscr{N}(t) - \frac{N_1^{\epsilon_2}}{2}\int_0^L g|\psi_x|^2\,dx + \epsilon_2\int_0^L (|\varphi_t|^2 + |\varphi_x|^2)\,dx. \tag{2.8}$$

Let us introduce the functional

$$K(t) := \int_0^L \rho_2\psi_t(\varphi_x + \psi)\,dx + \rho_2\int_0^L \psi_x\varphi_t\,dx - \frac{\rho_1}{k}\int_0^L (g * \psi_x)\varphi_t\,dx \tag{2.9}$$

which will provide us a negative term $-\int_0^L |\varphi + \psi_x|^2\,dx$, and it is the next lemma, where we shall use the essential condition (1.5) on the coefficients.

Lemma 2.4. *Assume* (1.5), *i.e.*

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}.$$

Then there exists for any $\epsilon > 0$ a constant $C_\epsilon > 0$ such that for $t \geqslant 0$:

$$\frac{d}{dt}K(t) \leqslant [(b\psi_x - g * \psi_x)\varphi_x]_{x=0}^{x=L} - k\int_0^L |\varphi_x + \psi|^2\,dx$$
$$+ \epsilon\int_0^L |\varphi_t|^2\,dx + C_\epsilon\int_0^L |g'|\Box\psi_x + g|\psi_x|^2\,dx + \rho_2\int_0^L |\psi_t|^2\,dx.$$

Proof. Multiplying Eq. (1.2) by $\psi + \varphi_x$ and using Eq. (1.1) we get

$$\begin{aligned}\frac{d}{dt}\int_0^L \rho_2\psi_t(\varphi_x + \psi)\,dx = {} & [(b\psi_x - g * \psi_x)\varphi_x]_{x=0}^{x=L} - b\frac{\rho_1}{k}\int_0^L \psi_x\varphi_{tt}\,dx \\ & - k\int_0^L |\varphi_x + \psi|^2\,dx + \frac{d}{dt}\left\{\frac{\rho_1}{k}\int_0^L (g * \psi_x)\varphi_t\,dx\right\} \\ & - \frac{\rho_1}{k}\int_0^L \{g(0)\psi_x + g' * \psi_x\}\varphi_t\,dx \\ & + \rho_2\int_0^L |\psi_t|^2\,dx + \rho_2\int_0^L \psi_t\varphi_{xt}\,dx.\end{aligned}$$

Noting that

$$\begin{aligned}\int_0^L \psi_t\varphi_{xt} &= \frac{d}{dt}\int_0^L \psi\varphi_{xt}\,dx - \int_0^L \psi\varphi_{xtt}\,dx \\ &= -\frac{d}{dt}\int_0^L \psi_x\varphi_t\,dx + \int_0^L \psi_x\varphi_{tt}\,dx,\end{aligned}$$

we are at the key point in using the basic assumption (1.5), because now

$$-b\frac{\rho_1}{k}\int_0^L \psi_x\varphi_{tt}\,dx + \rho_2\int_0^L \psi_t\varphi_{xt}\,dx = \rho_2\frac{d}{dt}\int_0^L \psi_x\varphi_t\,dx$$

and hence the terms of higher order cancel, and we have

$$\frac{d}{dt}K(t) = [(b\psi_x - g*\psi_x)\varphi_x]_{x=0}^{x=L} - k\int_0^L |\varphi_x+\psi|^2\,dx$$
$$-\frac{\rho_1}{k}\int_0^L (g\psi_x + g'\diamond\psi_x)\varphi_t\,dx + \rho_2\int_0^L |\psi_t|^2\,dx$$

from where our conclusion follows. $\square$

The last lemma implies the estimate

$$\begin{aligned}\frac{d}{dt}K(t) \leqslant\ & C_\epsilon\{|b\psi_x(L,t)-(g*\psi_x)(L,t)|^2 + |b\psi_x(0,t)-(g*\psi_x)(0,t)|^2\}\\ &+\epsilon\{|\varphi_x(L,t)|^2+|\varphi_x(0,t)|^2\} - k\int_0^L |\varphi_x+\psi|^2\,dx\\ &+\epsilon\int_0^L |\varphi_t|^2\,dx + C_\epsilon\int_0^L |g'|\Box\psi_x + g|\psi_x|^2\,dx + \rho_2\int_0^L |\psi_t|^2\,dx. \qquad (2.10)\end{aligned}$$

In order to deal with the boundary terms appearing we shall prove the following lemma using an extension q of the exterior normal into the domain, this being a well-known approach for dealing with this kind of boundary terms.

Lemma 2.5. *Let $q\in C^1([0,L])$ satisfy $q(0) = -q(L) = 2\gamma > 0$. Then there exist $C_1>0$ and for any $\tilde{\epsilon}>0$ a positive constant $C_{\tilde{\epsilon}}$ such that for $t\geqslant 0$ we have*

(i)
$$\begin{aligned}\frac{d}{dt}\int_0^L \rho_2\psi_t q(b\psi_x - g*\psi_x)\,dx \leqslant\ & -\gamma\{|b\psi_x(L,t)-(g*\psi_x)(L,t)|^2\\ &+|b\psi_x(0,t)-(g*\psi_x)(0,t)|^2\}\\ &+\tilde{\epsilon}\int_0^L |\varphi_x|^2\,dx + C_{\tilde{\epsilon}}\mathcal{N}(t).\end{aligned}$$

(ii)
$$\begin{aligned}\frac{d}{dt}\int_0^L \rho_1\varphi_t q\varphi_x\,dx \leqslant\ & -k\gamma\{|\varphi_x(L,t)|^2 + |\varphi_x(0,t)|^2\}\\ &+\ C_1\int_0^L |\varphi_t|^2 + |\varphi_x|^2 + |\psi_x|^2\,dx.\end{aligned}$$

Proof. With Eq. (1.2) we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^L \rho_2\psi_t q(b\psi_x - g*\psi_x)\,dx =& \frac{1}{2}[q(b\psi_x - g*\psi_x)^2]_{x=0}^{x=L} - \frac{1}{2}\int_0^L q_x(b\psi_x - g*\psi_x)^2\,dx \\ & - k\int_0^L (\varphi_x+\psi)q(b\psi_x - g*\psi_x)\,dx \\ & + \frac{1}{2}b\rho_2\int_0^L q\frac{d}{dx}|\psi_t|^2\,dx \\ & - \rho_2\int_0^L \psi_t q(g(0)\psi_x + g'*\psi_x)\,dx \\ \leqslant & -\gamma\{|b\psi_x(L,t) - (g*\psi_x)(L,t)|^2 \\ & + |b\psi_x(0,t) - (g*\psi_x)(0,t)|^2\} \\ & + \tilde{\epsilon}\int_0^L |\varphi_x|^2\,dx + C_{\tilde{\epsilon}}\mathcal{N}(t),\end{aligned}$$

where we used assumption (1.6) on g. This proves (i). Estimate (ii) is proved, using Eq. (1.1), as follows:

$$\begin{aligned}\frac{d}{dt}\int_0^L \rho_1\varphi_t q\varphi_x\,dx =& k\int_0^L q\varphi_{xx}\varphi_x\,dx + k\int_0^L q\psi_x\varphi_x\,dx \\ & + \frac{1}{2}\rho_1[q|\varphi_t|^2]_{x=0}^{x=L} - \rho_1\int_0^L q_x|\varphi_t|^2\,dx \\ \leqslant & -k\gamma\{|\varphi_x(L,t)|^2 + |\varphi_x(0,t)|^2\} \\ & + C_1\int_0^L |\varphi_t|^2 + |\varphi_x|^2 + |\psi_x|^2\,dx. \qquad \square\end{aligned}$$

For $\delta > 0$ and $N_4 > 1$ let

$$L(t) := K(t) + N_4\int_0^L \rho_2\psi_t q(b\psi_x - g*\psi_x)\,dx + \delta\int_0^L \rho_1\varphi_t q\varphi_x\,dx. \tag{2.11}$$

Observing

$$-\frac{k}{2}\int_0^L |\varphi_x+\psi|^2\,dx \leqslant -\frac{k}{4}\int_0^L |\varphi_x|^2\,dx + C\int_0^L |\psi_x|^2\,dx,$$

for some positive constant C, we conclude from Lemma 2.5 and (2.10) that for sufficiently large N_4 and sufficiently small δ we have for $0 < \tau < 1$ and some $C_\tau > 0$

and $C_2 > 0$ that

$$\frac{d}{dt}L(t) \leqslant -\frac{k}{2}\int_0^L |\varphi_x + \psi|^2\,dx + C_2\tau\int_0^L |\varphi_t|^2\,dx + C_\tau \mathcal{N}(t), \tag{2.12}$$

where we used (1.6) again. Here, one can choose first δ of order τ, then ϵ small enough, then N_4 large enough, then $\tilde{\epsilon}$ small enough.

Finally, let us introduce the functional

$$J_2(t) := \int_0^L \rho_1\varphi_t\varphi + \rho_2\psi_t\psi\,dx \tag{2.13}$$

to obtain, as usual, negative terms $-\int_0^L |\varphi_t|^2\,dx$ and $-\int_0^L |\psi_t|^2\,dx$; we easily get

Lemma 2.6. *There exists a positive constant c satisfying*

$$-\frac{d}{dt}J_2(t) \leqslant -\rho_1\int_0^L |\varphi_t|^2\,dx - \rho_2\int_0^L |\psi_t|^2\,dx + k\int_0^L |\varphi_x + \psi|^2\,dx + c\mathcal{N}(t).$$

Lemma 2.6 and (2.12) yield, choosing τ small enough,

$$\frac{d}{dt}\left\{L(t) - \frac{2C_2\tau}{\rho_1}J_2(t)\right\} \leqslant -\frac{k}{4}\int_0^L |\varphi_x + \psi|^2\,dx - C_2\tau\int_0^L |\varphi_t|^2\,dx + C_\tau\mathcal{N}(t). \tag{2.14}$$

Now we are in the position to show the main result of this section:

Theorem 2.7. *Let us suppose that the initial data satisfy*

$$\varphi_0, \psi_0 \in H_0^1((0,L)), \quad \varphi_1, \psi_1 \in L^2((0,L)),$$

and that the coefficients of system (1.1), (1.2) *satisfy* (1.5), *i.e.*,

$$\frac{\rho_1}{\rho_2} = \frac{k}{b}.$$

Moreover, assume that the kernel g is of exponential type satisfying (1.6). *Then the energy $E(t)$ decays exponentially as time tends to infinity, that is, there exist positive constants C and α, being independent of the initial data, such that for $t \geqslant 0$:*

$$E(t) \leqslant CE(0)e^{-\alpha t}.$$

Proof. To use the multiplicative techniques we need that the initial data satisfy $\varphi_0, \psi_0 \in H_0^1((0,L)) \cap H^2((0,L))$, $\varphi_1, \psi_1 \in H_0^1((0,L))$, but the conclusion of the theorem will then follow by density arguments.

Let the final Lyapunov functional be defined by

$$\mathscr{L}(t) := \mathscr{E}_1(t) + L(t) - \frac{2C_2\tau}{\rho_1} J_2(t),$$

where $\mathscr{E}_1(t)$, $L(t)$ and $J_2(t)$ were defined in (2.7), (2.11) and (2.13), respectively. With (2.8) and (2.14) we conclude for sufficiently small ϵ_2 and some $\beta_0 > 0$ that

$$\frac{d}{dt}\mathscr{L}(t) \leqslant -\beta_0 E(t).$$

Moreover, there are positive constants β_1, β_2 such that for $t \geqslant 0$

$$\beta_1 E(t) \leqslant \mathscr{L}(t) \leqslant \beta_2 E(t)$$

whence

$$\frac{d}{dt}\mathscr{L}(t) \leqslant -\alpha \mathscr{L}(t)$$

for $\alpha := \beta_0/\beta_2$, and hence our conclusion follows. □

3. Parameter optimality

Condition (1.5) assume out to be sufficient to prove the exponential stability in the previous section. Now we shall demonstrate that it is also a necessary condition in general. Since the convolution term, the memory type damping, in general does not generate a semigroup, but rather an evolution system, it is the first task for this new problem to get an appropriate semigroup approximation. Having obtained this, we shall find a compactly perturbed semigroup—not speaking of generators—for which the spectrum can be described explicitly and for which known methods and results for the essential spectrum apply.

3.1. Approximation of the problem

Let us denote the energy defined in (2.1) by

$$E(t) =: E_g(\varphi, \psi) =: E_g(t). \tag{3.1}$$

Let g be an exponential type kernel and let $k_1 > 0$ such that

$$g(t) \leqslant g(0)e^{-k_1 t}.$$

Let $k\in(0,\frac{k_1}{2})$ and let j be the bijection:

$$j:\begin{cases}[0,\infty)\mapsto(0,1],\\ t\to\ :j(t):=x=e^{-kt}.\end{cases}$$

With the kernel g, we associate the function f defined on $[0,1]$ by

$$f(x):=\frac{g\circ j^{-1}(x)}{kx},\quad x\in(0,1],\qquad f(0):=0.$$

Now we can approximate the function f by its *Bernstein polynomials* (*see* [5]) defined by

$$B_n(f,x):=\sum_{v=0}^{v=n} C_n^v f\left(\frac{v}{n}\right)x^v(1-x)^{n-v},$$

where $C_n^v:=\binom{n}{v}$.

Lemma 3.1. *Assume that* g *satisfies* (1.7) *and let*

$$g_n(t):=ke^{-kt}B_n\left(\frac{g\circ j^{-1}}{kId},e^{-kt}\right).$$

Then we have

$$\forall\epsilon>0,\exists N_\epsilon\in\mathbb{N}\quad n\geqslant N_\epsilon\Rightarrow||g-g_n||_{W^{1,1}((0,\infty))}\leqslant\epsilon.$$

Moreover, for all $n\in\mathbb{N}$, g_n *satisfies*:

(i) $g_n\geqslant 0$,
(ii) $g_n'\leqslant 0$,
(iii) $\lim_{n\to\infty}\int_0^\infty g_n(t)\,dt=\int_0^\infty g(t)\,dt$.

Proof. Using assumptions (1.7) on g, one gets

$$f(x)=\frac{g(-\frac{\ln x}{k})}{kx}\leqslant\frac{g(0)}{k}x^{\frac{k_1}{k}-1},\quad f'(x)\leqslant\frac{g(0)(k_0-k)}{k^2}x^{\frac{k_1}{k}-2}.$$

Then, if $k\in(0,\frac{k_1}{2})$ one deduces that $f\in C^1([0,1])$ and $f'(0)=0$. So, for all positive ϵ there exists N'_ϵ in $\mathbb{N}$ such that

$$n\geqslant N'_\epsilon\Rightarrow||f-B_n(f)||_{C^1([0,1])}\leqslant\epsilon$$

(see [5, Theorem 2.1, p. 306]), hence

$$n\geqslant N'_\epsilon\Rightarrow||f-B_n(f)||_{W^{1,1}((0,1))}\leqslant\epsilon.$$

Since

$$\int_0^{+\infty} |g(t) - g_n(t)|\, dt = \int_0^1 |f(x) - B_n(f,x)|\, dx,$$

hence

$$n \geqslant N'_\epsilon \Rightarrow ||g - g_n||_{L^1(0,1)} \leqslant \epsilon.$$

Moreover,

$$\int_0^\infty |g' - g'_n|(t) = k \int_0^1 |(f(x) + xf'(x)) - (B_n(f,x) + xB'_n(f,x))| \leqslant k\epsilon$$

and the first claim of the lemma follows.

The second part of the claim is a consequence of the definition of f and properties (1.7) of g. More precisely, one has, using assumptions on g,

$$f'(x) \geqslant \frac{(k_1 - k)}{k^2 x^2}\, g\left(\frac{-\ln x}{k}\right) > 0, \quad x > 0,$$

since $k \in (0, \frac{k_1}{2})$. Now, using the following simple properties of the Bernstein polynomials of f, $B_n(f,x)$ [5], one has also that for all $n \in \mathbb{N}$ and $x \in [0,1]$:

$$B_n(f,x) \geqslant 0, \quad B'_n(f,x) \geqslant 0.$$

Returning to g_n, one gets:

$$g'_n(t) = -k^2 e^{-kt}(B_n(f, e^{-kt}) + e^{-kt} B'_n(f, e^{-kt}))$$

which gives assertion (ii) of the lemma. To complete the proof, notice that, if we set

$$\theta_{v,n}(s) := k C_n^v e^{-(v+1)ks}(1 - e^{-ks})^{n-v}$$

one has

$$\int_0^{+\infty} g_n(s)\, ds = \frac{1}{n+1} \sum_{v=1}^{n} f\left(\frac{v}{n}\right) \to \int_0^1 f(x)\, dx = \int_0^{+\infty} g(s)\, ds. \qquad \square$$

Let us denote by $(\overline{\varphi}, \bar{\psi})$ the solution of the initial boundary value problem (1.1)–(1.4) with the same initial data but with g replaced by some approximation g_n corresponding to some fixed $\epsilon > 0$.

Lemma 3.2. *Assume that g satisfies* (1.7) *and that there exists a positive function $d \in L^1((0,\infty))$ such that $\lim_{t\to\infty} d(t) = 0$ and for all data and all $t \geqslant 0$*

$$E_g(\varphi, \psi)(t) \leqslant d^2(t) E_g(\varphi, \psi)(0). \tag{3.2}$$

Then, with the previous notations, there exists $C>0$, *such that for all* $\epsilon>0$ *sufficiently small and all* $t\geqslant 0$

(i) $$E_{g_n}(\varphi-\overline{\varphi},\psi-\bar{\psi})(t)\leqslant C\epsilon E_{g_n}(0).$$

(ii) $$|E_g(\varphi,\psi)(t)-E_{g_n}(\overline{\varphi},\bar{\psi})(t)|\leqslant C\epsilon^{1/2}E_{g_n}(0).$$

Proof. Let us denote by $z:=\varphi-\overline{\varphi}$ and by $w:=\psi-\bar{\psi}$. Then z and w satisfy

$$\begin{gathered}\rho_1 z_{tt}-k(z_x+w)_x=0,\\ \rho_2 w_{tt}-bw_{xx}+g_n*w_{xx}+k(z_x+w)=(g_n-g)*\psi_{xx},\\ z(0,t)=z(L,t)=w(0,t)=w(L,t)=0,\\ z(0)=0,\quad w(0)=0.\end{gathered}$$

Then we get

$$\begin{aligned}\frac{d}{dt}E_{g_n}(z,w)(t)=&-\frac{1}{2}g_n(t)\int_0^L w_x^2\,dx+\frac{1}{2}\int_0^L g_n'\Box w_x\,dx\\ &+\int_0^L((g_n-g)*\psi_{xx})w_t\,dx.\end{aligned}$$

Integration with respect to time yields

$$E_{g_n}(z,w)(t)\leqslant\int_0^t\int_0^L((g_n-g)*\psi_{xx})w_t\,dx\,ds=:I(t). \tag{3.3}$$

Integration by parts in time leads to

$$\begin{aligned}I(t)=&\int_0^L((g_n-g)*\psi_{xx})(t)w(t)\,dx-\int_0^t\int_0^L(g_n-g)(0)\psi_{xx}w\,dx\,ds\\ &-\int_0^t\int_0^L((g_n'-g')*\psi_{xx})w\,dx\,ds.\end{aligned}$$

Integration by parts in space yields

$$\begin{aligned}I(t)\leqslant&(|g_n-g|*||\psi_x||)||w_x||+\int_0^t\int_0^L(g_n-g)(0)\psi_x.w_x\,dx\,ds\\ &+\int_0^t(|g_n'-g'|*||\psi_x||)||w_x||\,ds\end{aligned} \tag{3.4}$$

where is introduced the notation

$$||f||^2(t) := \int_0^L |f(t,x)|^2\, dx.$$

On the other hand, there exists a constant $C>0$, independent of n, such that

$$||w_x||^2(t) \leqslant CE_{g_n}(z,w)(t), \quad ||\psi_x||^2(t) \leqslant CE_g(\varphi,\psi)(t). \tag{3.5}$$

Note also that, since $g - g_n \in W^{1,1}(0,\infty)$, for all $t>0$:

$$|(g-g_n)(t)| \leqslant ||g' - g_n'||_{L^1(0,\infty)} < \epsilon.$$

Thus

$$||g-g_n||_{L^\infty(0,\infty)} < \epsilon \tag{3.6}$$

Therefore, using the general decay of the energy (see (2.2)), the properties of g_n, (3.5) and (3.2), we arrive at (denoting by C various positive constants independent of t and ϵ all along this proof):

$$(|g_n - g| * ||\psi_x||(t))||w_x||(t) \leqslant C\epsilon(E_g(\varphi,\psi)(0) + E_{g_n}(z,w)(t)). \tag{3.7}$$

Using (3.2) and assumption on d, we also get

$$\begin{aligned}\int_0^t \int_0^L (g_n - g)(0)\psi_x w_x\, dx\, ds &\leqslant \epsilon\sqrt{E_g(\varphi,\psi)(0)} \int_0^t d(s).\sqrt{E_{g_n}(z,w)(s)}\, ds \\ &\leqslant \frac{\epsilon}{2}\left(||d||_{L^1(0,\infty)} E_g(\varphi,\psi)(0) + \int_0^t d(s)E_{g_n}(z,w)(s)\, ds\right) \\ &\leqslant C\epsilon\left(E_g(\varphi,\psi)(0) + \int_0^t d(s)E_{g_n}(z,w)(s)\, ds\right). \end{aligned} \tag{3.8}$$

Therefore, returning to the estimation of the right-hand member of inequality (3.4), using the previous estimate and (3.2), we also get

$$\begin{aligned}\int_0^t (|g_n' - g'| * ||\psi_x||)||w_x||\, ds \leqslant &\sqrt{E_g(\varphi,\psi)(0)}\left(\int_0^t |g_n' - g'| * d(s)\, ds\right)^{1/2} \\ &\times \left(\int_0^t |g_n' - g'| * d(s)E_{g_n}(z,w)(s)\, ds\right)^{1/2}\end{aligned}$$

for all $t \geqslant 0$. But now, we have by assumption on d:

$$\int_0^t \int_0^s |g_n' - g'|(s-\tau)\, d(\tau)\, d\tau\, ds \leqslant ||g_n' - g'||_{L^1(\mathbb{R}^+)} ||d||_{L^1(\mathbb{R}^+)} \leqslant C\epsilon. \tag{3.9}$$

Thus, setting

$$k(s) := \int_0^s |g_n' - g'|(s-\tau)\, d(\tau)\, d\tau, \quad s \geqslant 0,$$

we then have

$$\int_0^t (|g_n' - g'| * \|\psi_x\|)\|w_x\|\, ds \leqslant C\left(\epsilon E_g(\varphi,\psi)(0) + \int_0^t k(s) E_{g_n}(z,w)(s)\, ds\right). \quad (3.10)$$

Using (3.4), (3.7), (3.8) and (3.10), inequality (3.3) becomes

$$E_{g_n}(z,w)(t) \leqslant C\epsilon E_g(\varphi,\psi)(0) + C\epsilon E_{g_n}(z,w)(t) + C\int_0^t (d(s)+k(s)) E_{g_n}(z,w)(s)\, ds$$

and, for $\epsilon > 0$ sufficiently small,

$$E_{g_n}(z,w)(t) \leqslant \frac{C\epsilon}{1-C\epsilon} E_g(\varphi,\psi)(0) + \frac{C}{1-C\epsilon}\int_0^t (d(s)+k(s)) E_{g_n}(z,w)(s)\, ds.$$

Applying Gronwall's inequality to $E_{g_n}(z,w)(t)$ yields

$$E_{g_n}(z,w)(t) \leqslant \frac{C\epsilon}{1-C\epsilon} E_g(\varphi,\psi)(0) \exp\left(\frac{C}{1-C\epsilon}\int_0^t (d(s)+k(s))\, ds\right).$$

From (3.9), it follows that $\int_0^t (d(s)+k(s))\, ds \leqslant C(\epsilon+1)$ uniformly in t and then we conclude that

$$\begin{aligned} E_{g_n}(z,w)(t) &\leqslant \frac{C\epsilon}{1-C\epsilon} E_g(\varphi,\psi)(0) \exp\left(\frac{C(\epsilon+1)}{1-C\epsilon}\right) \\ &\leqslant C\epsilon E_g(\varphi,\psi)(0) = C\epsilon E_{g_n}(0) \end{aligned}$$

which is claim (i) of the lemma (notice that $E_g(0)$ does not depend on g: $E_g(0) = E_{g_n}(0) = E(0)$ for the same initial data).

Let us prove claim (ii) of our lemma. Using the definition, the general decay of the energy, we get

$$\begin{aligned} |E_g(\varphi,\psi) - E_{g_n}(\varphi,\psi)| &= \frac{1}{2}\left|\int_0^L \left(\int_0^t (g_n - g)(s)\, ds\, |\psi_x|^2 + (g - g_n)\Box\psi_x\right) dx\right| \\ &\leqslant C\left(\int_0^t |g_n - g|(s)\, ds\, E_g(\varphi,\psi)(t) + |g_n - g| * E_g(\varphi,\psi)(t)\right) \\ &\leqslant C\left(\int_0^t |g_n - g|(s) d(s)\, ds + |g_n - g| * d(t)\right) E_g(\varphi,\psi)(0) \\ &\leqslant C\epsilon E_g(\varphi,\psi)(0). \end{aligned}$$

So, from this last inequality and claim (i), we get

$$\begin{aligned}
|E_g(\varphi,\psi)(t) - E_{g_n}(\overline{\varphi},\bar{\psi})(t)| &\leqslant |E_g(\varphi,\psi) - E_{g_n}(\varphi,\psi)| + |E_{g_n}(\varphi,\psi)(t) - E_{g_n}(\overline{\varphi},\bar{\psi})(t)| \\
&\leqslant C\epsilon E_g(0) + \left|\sqrt{E_{g_n}(\varphi,\psi)(t)} - \sqrt{E_{g_n}(\overline{\varphi},\bar{\psi})(t)}\right| \\
&\quad \times \left|\sqrt{E_{g_n}(\varphi,\psi)(t)} + \sqrt{E_{g_n}(\overline{\varphi},\bar{\psi})(t)}\right| \\
&\leqslant C\epsilon E_g(0) + \sqrt{E_{g_n}(\varphi-\overline{\varphi},\psi-\bar{\psi})(t)} \\
&\quad \times \left(\sqrt{E_{g_n}(\varphi,\psi)(0)} + \sqrt{E_{g_n}(\overline{\varphi},\bar{\psi})(0)}\right) \\
&\leqslant C\epsilon^{1/2} E_g(0)
\end{aligned}$$

which is exactly claim (ii). □

3.2. Nonuniform decay for the approximated problem

In this section we will prove that, whenever the wave speeds are different, i.e. when

$$\frac{k}{\rho_1} \neq \frac{b}{\rho_2}$$

then the associated energy does not decay uniformly.

Recall that

$$\theta_{n,v}(t) = kC_n^v e^{-(v+1)kt}(1-e^{-kt})^{n-v}.$$

Thus

$$g_n * \bar{\psi}_{xx} = \sum_{v=1}^{v=n} f\left(\frac{v}{n}\right)\theta_{n,v} * \bar{\psi}_{xx},$$

and if we denote by

$$y_{n,v}(.,t) := \int_0^t \theta_{n,v}(t-\tau)\bar{\psi}_x(.,\tau)\,d\tau,$$

we obtain

$$g_n * \bar{\psi}_{xx} = \sum_{v=1}^{v=n} f\left(\frac{v}{n}\right)(y_{n,v})_x.$$

Let us define the vector-valued function in $\mathbb{R}^n$

$$Y_n := \begin{pmatrix} y_{n,1} \\ \vdots \\ y_{n,n} \end{pmatrix}.$$

One has

$$\left.\begin{aligned} Y_n' = A_n Y_n + D_n \bar{\psi}_x, \\ Y_n(\cdot, 0) = 0, \end{aligned}\right\}$$

where

$$D_n = (0, \dots, 0, k)', \quad A_n = (a_{ij})_{1 \leqslant i,j \leqslant n},$$

$$a_{ij} = \begin{cases} -k(i+1) & \text{if} \quad j = i, \\ k(i+1) & \text{if} \quad j = i+1, \\ 0 & \text{if not.} \end{cases}$$

Lemma 3.3. *A_n is the generator of a C^0 semigroup in $H := L^2((0,L))^n$. Moreover this semigroup satisfies the following uniform estimate*:

$$\forall t \geqslant 0: \quad ||e^{tA_n}|| \leqslant e^{-\frac{k}{2}t}.$$

Proof. Observing that for the scalar product in $\mathbb{R}^n$: $A_n Y.Y$ we have

$$\begin{aligned} A_n Y.Y &= -k \sum_{i=1}^{n-1} \frac{(i+1)}{2} (y_i - y_{i+1})^2 - \frac{k}{2} \sum_{i=1}^{n} y_i^2 - \frac{k}{2} y_1^2 - \frac{k(n+1)}{2} y_n^2 \\ &\leqslant -\frac{k}{2} |Y|^2 \end{aligned}$$

the assertion follows. □

The initial–boundary value problem for $(\overline{\varphi}, \bar{\psi})$ is then equivalent to the following one:

$$\left.\begin{aligned} \rho_1 \overline{\varphi}_{tt} - k(\overline{\varphi}_x + \bar{\psi})_x = 0, \\ \rho_2 \bar{\psi}_{tt} - b\bar{\psi}_{xx} + k(\overline{\varphi}_x + \bar{\psi}) + B_n Y_x = 0, \\ Y_n'(t) = A_n Y(t) + D_n \bar{\psi}_x, \\ \overline{\varphi}(0,t) = \overline{\varphi}(L,t) = \bar{\psi}(0,t) = \bar{\psi}(L,t) = 0, \\ Y_n(0) = 0, \end{aligned}\right\} \tag{3.11}$$

where

$$B_n := \left(f\left(\frac{1}{n}\right), \dots, f\left(\frac{n}{n}\right)\right).$$

To system (3.11), we associate the energy:

$$E_n(\overline{\varphi}, \bar{\psi}, Y)(t) := \frac{1}{2}\int_0^L \{\rho_1 \overline{\varphi}_t^2 + \rho_2 \bar{\psi}_t^2 + b\bar{\psi}_x^2 + k(\overline{\varphi}_x + \bar{\psi})^2 + |Y_n|^2\}\, dx(t). \tag{3.12}$$

In the energy space $H = (H_0^1 \times L^2)^2 \times (L^2)^n$, it is not difficult to associate a semigroup to system (3.11).

Now, the new variables:

$$Z \equiv \begin{pmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_5 \end{pmatrix} := \begin{pmatrix} \sqrt{\rho_1}\,\overline{\varphi}_t - \sqrt{k}(\overline{\varphi}_x + \bar{\psi}) \\ \sqrt{\rho_2}\,\bar{\psi}_t - \sqrt{b}\,\bar{\psi}_x + \frac{1}{\sqrt{b}} B_n Y_n \\ \sqrt{\rho_1}\,\overline{\varphi}_t + \sqrt{k}(\overline{\varphi}_x + \bar{\psi}) \\ \sqrt{\rho_2}\,\bar{\psi}_t + \sqrt{b}\,\bar{\psi}_x - \frac{1}{\sqrt{b}} B_n Y_n \\ Y_n \end{pmatrix} \tag{3.13}$$

satisfy the system

$$Z_t = \Lambda Z_x + MZ \tag{3.14}$$

with the boundary conditions

$$(z_i + z_{i+2})(0, t) = (z_i + z_{i+2})(L, t) = 0, \quad i = 1, 2. \tag{3.15}$$

in the space $G = (L^2)^4 \times (L^2)^n$. Here

$$\Lambda := diag\left(-\sqrt{\frac{k}{\rho_1}}, -\sqrt{\frac{b}{\rho_2}}, \sqrt{\frac{k}{\rho_1}}, \sqrt{\frac{b}{\rho_2}}, 0_n\right),$$
$$0_n := (0, \dots, 0) \in \mathbb{R}^n$$

and

$$M := \begin{pmatrix} M_4 & M_{4n} \\ N_{n4} & A_n + \frac{1}{b} D_n B_n \end{pmatrix},$$

where

$$M_4 := \begin{pmatrix} 0 & -\frac{1}{2}\sqrt{\frac{k}{\rho_2}} & 0 & -\frac{1}{2}\sqrt{\frac{k}{\rho_2}} \\ \frac{1}{2}\sqrt{\frac{k}{\rho_2}} & -\frac{\sum_{\nu=1}^{\nu=n} f(\frac{\nu}{n})}{2b} & -\frac{1}{2}\sqrt{\frac{k}{\rho_2}} & \frac{\sum_{\nu=1}^{\nu=n} f(\frac{\nu}{n})}{2b} \\ 0 & \frac{1}{2}\sqrt{\frac{k}{\rho_2}} & 0 & \frac{1}{2}\sqrt{\frac{k}{\rho_2}} \\ \frac{1}{2}\sqrt{\frac{k}{\rho_2}} & \frac{\sum_{\nu=1}^{\nu=n} f(\frac{\nu}{n})}{2b} & -\frac{1}{2}\sqrt{\frac{k}{\rho_2}} & -\frac{\sum_{\nu=1}^{\nu=n} f(\frac{\nu}{n})}{2b} \end{pmatrix}.$$

The matrices M_{4n} and N_{n4} will not play any role in the sequel but we give their expressions for completeness:

$$M_{4n} := \begin{pmatrix} 0 \\ \frac{1}{b^{3/2}}(B_n D_n B_n + bB_n A_n) \\ 0 \\ -\frac{1}{b^{3/2}}(B_n D_n B_n + bB_n A_n) \end{pmatrix}; \quad N_{n4} := \begin{pmatrix} 0 & -\frac{D_n}{2b^{1/2}} & 0 & \frac{D_n}{2b^{1/2}} \end{pmatrix}.$$

Note here that (3.13) can be written as

$$Z = P_n^{-1} \begin{pmatrix} \sqrt{\rho_1}\,\overline{\varphi}_t \\ \sqrt{k}(\overline{\varphi}_x + \bar{\psi}) \\ \sqrt{\rho_2}\,\bar{\psi}_t \\ \sqrt{b}\,\bar{\psi}_x \\ Y_n \end{pmatrix}$$

with

$$P_n^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & \frac{1}{\sqrt{b}}B_n \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -\frac{1}{\sqrt{b}}B_n \\ 0 & 0 & 0 & 0 & I_n \end{pmatrix},$$

$$P_n = \begin{pmatrix} 1/2 & 0 & 1/2 & 0 & 0 \\ -1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 & \frac{1}{\sqrt{b}}B_n \\ 0 & 0 & 0 & 0 & I_n \end{pmatrix}. \tag{3.16}$$

If we forget for a moment the condition $Y_n(x,0) = 0$ and replace it by any initial data, we may associate to (3.11) a C_0-semigroup $e^{(\Lambda\partial_x + M)t}$. A result of Neves et al. [17] (actually, an extension of their result to the case where there are null and multiple eigenvalues in the diagonal matrix Λ, see [1,2]) asserts that, since $\frac{k}{\rho_1} \neq \frac{b}{\rho_2}$, if we consider the semigroup $e^{(\Lambda\partial_x + M_0)t}$ associated with the following system:

$$Z_t = \Lambda Z_x + M_0 Z,$$

$$(z_i + z_{i+2})(0,t) = (z_i + z_{i+2})(L,t) = 0, \quad i = 1,2,$$

$$M_0 := diag\left(0, -\frac{\sum_{v=1}^{v=n} f\left(\frac{v}{n}\right)}{2b}, 0, -\frac{\sum_{v=1}^{v=n} f\left(\frac{v}{n}\right)}{2b}, A_n + \frac{1}{b} D_n B_n\right) \tag{3.17}$$

with the same boundary conditions, then $e^{(\Lambda\partial_x + M)t} - e^{(\Lambda\partial_x + M_0)t}$ is a compact operator. Now, for this last system, the eigenvalues can be easily computed by solving the diagonal differential system:

$$\lambda Z = \Lambda Z_x + M_0 Z,$$

$$(z_i + z_{i+2})(0) = (z_i + z_{i+2})(L) = 0, \quad i = 1,2.$$

Indeed we get

$$\sigma(\Lambda\partial_x + M_0) = \sigma\left(A_n + \frac{1}{b} D_n B_n\right) \cup \left\{ m\sqrt{\frac{k}{\rho_1}} \frac{\pi}{L} i, m \in \mathbb{Z} \right\}$$
$$\cup \left\{ -\frac{\sum_{v=1}^{v=n} f\left(\frac{v}{n}\right)}{2b} + m\sqrt{\frac{b}{\rho_2}} \frac{\pi}{L} i, m \in \mathbb{Z} \right\}. \tag{3.18}$$

Now we compare the energy $E_{g_n}(\overline{\varphi}, \bar{\psi})(t)$ and the norm of the solution $Z(t) := e^{(\Lambda\partial_x + M)t} Z_0$ associated to the corresponding initial data.

Lemma 3.4. *There exists a constant C such that for all $n \in \mathbb{N}$ and all $((\varphi_0, \varphi_1), (\psi_0, \psi_1)) \in [H_0^1((0,L)) \times L^2((0,L))]^2$ one has*

$$||Z(t)||^2 = ||e^{(\Lambda\partial_x + M)t} Z_0||^2 \leqslant C\left(E_{g_n}(\overline{\varphi}, \bar{\psi})(t) + \int_0^t e^{\frac{-k(t-\tau)}{2}} E_{g_n}(\overline{\varphi}, \bar{\psi})(\tau)\, d\tau \right),$$

where

$$Z_0 := P_n^{-1} \begin{pmatrix} \sqrt{\rho_1}\,\varphi_1 \\ \sqrt{k}(\varphi_{0x} + \psi_0) \\ \sqrt{\rho_2}\,\psi_1 \\ \sqrt{b}\,\psi_{0x} \\ 0 \end{pmatrix}.$$

Proof. Notice that with the particular choice of the initial data, Z_0 is independent of n. Now, computing $Z(t)$ by using the matrix P_n^{-1}, one gets

$$||Z(t)||^2 \leqslant CE_{g_n}(\overline{\varphi}, \bar{\psi})(t) + \frac{1}{2b}||B_n Y_n(t)||^2 + ||Y_n(t)||^2.$$

Let's compute $||B_n Y_n(t)||^2$ using the fact that $Y_n(0) = 0$:

$$\begin{aligned}
||B_n Y_n(t)||^2 &= \int_0^L \left(\int_0^t \sum_{i=1}^n f\left(\frac{v}{n}\right) \theta_{v,n}(t-s) \bar{\psi}_x(s)\, ds \right)^2 dx \\
&\leqslant \int_0^L \left\{ \int_0^t \sum_{i=1}^n f\left(\frac{v}{n}\right) \theta_{v,n}(s)\, ds \int_0^t \sum_{i=1}^n f\left(\frac{v}{n}\right) \theta_{v,n}(t-s) \bar{\psi}_x^2(s)\, ds\, dx \right\} \\
&\leqslant \left(\int_0^t g_n(s)\, ds \right) . \left\{ \int_0^L \int_0^t g_n(t-s) \bar{\psi}_x^2(s)\, ds\, dx \right\} \\
&\leqslant 2b \int_0^L \int_0^t g_n(t-s)(\bar{\psi}_x(s) - \bar{\psi}_x(t))^2\, ds + 2\left(\int_0^t g_n(s)\, ds \right)^2 ||\bar{\psi}_x(t)||^2 \\
&\leqslant CE_{g_n}(\overline{\varphi}, \bar{\psi}).
\end{aligned}$$

Recall that

$$\lim_{n \to +\infty} \int_0^{+\infty} g_n(s)\, ds = \int_0^{+\infty} g(t)\, dt = \lambda < b.$$

It remains to estimate the norm of Y_n:

$$||Y_n(t)||^2 := \left|\left| \int_0^t e^{(t-s)A_n} D_n \bar{\psi}_x(s)\, ds \right|\right|^2.$$

But, using the definition of D_n, we get

$$\left|\left| \int_0^t e^{(t-s)A_n} D_n \bar{\psi}_x(s)\, ds \right|\right| \leqslant \int_0^t ||e^{(t-s)A_n}||\, ||\bar{\psi}_x(s)||\, ds$$

and using Lemma 3.3, one gets

$$||Y_n(t)||^2 \leqslant \int_0^t e^{-\frac{k(t-s)}{2}} ||\bar{\psi}_x(s)||^2\, ds.$$

But, for all $s \in (0, \infty)$ one has

$$||\bar{\psi}_x(s)||^2 \leqslant \frac{1}{(b - \int_0^s g_n(\tau)\, d\tau)} E_{g_n}(\overline{\varphi}, \bar{\psi})(s).$$

Consequently, we get

$$||Y_n(t)||^2 \leqslant \frac{1}{b-\lambda}\int_0^t e^{-\frac{k(t-s)}{2}} E_{g_n}(\overline{\varphi},\bar{\psi})(s)\,ds.$$

Then, collecting the previous estimates, we obtain

$$||Z(t)||^2 \leqslant CE_{g_n}(\overline{\varphi},\bar{\psi})(t) + C\int_0^t e^{-\frac{k(t-s)}{2}} E_{g_n}(\overline{\varphi},\bar{\psi})(s)\,ds$$

which is the claim of the lemma. □

We are now ready to state and prove the main result of this section:

Theorem 3.5. *Assume that g satisfies* (1.7). *Assume moreover that*

$$\frac{\rho_1}{\rho_2} \neq \frac{k}{b}.$$

Then the energy $E_g(\varphi,\psi)$ does not decay uniformly in the initial data as time tends to infinity, i.e., there does not exist $d \in L^1((0,\infty)) \cap L^2_{loc}([0,\infty))$ such that for all $t \geqslant 0$: $\lim_{t\to\infty} d(t) = 0$ and

$$E_g(t) \leqslant d^2(t)E_g(0).$$

Proof. Assume to the contrary that there exists $L^1((0,\infty)) \cap L^2_{loc}([0,\infty))$ such that for all $t \geqslant 0$ $\lim_{t\to\infty} d(t) = 0$ and

$$E_g(t) \leqslant d^2(t)E_g(0). \tag{3.19}$$

Using (ii) in Lemma 3.2, we get, g_n being the ϵ-approximate function of g,

$$\begin{aligned} E_{g_n}(\overline{\varphi},\bar{\psi})(t) &\leqslant E_g(\varphi,\psi)(t) + C\epsilon^{1/2}E_g(\varphi,\psi)(0) \\ &\leqslant (d^2(t) + C\epsilon^{1/2})E_g(\varphi,\psi)(0). \end{aligned} \tag{3.20}$$

From Lemma 3.4, it follows for $Y_n(t,x) = \int_0^t e^{A(t-s)}D_n\bar{\psi}_x\,ds$

$$\begin{aligned} ||Z(t)||^2 &\leqslant CE_{g_n}(\overline{\varphi},\bar{\psi})(t) + C\int_0^t e^{-\frac{k(t-s)}{2}} E_{g_n}(\overline{\varphi},\bar{\psi})(s)\,ds \\ &\leqslant C\left(d^2(t) + C\epsilon^{1/2} + \int_0^t (d^2(s) + C\epsilon^{1/2})e^{-\frac{k(t-s)}{2}}\,ds\right)E_g(\varphi,\psi)(0). \end{aligned} \tag{3.21}$$

Now, let us choose as initial data the particular sequence $(Z_0^m)_{m\in\mathbb{Z}}$ of the eigenfunctions of the operator $\Lambda\partial_x + M_0$, associated with the eigenvalues $im\sqrt{\frac{k}{\rho_1}}\frac{\pi}{L}$,

$m \in \mathbb{Z}$ (see (3.18)), which are given by

$$Z_0^m = \frac{1}{\sqrt{2L}}(e^{-im\frac{\pi}{L}x}, 0, -e^{im\frac{\pi}{L}x}, 0, 0), \quad \forall m \in \mathbb{Z}.$$

Clearly, we have

$$||Z_0^m||^2 = 1, \quad \forall m \in \mathbb{Z}, \quad Z_0^m \overset{\text{weakly}}{\rightharpoonup} 0 \text{ in } (L^2)^4 \times (L^2)^n. \tag{3.22}$$

Now, using (3.21) and Lemma 3.2 yields

$$\begin{aligned} ||e^{(\Lambda\partial_x + M_0)t} Z_0^m||^2 &\leqslant 2||(e^{(\Lambda\partial_x + M_0)t} - e^{(\Lambda\partial_x + M)t}) Z_0^m||^2 + 2||e^{(\Lambda\partial_x + M)t} Z_0^m||^2 \\ &\leqslant 2||(e^{(\Lambda\partial_x + M_0)t} - e^{(\Lambda\partial_x + M)t}) Z_0^m||^2 \\ &\quad + 2C\left(d^2(t) + C\epsilon^{1/2} + \int_0^t (d^2(s) + C\epsilon^{1/2}) e^{-\frac{k(t-s)}{2}} ds\right) \\ &\quad \times E_g(\varphi, \psi)(0). \end{aligned} \tag{3.23}$$

But

$$\int_0^t (d^2(s) + C\epsilon^{1/2}) e^{-\frac{k(t-s)}{2}} ds = \frac{C}{2k}\epsilon^{1/2} + \int_0^t d^2(s) e^{-\frac{k(t-s)}{2}} ds.$$

Hence

$$\begin{aligned} ||e^{(\Lambda\partial_x + M_0)t} Z_0^m||^2 &\leqslant 2||(e^{(\Lambda\partial_x + M_0)t} - e^{(\Lambda\partial_x + M)t}) Z_0^m||^2 \\ &\quad + C\left(d^2(t) + C\epsilon^{1/2} + \int_0^t d^2(s) e^{-\frac{k(t-s)}{2}} ds\right) E_g(\varphi, \psi)(0). \end{aligned}$$

On the other hand, since Z_0^m is an eigenfunction of the operator $\Lambda\partial_x + M_0$ associated to the eigenvalue $im\sqrt{\frac{k}{\rho_1}}\frac{\pi}{L}$,

$$||e^{(\Lambda\partial_x + M_0)t} Z_0^m|| = ||e^{im\sqrt{\frac{k}{\rho_1}}\frac{\pi}{L}t} Z_0^m|| = 1 \quad \forall m \in \mathbb{Z}, \ \forall t \geqslant 0.$$

Now, we arrive at a contradiction, choosing ϵ sufficiently small and observing

1. $||(e^{(\Lambda\partial_x + M_0)t} - e^{(\Lambda\partial_x + M)t}) Z_0^m||$ as $|m| \to \infty$, since $e^{(\Lambda\partial_x + M_0)t} - e^{(\Lambda\partial_x + M)t}$ is a compact operator and (3.22) holds,
2. $d(t) \to 0$ as $t \to \infty$,
3. $\int_0^t d^2(s) e^{-\frac{k(t-s)}{2}} ds \to 0$ as $t \to \infty$. $\square$

4. Polynomial decay

Here we shall show the polynomial decay of the solution when the kernel g decays polynomially. More precisely, we use assumption (1.7) to prove the polynomial rate of decay of the first-order energy. The method used is essentially the same as in Section 2, but there exist some major points in some estimates which demand a different procedure. Therefore the proof has to be adapted to the case of polynomially decaying kernels, and we have to discuss the points that need a different argument. We follow the approach in [16] and shall prove the following theorem.

Theorem 4.1. *Let us suppose that the initial data satisfy*

$$\varphi_0, \psi_0 \in H_0^1((0,L)), \quad \varphi_1, \psi_1 \in L^2((0,L)),$$

and that the coefficients of system (1.1), (1.2) *satisfy* (1.5). *Moreover assume that the kernel g is of polynomial type satisfying* (1.7) *with $p>2$. Then the energy $E(t)$ decays polynomially as time tends to infinity, that is, there exists a positive constant C, being independent of the initial data, such that for $t \geqslant 0$:*

$$E(t) \leqslant \frac{C}{(1+t)^p} E(0).$$

For the proof we need the following versions of three lemmas from [16] (based on [12]) which we state for the sake of completeness with the short proofs.

Lemma 4.2. *Let m and h be integrable functions, and let $0 \leqslant r < 1$ and $q > 0$. Then, for $t \geqslant 0$:*

$$\int_0^t |m(t-\tau)\, h(\tau)|\, d\tau \leqslant \left(\int_0^t |m(t-\tau)|^{1+\frac{1-r}{q}} |h(\tau)|\, d\tau \right)^{\frac{q}{q+1}} \left(\int_0^t |m(t-\tau)|^r |h(\tau)|\, d\tau \right)^{\frac{1}{q+1}}.$$

Proof. Define

$$v(\tau) := |m(t-\tau)|^{1-\frac{r}{q+1}} |h(\tau)|^{\frac{q}{q+1}}, \quad w(\tau) := |m(t-\tau)|^{\frac{r}{q+1}} |h(\tau)|^{\frac{1}{q+1}}.$$

An application of Hölder's inequality with exponents

$$\delta = \frac{q}{q+1} \quad \text{for } v, \qquad \delta^* = q+1 \quad \text{for } w$$

gives the assertion of Lemma 4.2. $\square$

Lemma 4.3. *Let $p>1$, $0\leqslant r<1$, $t\geqslant 0$ and $z\in L^\infty((0,T),H^1((0,L)))$ for any $T>0$. Then we have for $r>0$:*

$$\int_0^L g\square z_x\,dx\leqslant 2\left(\int_0^t |g(\tau)|^r\,d\tau ||z||^2_{L^\infty((0,t),H^1((0,L)))}\right)^{\frac{1}{1+(1-r)p}}\left(\int_0^L |g|^{1+\frac{1}{p}}\square z_x\,dx\right)^{\frac{(1-r)p}{1+(1-r)p}},$$

and for $r=0$:

$$\int_0^L g\square z_x\,dx\leqslant 2\left(\int_0^t ||z_x(\tau,\cdot)||^2\,d\tau+t||z_x(t,\cdot)||^2\right)^{\frac{1}{p+1}}\left(\int_0^L |g|^{1+\frac{1}{p}}\square z_x\,dx\right)^{\frac{p}{p+1}}.$$

Proof. Apply Lemma 4.2 with $m(\tau):=|g(\tau)|$, $h(\tau):=\int_0^L |z_x(t)-z_x(\tau)|^2\,dx$ and $q:=(1-r)p$, for fixed t. This proves Lemma 4.3. $\square$

Lemma 4.4. *Let $f\geqslant 0$ be differentiable, let $\alpha>0$ and let f satisfy*

$$f'(t)\leqslant\frac{-\bar{c}_1}{f(0)^{1/\alpha}}f(t)^{1+\frac{1}{\alpha}}+\frac{\bar{c}_2}{(1+t)^\beta}f(0)$$

for $t\geqslant 0$, positive constants $\bar{c}_1,\bar{c}_2$ and

$$\beta\geqslant\alpha+1.$$

Then there exists a constant $\bar{c}_3>0$ such that for $t\geqslant 0$:

$$f(t)\leqslant\frac{\bar{c}_3}{(1+t)^\alpha}f(0).$$

Proof. Let $t\geqslant 0$ and

$$F(t):=f(t)+\frac{2\bar{c}_2}{\alpha}(1+t)^{-\alpha}f(0).$$

Using $\beta\geqslant\alpha+1$ we get

$$F'\leqslant\frac{-c}{f(0)^{1/\alpha}}\left(f^{1+\frac{1}{\alpha}}+(1+t)^{-(\alpha+1)}f(0)^{1+\frac{1}{\alpha}}\right)\leqslant\frac{-c}{F(0)^{1/\alpha}}F^{1+\frac{1}{\alpha}}.$$

Integration yields

$$F(t)\leqslant\frac{F(0)}{(1+ct)^\alpha}\leqslant\frac{c}{(1+t)^\alpha}f(0)$$

from where our conclusion follows. $\square$

Proof of Theorem 4.1. Lemma 4.3 yields

$$\int_0^L g\Box\psi_x\,dx \leqslant cE(0)^{\frac{1}{1+(1-r)p}}\left(\int_0^L g^{1+\frac{1}{p}}\Box\psi_x\,dx\right)^{\frac{(1-r)p}{1+(1-r)p}}, \tag{4.1}$$

for $0<r<1$ with $rp>1$. From the proof of Lemma 2.2 we get

$$\begin{aligned}-\frac{d}{dt}I(t) = &-\rho_2 g(0)\int_0^L |\psi_t|^2\,dx - \rho_2\int_0^L g'\psi_t\psi\,dx\\ &+\rho_2\int_0^L \psi_t(g''\diamond\psi)\,dx - k\int_0^L \varphi_x(g\psi - g'\diamond\psi)\,dx\\ &+\frac{b}{2}\int_0^L g|\psi_x|^2\,dx - \frac{b}{2}\int_0^L g'\Box\psi_x\,dx + \frac{k}{2}\int_0^L g|\psi|^2\,dx - \frac{k}{2}\int_0^L g'\Box\psi\,dx.\end{aligned}$$

Hypothesis (1.7) implies that

$$|g'(t)|\leqslant cg^{1+\frac{1}{p}}(t), \quad |g''(t)|\leqslant cg^{1+\frac{1}{p}}(t).$$

Therefore we have that

$$\rho_2\int_0^L \psi_t(g''\diamond\psi)\,dx \leqslant c\left(\int_0^L g^{1+\frac{1}{p}}\Box\psi\,dx\right)^{1/2}\left(\int_0^L |\psi_t|^2\,dx\right)^{1/2}.$$

Similarly

$$k\int_0^L \varphi_x(g'\diamond\psi)\,dx \leqslant c\left(\int_0^L g^{1+\frac{1}{p}}\Box\psi\,dx\right)^{1/2}\left(\int_0^L |\varphi_x|^2\,dx\right)^{1/2}.$$

Using these relations and Poincaré's inequality we have

$$\begin{aligned}-\frac{d}{dt}I(t)\leqslant &-\frac{1}{2}\rho_2 g(0)\int_0^L |\psi_t|^2\,dx + c_\varepsilon(|g'|+|g|)\int_0^L |\psi_x|^2\,dx + \varepsilon\int_0^L |\varphi_x|^2\,dx\\ &+ c_\varepsilon\int_0^L g^{1+\frac{1}{p}}\Box\psi_x\,dx.\end{aligned}$$

On the other hand, from the proof of Lemma 2.3 we have that

$$\begin{aligned}\frac{d}{dt}J_1(t) = &\rho_2\int_0^L |\psi_t|^2\,dx - \left(b-\int_0^t g\,d\tau\right)\int_0^L |\psi_x|^2\,dx - k\int_0^L |\psi|^2\,dx\\ &+ k\int_0^L |w_x|^2\,dx + \rho_1\int_0^L \varphi_t w_t\,dx - \int_0^L (g\diamond\psi_x)\psi_x\,dx.\end{aligned} \tag{4.2}$$

Since

$$g \diamondsuit \psi_x(x,t) \leqslant \left(\int_0^t g^{1-\frac{1}{p}}\, ds\right)^{1/2} \left\{ g^{1+\frac{1}{p}} \Box \psi_x \right\}^{1/2}$$

which implies

$$\left| \int_0^L (g \diamondsuit \psi_x)\psi_x\, dx \right| \leqslant C_\delta \int_0^L g^{1+\frac{1}{p}} \Box \psi_x\, dx + \delta \int_0^L |\psi_x|^2\, dx$$

identity (4.2) can be rewritten as

$$\begin{aligned} \frac{d}{dt} J_1(t) \leqslant C_{\epsilon_1} \int_0^L |\psi_t|^2\, dx - \frac{\lambda}{2} \int_0^L |\psi_x|^2\, dx \\ + C_{\epsilon_1} \int_0^L g^{1+\frac{1}{p}} \Box \psi_x\, dx + \epsilon_1 \int_0^L |\varphi_t|^2\, dx. \end{aligned} \tag{4.3}$$

As in Section 2 we consider

$$\mathscr{E}_1(t) = N_1 E(t) - N_2 I(t) + N_3 J_1(t).$$

Let

$$\mathscr{N}_p(t) := \left\{ \int_0^L |\psi_t|^2 + \left(b - \int_0^t g\, ds\right)|\psi_x|^2 + g^{1+\frac{1}{p}} \Box \psi_x\, dx \right\}.$$

From the inequalities above we conclude that

$$\frac{d}{dt}\mathscr{E}_1(t) \leqslant -\frac{N_2^{\epsilon_2}}{2}\mathscr{N}_p(t) - \frac{N_1^{\epsilon_2}}{2}\int_0^L g|\psi_x|^2\, dx + \epsilon_2 \int_0^L (|\varphi_t|^2 + |\varphi_x|^2)\, dx. \tag{4.4}$$

Using the same reasoning as above we can show that Lemmas 2.4 and 2.5 imply

$$\begin{aligned} \frac{d}{dt}K(t) \leqslant{}& [(b\psi_x - g * \psi_x)\varphi_x]_{x=0}^{x=L} - k\int_0^L |\varphi_x + \psi|^2\, dx \\ &+ \epsilon \int_0^L |\varphi_t|^2\, dx + C_\epsilon \int_0^L g^{1+\frac{1}{p}} \Box \psi_x + g|\psi_x|^2\, dx + \rho_2 \int_0^L |\psi_t|^2\, dx, \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\int_0^L \rho_2 \psi_t q(b\psi_x - g * \psi_x)\, dx \leqslant{}& -\gamma\{|b\psi_x(L,t) - (g * \psi_x)(L,t)|^2 \\ &+ |b\psi_x(0,t) - (g * \psi_x)(0,t)|^2\} + \tilde{\epsilon}\int_0^L |\varphi_x|^2\, dx + C_{\tilde{\epsilon}} \\ &+ C_\epsilon \int_0^L |\psi_t|^2 + \left(b - \int_0^t g\, ds\right)|\psi_x|^2 \\ &+ g^{1+\frac{1}{p}} \Box \psi_x\, dx\, dx, \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\int_0^L \rho_1 \varphi_t q \varphi_x\, dx \leqslant{}& -k\gamma\{|\varphi_x(L,t)|^2 + |\varphi_x(0,t)|^2\} \\ &+ C_1 \int_0^L |\varphi_t|^2 + |\varphi_x|^2 + |\psi_x|^2\, dx. \end{aligned}$$

Denoting by $L(t)$ again the functional

$$L(t) = K(t) + N_3 \int_0^L \rho_2 \psi_t q(b\psi_x - g * \psi_x)\,dx + \delta \int_0^L \rho_1 \varphi_t q \varphi_x \,dx,$$

we get

$$\frac{d}{dt} L(t) \leqslant -\frac{k}{2} \int_0^L |\varphi_x + \psi|^2\,dx + C_2\tau \int_0^L |\varphi_t|^2\,dx + + C_\tau \mathcal{N}_p(t). \tag{4.5}$$

Finally, the functional J_2 defined in Section 2 satisfies

$$-\frac{d}{dt} J_2(t) \leqslant -\rho_1 \int_0^L |\varphi_t|^2\,dx - \rho_2 \int_0^L |\psi_t|^2\,dx + k \int_0^L |\varphi_x + \psi|^2\,dx + C_\tau \mathcal{N}_p(t). \tag{4.6}$$

From inequalities (4.5) and (4.6) we get

$$\frac{d}{dt}\left\{ L(t) - \frac{2C_2\tau}{\rho_1} J_2(t) \right\} \leqslant -\frac{k}{4} \int_0^L |\varphi_x + \psi|^2\,dx - C_2\tau \int_0^L |\varphi_t|^2\,dx + C_\tau \mathcal{N}_p(t).$$

Now using again the functional

$$\mathcal{L}(t) = \mathcal{E}_1(t) + L(t) - \frac{2C_2\tau}{\rho_1} J_2(t)$$

it is not difficult to see that

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\beta_0 \left\{ \frac{1}{2} \int_0^L \rho_1 |\varphi_t|^2 + \rho_2 |\psi_t|^2 + |\psi_x|^2 + k|\varphi_x + \psi|^2 + g^{1+\frac{1}{p}} \Box \psi_x \,dx \right\}.$$

Let us denote by $\mathcal{E}_0(t)$ the functional

$$\mathcal{E}_0(t) := \frac{1}{2} \int_0^L \rho_1 |\varphi_t|^2 + \rho_2 |\psi_t|^2 + |\psi_x|^2 + k|\varphi_x + \psi|^2\,dx.$$

Since the energy is bounded, Lemma 4.3 implies

$$\mathcal{E}_0(t) \geqslant c\mathcal{E}_0(t)^{\frac{1+(1-r)p}{(1-r)p}} E(0)^{\frac{-1}{(1-r)p}},$$

$$\int_0^L g^{1+\frac{1}{p}} \Box \psi_x \,dx \geqslant c \left\{ \int_0^L g \Box \psi_x \,dx \right\}^{\frac{1+(1-r)p}{(1-r)p}} E(0)^{\frac{-1}{(1-r)p}}.$$

Observing that $\mathcal{L}$ satisfies (cf. Section 2)

$$\beta_1 E(t) \leqslant \mathcal{L}(t) \leqslant \beta_3 \left\{ \mathcal{E}_0(t) + \int_0^L g^{1+\frac{1}{p}} \Box \psi_x \,dx \right\}^{\frac{(1-r)p}{1+(1-r)p}} E(0)^{\frac{1}{1+(1-r)p}} \tag{4.7}$$

with some $\beta_3 > 0$, it follows that

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -c\mathcal{L}(t)^{\frac{1+(1-r)p}{(1-r)p}} \mathcal{L}(0)^{\frac{-1}{(1-r)p}}$$

and hence, by Lemma 4.4,

$$\mathcal{L}(t) \leqslant C \frac{1}{(1+t)^{(1-r)p}} \mathcal{L}(0).$$

This implies for $r>0$

$$\int_0^t ||\psi(\tau)||^2_{H^1((0,L))}\, d\tau \leqslant c \int_0^t \mathscr{L}(\tau)\, d\tau \leqslant \int_0^t \frac{1}{(1+\tau)^{(1-r)p}}\, d\tau\, \mathscr{L}(0) \leqslant c\mathscr{L}(0)$$

provided $(1-r)p>1$ which together with the previous condition $rp>1$ can be satisfied since $p>2$. Moreover, we conclude

$$t||\psi(\cdot,t)||^2_{H^1((0,L))} \leqslant ct\mathscr{L}(t) \leqslant c\mathscr{L}(0)$$

and hence, using Lemma 4.3 now with $r=0$,

$$\mathscr{E}_0(t) \geqslant c\mathscr{E}_0(t)^{\frac{1+p}{p}} E(0)^{\frac{-1}{p}}$$

and

$$\int_0^L g\Box\psi_x\, dx \geqslant c\left\{\int_0^L g\Box\psi_x\, dx\right\}^{\frac{1+p}{p}} E(0)^{\frac{-1}{p}}.$$

Repeating the same reasoning as before we now get

$$\frac{d}{dt}\mathscr{L}(t) \leqslant -c\mathscr{L}(t)^{1+\frac{1}{p}}\mathscr{L}(0)^{\frac{-1}{p}}$$

which implies by Lemma 4.4

$$\mathscr{L}(t) \leqslant \frac{C}{(1+t)^p}\mathscr{L}(0)$$

from where our result follows. The proof is now complete. □

5. Decay rate optimality

Already for the system of (magneto-thermo-) elasticity with memory type boundary conditions it was shown in [16] that a merely polynomial kernel cannot lead to an exponential decay result for the energy in general. In a similar manner, we are now able to prove that the decay rate for polynomial kernels cannot be of exponential type.

We take the kernel

$$g(t) = \frac{1}{(1+t)^p}$$

for some $p>1$.

For the initial data we assume

$$\psi_0 = 0, \quad \varphi_0, \varphi_1, \psi_1 \in C_0^\infty((0,L)), \quad \int_0^L \psi_1\, dx \neq 0. \tag{5.1}$$

Then we shall demonstrate that assumption of exponential decay,

$$\exists c > 0 \ \exists \delta > 0 \ \forall t \geqslant 0: \quad E(t) \leqslant c e^{-\delta t} E(0) \tag{5.2}$$

leads to a contradiction. With the choice of the initial conditions as in (5.1), $(v, w) := (\varphi_t, \psi_t)$ satisfies the same differential equations and boundary conditions as (φ, ψ). Hence also the energy associated to (v, w) decays exponentially, which implies, using the differential equation, that there is a constant c_0 depending on the initial data such that for all $t \geqslant 0$:

$$\left| \int_0^L (g * \psi_{xx}(x, \cdot))(t) \, dx \right| \leqslant c_0 e^{-\delta t/2}$$

which is equivalent to

$$\left| \int_0^t \frac{1}{(1+t-s)^p} \underbrace{(\psi_x(L, s) - \psi_x(0, s))}_{=:h(s)} \, ds \right| \leqslant c_0 e^{-\delta t/2}. \tag{5.3}$$

On the other hand, since h decays exponentially by Sobolev's imbedding theorem and the assumption on exponential decay of the energy (applied to (φ_t, ψ_t)), it can be easily seen that for any $m > 1$

$$\left| \int_0^t \frac{1}{(1+t-s)^m} h(s) \, ds \right| \leqslant \frac{c_m}{(1+t)^m} \tag{5.4}$$

for some constant c_m. For $t \geqslant 0$ and $\beta \geqslant 0$ let

$$G_\beta(t) := \int_{t+\beta}^\infty h(s) \, ds.$$

Then

$$\int_0^t \frac{1}{(1+t-s)^p} h(s) \, ds = \frac{G_\beta(0)}{(1+t)^p} - G_\beta(t) + \mathcal{O}\left(\frac{1}{(1+t)^{p+1}} \right), \tag{5.5}$$

where we used (5.4) for $m = p + 1$.

Case 1: $\exists \tilde{\beta} \in [0, \infty]$: $G_{\tilde{\beta}}(0) \neq 0$. Thus, from (5.5),

$$\lim_{t \to \infty} \left| \int_0^t \frac{1}{(1+t-s)^p} h(s) \, ds \right| (1+t)^p = G_{\tilde{\beta}}(0) \neq 0$$

which is a contradiction to (5.3).

Case 2: $\forall \beta \in [0, \infty]$: $G_\beta(0) = 0$. This implies

$$\forall t \geqslant 0: \quad h(t) = 0$$

which means that ψ satisfies additionally the boundary conditions

$$\psi_x(L,t) = \psi_x(0,t), \quad t \geqslant 0.$$

Using this we conclude after integration of both sides of the differential equation (1.2) that

$$\rho_2 \frac{d^2}{dt^2} \int_0^L \psi(x,t)\,dx + k \int_0^L \psi(x,t)\,dx = 0$$

which implies, for $v := \sqrt{k/\rho_2}$,

$$\int_0^L \psi(x,t)\,dx = \int_0^L \psi_1(x)\,dx \sin(vt)$$

which is a contradiction to the assumption of exponential decay of the energy.

Remark. After the submission of our paper there appeared a paper by Fabrizio and Polidoro [6] where, for a special class of integro-differential equations, the question of optimality is analyzed in great generality.

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