Stability of coupled second order equations.

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1 Introduction

In [13], D. L. Russell considered a linear oscillator in a Hilbert space H represented in the form

$$u''(t) + \mathcal{A}u(t) = h \tag{1}$$

where $\mathcal{A}$ is a (generally unbounded) positive self-adjoint operator on H. He proposed to introduce "certain *indirect damping* mechanisms which arise, not from insertion of damping terms into the original equations describing the mechanical motion, but by coupling those equations to further equations describing other processes in the structure..." (see [13, p. 340]). In the same work, two types of *indirect damping* were described: the velocity coupled dissipator and the displacement coupled dissipator. Works on *indirect damping* mechanisms of the first type leading to exponential decay of the total energy may be found for instance in [1], [2], [3], [9] and of course in [13]. In this paper, we focus our attention on the second type of *indirect damping* mechanisms. Let's recall the description given in [13]. It consists in considering a system with displacement vector (w, z), velocity (w', z') and energy form:

$$E(w,z)(t) = \frac{1}{2}\left(< \begin{pmatrix} w \\ z \end{pmatrix}, S \begin{pmatrix} w \\ z \end{pmatrix} >_{H\times G} + \| w' \|_H^2 + \| z' \|_G^2 \right) \tag{2}$$

where G is a second Hilbert space and S is a positive self-adjoint operator on $H \times G$ representable in operator matrix form as

$$S = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B}^* & \mathcal{C} \end{pmatrix} \tag{3}$$

The energy $E(w, z)(t)$ is conserved for the second order system:

$$\begin{pmatrix} w'' \\ z'' \end{pmatrix} + S \begin{pmatrix} w \\ z \end{pmatrix} = 0$$

Thus, damping is introduced in the second equation of the system:

$$\begin{pmatrix} w'' \\ z'' \end{pmatrix} + S \begin{pmatrix} w \\ z \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ z' \end{pmatrix} = 0$$

At this step, D. L. Russell supposes that inertial forces in the "z" system are small in comparison with the damping and, then, z'' is discarded. In our work, we do not adopt this last assumption. Moreover, we replace the constant γ by an (eventually unbounded) positive self-adjoint operator D acting on G:

$$\begin{pmatrix} w'' \\ z'' \end{pmatrix} + S \begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} 0 \\ Dz' \end{pmatrix} = 0$$

Our problem is then to find, among all these damping mechanisms, those for which the energy of the resulting system has an exponential decay to zero (D. L. Russell was interested by the analyticity of the associated semigroup).

Let us note that these equations are coupled through displacements. However, under suitable assumptions, they may be transformed into two other equations coupled through velocities in such a way that these two systems are "equivalent". Indeed, we transform our system by using the following operator (introduced in [5]):

$$P = \begin{pmatrix} 0 & \mathcal{A}^{-1/2} & -\mathcal{A}^{-1}\mathcal{B} & 0 \\ -\mathcal{A}^{1/2} & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

and then

$$P^{-1} = \begin{pmatrix} 0 & -\mathcal{A}^{-1/2} & 0 & 0 \\ \mathcal{A}^{1/2} & 0 & \mathcal{A}^{-1/2}\mathcal{B} & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

Setting

$$(w, w', z, z') = P(u, u', v, v')$$

one gets the following system

$$\begin{cases} u''(t) = & -Au(t) + Bv'(t) \\ v''(t) = & -B^*u'(t) - Cv(t) - Dv'(t) \end{cases} \tag{4}$$

where $A = \mathcal{A}$, $B = -\mathcal{A}^{-1/2}\mathcal{B}$ and $C = \mathcal{C} - B^*B$. Of course, C will be a positive self-adjoint operator under suitable assumptions on the entries of S (see [5] for precise assumptions). We will study this last system.

Russell's point of view is closely related to the one of automaticians. The second author adopted this last point of view in previous works (see [1], [2]). Dissipation mechanisms which are refered to as *indirect damping* are often called *dynamical stabilizers* while *direct dampings* correspond to *static stabilizers* (see for instance [7], [6], [10]...).

Our last remark is that, from the mathematical point of view, the well-posedness of the system we consider leads to problems on matrix operators like, for example, to find conditions on the entries which allow to prove closedness, to study the spectrum or the generation property of semigroups (see for instance [11], [4]).

The paper is organized as follows. In the second section, we give some remarks related to the closure of the matrix operator associated to System (4) and to the well-posedness of the associated abstract Cauchy problem. The third section is devoted to (uniform) stability results and, in the fourth and last section, some examples are considered in which a description of the spectrum is given.

2 Well-posedness

Let H and G be two Hilbert spaces. We set

$$S = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}; \quad D(S) = D(A) \times D(C) \subset H \times G, \tag{5}$$

where A (resp. C) is a positive self-adjoint operator on H (resp. G), and

$$T = \begin{pmatrix} 0 & B \\ -B^* & -D \end{pmatrix}; \quad D(T) = D(B^*) \times (D(B) \cap D(D)) \tag{6}$$

where B is a densely defined operator from G to H and D is a positive self-adjoint operator on G. System (4) may be written

$$w''(t) = -Sw(t) + Tw'(t) \qquad t > 0 \tag{7}$$

with $w = (u, v)$ or in matrix form

$$y'(t) = \mathcal{A}_T\, y(t) \quad t > 0 \tag{8}$$

with $y = (w, w')$ and

$$\mathcal{A}_T = \begin{pmatrix} 0 & I \\ -S & T \end{pmatrix}; \quad D(\mathcal{A}_T) = D(S) \times (D(S^{1/2}) \cap D(T)) \tag{9}$$

is defined on the Hilbert space $X = D(S^{1/2}) \times Y$ where $Y = H \times G$. In view of the study of the well-posedness of the abstract Cauchy problem associated to (8),we have the following general propositions

Proposition 1 *Suppose that* $\overline{D(\mathcal{A}_T)} = X$ *and* $S^{-\frac{1}{2}}TS^{-\frac{1}{2}} \in L(Y)$ *(or extends to a bounded operator). Then*

$$\begin{aligned} \mathcal{A}_T^* &= \begin{pmatrix} 0 & I \\ -S & T \end{pmatrix}^* (u, v) = \Big(-v, S(u + S^{-1}T^*v)\Big) \\ D(\mathcal{A}_T^*) &= \Big\{(u, v) \in D(S^{\frac{1}{2}}) \times D(S^{\frac{1}{2}}),\ u + S^{-1}T^*v \in D(S)\Big\} \end{aligned} \tag{10}$$

Proof. Let $(u, v) \in D(\mathcal{A}_T)$ and $(f, g) \in X$. Then

$$\begin{aligned} \langle \mathcal{A}_T(u, v), (f, g)\rangle &= \left\langle S^{\frac{1}{2}}v, S^{\frac{1}{2}}f \right\rangle + \langle -Su + Tv, g\rangle \\ &= \langle -Su, g\rangle + \left\langle S^{\frac{1}{2}}v, S^{\frac{1}{2}}f \right\rangle + \langle Tv, g\rangle \end{aligned}$$

Now, necessarily g is in $D(S^{\frac{1}{2}})$ (take $(u, 0) \in D(\mathcal{A}_T)$). Thus, thanks to the boundedness of $S^{-\frac{1}{2}}TS^{-\frac{1}{2}}$, one has

$$\begin{aligned} \langle Tv, g\rangle &= \left\langle (S^{-\frac{1}{2}}TS^{-\frac{1}{2}})S^{\frac{1}{2}}v, S^{\frac{1}{2}}g \right\rangle \\ &= \left\langle S^{\frac{1}{2}}v, (S^{-\frac{1}{2}}TS^{-\frac{1}{2}})^* S^{\frac{1}{2}}g \right\rangle \\ &= \left\langle S^{\frac{1}{2}}v, S^{-\frac{1}{2}}T^*g \right\rangle \end{aligned}$$

So, we have

$$\begin{aligned} \left\langle S^{\frac{1}{2}}v, S^{\frac{1}{2}}f \right\rangle + \langle Tv, g\rangle &= \left\langle S^{\frac{1}{2}}v, S^{\frac{1}{2}}f + S^{-\frac{1}{2}}T^*g \right\rangle \\ &= \left\langle S^{\frac{1}{2}}v, S^{\frac{1}{2}}(f + S^{-1}T^*g) \right\rangle \\ &= \left\langle v, S(f + S^{-1}T^*g) \right\rangle \end{aligned}$$

with, of course $f + S^{-1}T^*g \in D(S)$. This proves the assertion of the proposition. ■

Proposition 2 *Under the assumptions of Proposition 1, if T and $\mathcal{A}_T$ are closable then*

$$\begin{aligned} \overline{\mathcal{A}_T}(u,v) &= \left(v, S(-u + S^{-1}\overline{T}v)\right) \\ D(\overline{\mathcal{A}_T}) &= \left\{(u,v) \in D(S^{\frac{1}{2}}) \times D(S^{\frac{1}{2}}),\ -u + S^{-1}\overline{T}v \in D(S)\right\} \end{aligned} \tag{11}$$

Proof. Since $\mathcal{A}_T$ is closable, its closure is given by $\overline{\mathcal{A}_T} = (\mathcal{A}_T^*)^*$. We proceed as in the proof of Proposition 1. Let $(u,v) \in D(\mathcal{A}_T^*)$ and $(f,g) \in X$,

$$\begin{aligned} \langle \mathcal{A}_T^*(u,v), (f,g)\rangle &= \left\langle -S^{\frac{1}{2}}v, S^{\frac{1}{2}}f \right\rangle + \left\langle S(u + S^{-1}T^*v), g \right\rangle \\ &= \left\langle -S^{\frac{1}{2}}v, S^{\frac{1}{2}}f \right\rangle + \left\langle S^{\frac{1}{2}}(S^{\frac{1}{2}}u + S^{-\frac{1}{2}}T^*v), g \right\rangle \end{aligned}$$

Now, if $g \in D(S^{\frac{1}{2}})$, then

$$\begin{aligned} \langle \mathcal{A}_T^*(u,v), (f,g)\rangle &= \left\langle -S^{\frac{1}{2}}v, S^{\frac{1}{2}}f \right\rangle + \left\langle S^{\frac{1}{2}}u + S^{-\frac{1}{2}}T^*v, S^{\frac{1}{2}}g \right\rangle \\ &= \left\langle S^{\frac{1}{2}}u, S^{\frac{1}{2}}g \right\rangle + \left\langle S^{\frac{1}{2}}v, -S^{\frac{1}{2}}f + S^{-\frac{1}{2}}\overline{T}g \right\rangle \\ &= \left\langle S^{\frac{1}{2}}u, S^{\frac{1}{2}}g \right\rangle + \left\langle v, S^{\frac{1}{2}}\left(-S^{\frac{1}{2}}f + S^{-\frac{1}{2}}\overline{T}g\right) \right\rangle \end{aligned}$$

if $-S^{\frac{1}{2}}f + S^{-\frac{1}{2}}Tg \in D(S^{\frac{1}{2}})$ or equivalently if $-f + S^{-1}\overline{T}g \in D(S)$. Let us define on X, the operator

$$\begin{aligned} \breve{A}(u,v) &= \left(v, S(-f + S^{-1}\overline{T}g)\right) \\ D(\breve{A}) &= \left\{(u,v) \in D(S^{\frac{1}{2}}) \times D(S^{\frac{1}{2}}),\ -u + S^{-1}\overline{T}v \in D(S)\right\} \end{aligned}$$

From the previous computations, it follows that $\overline{\mathcal{A}_T}$ is an extension of $\breve{A}$. On the other hand, it is easily seen that $D(\mathcal{A}_T) \subset D(\breve{A})$ and that $\breve{A} = \mathcal{A}_T$ on $D(\mathcal{A}_T)$. If $\breve{A}$ is closed, it will follow from the definition of $\overline{\mathcal{A}_T}$ that $\breve{A} = \overline{\mathcal{A}_T}$. Let $(u_n, v_n) \in D(\breve{A})$ be a sequence such that

$$(u_n, v_n) \underset{n\to\infty}{\to} (u, v) \tag{12}$$

$$\breve{A}(u_n, v_n) \underset{n\to\infty}{\to} (f, g) \tag{13}$$

in $D(S^{\frac{1}{2}}) \times H$. It follows first that

$$v_n \underset{n\to\infty}{\to} v \quad \text{and} \quad S^{\frac{1}{2}} v_n \underset{n\to\infty}{\to} S^{\frac{1}{2}} f$$

and thus $v = f \in D(S^{\frac{1}{2}})$. We also have, using again the boundedness of $S^{-\frac{1}{2}} T S^{-\frac{1}{2}}$

$$S^{\frac{1}{2}} \left(-S^{\frac{1}{2}} u_n + S^{-\frac{1}{2}} \overline{T} v_n \right) \underset{n\to\infty}{\to} g$$

$$-S^{\frac{1}{2}} u_n + S^{-\frac{1}{2}} \overline{T} v_n \underset{n\to\infty}{\to} -S^{\frac{1}{2}} u + S^{-\frac{1}{2}} T v$$

Since $S^{\frac{1}{2}}$ is closed, $-S^{\frac{1}{2}} u + S^{-\frac{1}{2}} \overline{T} v \in D(S^{\frac{1}{2}})$ and $g = S^{\frac{1}{2}} \left(-S^{\frac{1}{2}} u + S^{-\frac{1}{2}} \overline{T} v \right)$, proving the closedness of $\breve{A}$ and the assertion of this lemma. ■

As a consequence, one has

Corollary 3 *With the assumptions of Proposition 2, if moreover T is dissipative, then $\overline{\mathcal{A}_T}$ is m-dissipative.*

Proof. If T is dissipative then so is $\mathcal{A}_T$ and, consequently, $\overline{\mathcal{A}_T}$. On the other hand $\overline{\mathcal{A}_T}$ is boundedly invertible as it may be seen by using Proposition 2 and

$$\overline{\mathcal{A}_T}^{-1} = \begin{pmatrix} S^{-1}\overline{T} & -S^{-1} \\ I & 0 \end{pmatrix}$$

Thus $\overline{\mathcal{A}_T}$ is m-dissipative. ■

Let's now return to our operators as defined in (5), (6) and (9). The assumptions in propositions 1 and 2 are interpreted as follows. $\overline{D(\mathcal{A}_T)} = X$ if and only if

$$\overline{D(S^{1/2}) \cap D(T)} = Y \tag{14}$$

which amounts to

$$\overline{D(A^{1/2}) \cap D(B^*)} \quad = \quad H; \tag{15}$$

$$\overline{D(C^{1/2}) \cap D(B) \cap D(D)} \quad = \quad G \tag{16}$$

Since

$$S^{-\frac{1}{2}} T S^{-\frac{1}{2}} = \begin{pmatrix} 0 & A^{-1/2} B C^{-1/2} \\ -C^{-1/2} B^* A^{-1/2} & -C^{-1/2} D C^{-1/2} \end{pmatrix}$$

the second assumption in Proposition 1 is satisfied if

$$A^{-1/2} B C^{-1/2} \quad \in \quad L(G, H) \tag{17}$$

$$C^{-1/2} D C^{-1/2} \quad \in \quad L(G) \tag{18}$$

By (15) and (16), T and $\mathcal{A}_T$ are densely defined and a simple computation shows that they are both dissipative. Thus, they are both closable and the assumptions in Proposition 2 are also satisfied. As a result, Corollary 3 implies that $\overline{\mathcal{A}_T}$ is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t \geq 0}$. We summarize all these remarks in the

Theorem 4 *Under the assumptions (15), (16), (17) and (18), $\mathcal{A}_T$ defined in (5), (9) and (6) is closable and its closure $\overline{\mathcal{A}_T}$ is the infinitesimal generator of a C_0-semigroup of contractions $(S(t))_{t \geq 0}$.*

Moreover, $\overline{\mathcal{A}_T}$ is given by (11) and $\mathcal{A}_T^*$ by (10).

Remark 1 *This result is probably known but we were unable to find it in the litterature. The nearest result we know is the one proposed by K.-J. Engel [8] where T is assumed to be m-dissipative, invertible and satisfying*

$$\operatorname{Re}(Tu, S^{-1}u) \leq 0 \quad u \in D(T); \qquad \operatorname{Re}(T^*u, S^{-1}u) \leq 0 \qquad u \in D(T^*)$$

Remark 2 *The precise expression of $\overline{\mathcal{A}_T}$ and of its domain allows to know where do the solutions of (8) "live" when we start with an initial data in $D(\mathcal{A}_T)$. This was a question asked by D. L. Russell in [13, Remarks, p. 345].*

3 Uniform stability.

Our main result is the following:

Theorem 5 *Let the assumptions of Theorem 4 hold and assume moreover that:*

(i) $D(C^{\frac{1}{2}}) \subset D(D^{\frac{1}{2}})$ *and* D *is boundedly invertible,*

(ii) B *is boundedly invertible and the operators* $D^{-\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$, $D^{-\frac{1}{2}}CB^{-1}A^{-\frac{1}{2}}$, $D^{\frac{1}{2}}B^{-1}$ *and* $A^{-\frac{1}{2}}BD^{-\frac{1}{2}}$ *all extend to bounded operators.*

Then $(S(t))_{t\geq 0}$ *is uniformly stable: there exist* $M \geq 1$ *and* $\omega > 0$ *such that*

$$\| S(t) \| \leq Me^{-\omega t} \qquad t \geq 0$$

Proof. . Let $\varepsilon > 0$ a real number which will be precisely chosen later and $y = ((u,v),(u_t,v_t))$ a solution of (8) corresponding to an initial data $y_0 = ((u_0,v_0),(u_1,v_1)) \in D(\mathcal{A}_T)$. We introduce the function

$$\begin{aligned} \rho_\varepsilon(y)(t) \quad = \quad & \frac{1}{2} \| y(t) \|^2 + \varepsilon \operatorname{Re}(\frac{1}{2}(u_t,u) + \frac{1}{4} \| D^{\frac{1}{2}}v \|^2 + \frac{1}{2}(v_t,v) + \\ & \frac{1}{2}(u,Bv) + (v_t, B^{-1}u_t) + (Cv, B^{-1}u) \end{aligned} \tag{19}$$

defined for all $t \geq 0$. We begin by proving that

Lemma 6 *For* $\varepsilon > 0$ *sufficiently small, there exist* $M_\varepsilon > 0$ *and* $N_\varepsilon > 0$ *such that*

$$M_\varepsilon \| y(t) \|^2 \leq \rho_\varepsilon(y(t)) \leq N_\varepsilon \| y(t) \|^2 \qquad \forall t \geq 0 \tag{20}$$

y *being a solution of (8) corresponding to an initial data* $y_0 \in D(\mathcal{A}_T)$.

Proof. By using Cauchy-Schwartz inequality and the assumptions on A and C, one gets (η will denote various positive constants)

$$| \frac{1}{2}(u_t,u) + \frac{1}{2}(v_t,v) | \leq \eta \| y \|^2$$

On the other hand

$$| (u,Bv) | = | (A^{\frac{1}{2}}u, A^{-\frac{1}{2}}BC^{-\frac{1}{2}}C^{\frac{1}{2}}v) |$$

From assumption (17), $A^{-\frac{1}{2}}BC^{-\frac{1}{2}}$ is bounded and thus

$$| (u, Bv) | \leq \eta \parallel y \parallel^2$$

In the same way, since B^{-1} is bounded

$$| (v_t, B^{-1}u_t) | \leq \eta \parallel y \parallel^2$$

Since $P := D^{-\frac{1}{2}}CB^{-1}A^{-\frac{1}{2}}$ is bounded (assumption (ii)), it follows that $C^{-\frac{1}{2}}D^{\frac{1}{2}}P = C^{\frac{1}{2}}B^{-1}A^{-\frac{1}{2}}$ is bounded and

$$| (Cv, B^{-1}u) | = | (C^{\frac{1}{2}}u, C^{\frac{1}{2}}B^{-1}A^{-\frac{1}{2}}A^{\frac{1}{2}}v) | \leq \eta \parallel y \parallel^2$$

Last, from assumption (i),

$$\parallel D^{\frac{1}{2}}v \parallel^2 \leq \alpha \parallel C^{\frac{1}{2}}v \parallel^2$$

All of these estimates imply

$$(\frac{1}{2} - \varepsilon\eta) \parallel y(t) \parallel^2 \leq \rho_\varepsilon(y(t)) \leq (\frac{1}{2} + \varepsilon\eta) \parallel y(t) \parallel^2$$

and this proves the lemma. ■

For $\rho_\varepsilon(y(t))$, we have the following differential inequality

Lemma 7 *For $\varepsilon > 0$ sufficiently small, there exists $\omega_\varepsilon > 0$ such that*

$$\frac{d}{dt}\rho_\varepsilon(y(t)) \leq -\omega_\varepsilon \rho_\varepsilon(y(t)) \qquad t > 0 \tag{21}$$

Proof. One can readily see that

$$\begin{aligned} \tfrac{d}{dt}\rho_\varepsilon(y(t)) = \;& - \parallel D^{\frac{1}{2}}v_t \parallel^2 + \varepsilon \operatorname{Re}(-\tfrac{1}{2} \parallel u_t \parallel^2 - \tfrac{1}{2} \parallel A^{\frac{1}{2}}u \parallel^2 + \\ & \tfrac{3}{2} \parallel v_t \parallel^2 - \tfrac{1}{2} \parallel C^{\frac{1}{2}}v \parallel^2 - (v_t, B^{-1}Au) + \\ & (Cv_t, B^{-1}u) - (Dv_t, B^{-1}u_t) + (Bv_t, u)). \end{aligned} \tag{22}$$

From assumption (ii), one derives the following estimates

$$| (v_t, B^{-1}Au) | \leq \frac{1}{2} \parallel D^{-\frac{1}{2}}B^{-1}A^{\frac{1}{2}} \parallel (\frac{1}{\alpha} \parallel D^{\frac{1}{2}}v_t \parallel^2 + \alpha \parallel A^{\frac{1}{2}}u \parallel^2)$$

$$| (Cv_t, B^{-1}u) | \leq \frac{1}{2} \parallel D^{-\frac{1}{2}}CB^{-1}A^{-\frac{1}{2}} \parallel (\frac{1}{\alpha} \parallel D^{\frac{1}{2}}v_t \parallel^2 + \alpha \parallel A^{\frac{1}{2}}u \parallel^2)$$

$$| (Dv_t, B^{-1}u_t) | \leq \frac{1}{2} \parallel D^{\frac{1}{2}}B^{-1} \parallel (\frac{1}{\alpha} \parallel D^{\frac{1}{2}}v_t \parallel^2 + \alpha \parallel u_t \parallel^2)$$

$$| (Bv_t, u) | \leq \frac{1}{2} \parallel A^{-\frac{1}{2}}BD^{-\frac{1}{2}} \parallel (\frac{1}{\alpha} \parallel D^{\frac{1}{2}}v_t \parallel^2 + \alpha \parallel A^{\frac{1}{2}}u \parallel^2)$$

where α is a positive constant which will be chosen later.

By inserting these inequalities in (22), one gets

$$\begin{aligned} \tfrac{d}{dt}\rho_\varepsilon(y(t)) \leq \ & -(1 - \tfrac{a}{\alpha}\varepsilon) \parallel D^{\frac{1}{2}}v_t \parallel^2 + \varepsilon(-(\tfrac{1}{2} - a\alpha) \parallel u_t \parallel^2 - (\tfrac{1}{2} - a\alpha) \parallel A^{\frac{1}{2}}u \parallel^2 \\ & + \tfrac{3}{2} \parallel v_t \parallel^2 - \tfrac{1}{2} \parallel C^{\frac{1}{2}}v \parallel^2). \end{aligned}$$

Now, using the invertibility of D,

$$\begin{aligned} \tfrac{d}{dt}\rho_\varepsilon(y(t)) \leq \ & -(\tfrac{1}{2} - a\alpha)\varepsilon \parallel u_t \parallel^2 - (\tfrac{1}{2} - a\alpha)\varepsilon \parallel A^{\frac{1}{2}}u \parallel^2 \\ & -(1 - (\tfrac{a}{\alpha} + \tfrac{3}{2b})\varepsilon) \parallel D^{\frac{1}{2}}v_t \parallel^2 - \tfrac{\varepsilon}{2} \parallel C^{\frac{1}{2}}v \parallel^2). \end{aligned} \tag{23}$$

with

$$a = \frac{1}{2}\max(\parallel D^{-\frac{1}{2}}B^{-1}A^{\frac{1}{2}} \parallel, \parallel D^{-\frac{1}{2}}CB^{-1}A^{-\frac{1}{2}} \parallel, \parallel D^{\frac{1}{2}}B^{-1} \parallel, \parallel A^{-\frac{1}{2}}BD^{-\frac{1}{2}} \parallel)$$

and

$$b = 1/ \parallel D^{-\frac{1}{2}} \parallel^2$$

Let $\alpha < \frac{1}{2a}$ and choose $\varepsilon < 1/(\frac{a}{\alpha} + \frac{3}{2b})$. Inequality (23) becomes, by using the invertibility of D and (20)

$$\frac{d}{dt}\rho_\varepsilon(y(t)) \leq -\theta_\varepsilon \parallel y(t) \parallel^2 \leq -\frac{\theta_\varepsilon}{N_\varepsilon}\rho_\varepsilon(y(t))$$

which proves (21) with $\omega_\varepsilon = \frac{\theta_\varepsilon}{N_\varepsilon}$. ■

The proof of the theorem is now quickly derived from (20) and (21). In fact, from (21) it follows that

$$\rho_\varepsilon(y(t)) \leq \exp(-\omega_\varepsilon\, t)\rho_\varepsilon(y_0)$$

and from (20)

$$\| y(t) \|^2 \leq \frac{N_\varepsilon}{M_\varepsilon} \exp(-\omega_\varepsilon\, t) \| y_0 \|^2 \qquad \text{for all } y_0 \in D(L)$$

■

Remark 3 *Whenever $A = C$ (with of course $H = G$) and $B^{-1}A = AB^{-1}$, the derivative of $\rho_\varepsilon(y(t))$ reduces to*

$$\begin{aligned} \tfrac{d}{dt}\rho_\varepsilon(y(t)) = \;& - \| D^{\frac{1}{2}} v_t \|^2 + \varepsilon \operatorname{Re}(-\tfrac{1}{2} \| u_t \|^2 - \tfrac{1}{2} \| A^{\frac{1}{2}} u \|^2 + \tfrac{3}{2} \| v_t \|^2 \\ & -\tfrac{1}{2} \| C^{\frac{1}{2}} v \|^2 - (Dv_t, B^{-1}u_t) + (Bv_t, u)). \end{aligned}$$

and then the operators $D^{-\frac{1}{2}}CB^{-1}A^{-\frac{1}{2}}$ and $D^{-\frac{1}{2}}B^{-1}A^{\frac{1}{2}}$ need not to be bounded.
■

Remark 4 *The invertibility of B is a restrictive assumption but, on the other hand, our result does not require that the entries A, B, C and D commute with each other.* ■

4 Examples.

In this section, we will set $H = G$ and assume that B, C and D are powers of the positive self-adjoint operator A, namely

$$B = A^\alpha, \;\; C = A^\beta, \;\; D = A^\gamma$$

and we will assume that

$$A^{-1} \text{ is } \textit{compact}. \tag{24}$$

Our goal is then to find *all* $(\alpha, \beta, \gamma) \in R^3$ for which exponential stability occurs for our system, namely

$$\begin{aligned} u''(t) &= -Au(t) + A^\alpha v'(t) \qquad t > 0 \\ v''(t) &= -A^\alpha u'(t) - A^\beta v(t) - A^\gamma v'(t) \qquad t > 0 \end{aligned} \tag{25}$$

and initial datas. The assumptions of Theorem 4 are satisfied if (and only if)

$$2\alpha \leq 1 + \beta; \quad \gamma \leq \beta. \tag{26}$$

In view of Theorem 5 and Remark 3, the semigroup associated to System (25) is uniformly stable if either

$$\beta \neq 1, \;\; \gamma \in [\max(1-2\alpha, 2\alpha-1, 2\beta-2\alpha-1), 2\alpha] \tag{27}$$

or

$$\beta = 1, \, \gamma \in [\max(0, 2\alpha-1), 2\alpha] \tag{28}$$

Note that since B and D are boundedly invertible in Theorem 5 and A is unbounded from (24), it follows that $\alpha \geq 0$ and $\gamma \geq 0$.

A natural question is then: through this example, are the assumptions of Theorem 5 sharp? The answer is given by the following result:

Theorem 8 *Assume that (26) and (24) hold and that $\beta = 1$ in the previous example. Then the semigroup $S(t)$ is uniformly stable if and only if (28) holds.*

Proof. It remains only to prove the necessity part since, by Theorem 5, (28) is sufficient for uniform stability of the semigroup. The necessity of condition (28) will follow from the Hille-Yosida theorem if we prove that

$$\sup \mathrm{Re}\, \sigma(\mathcal{A}_T) = 0$$

($\sigma(\mathcal{A}_T)$ stands for the spectrum of $\mathcal{A}_T$) whenever α and γ do not satisfy it. Recall that in this example

$$\mathcal{A}_T = \begin{pmatrix} 0 & I \\ -S & T \end{pmatrix}$$

with

$$S = \begin{pmatrix} A & 0 \\ 0 & A^{\beta} \end{pmatrix}, \quad T = \begin{pmatrix} 0 & A^{\alpha} \\ -A^{\alpha} & -A^{\gamma} \end{pmatrix}$$

The eigenvalues of $\mathcal{A}_T$ satisfy, as a direct computation shows, the algebraic equations

$$\lambda^4 + \mu_k^{\gamma}\lambda^3 + (\mu_k^{2\alpha} + \mu_k^{\beta} + \mu_k)\lambda^2 + \mu_k^{1+\gamma}\lambda + \mu_k^{1+\beta} = 0 \qquad k \geq 1 \tag{29}$$

where $(\mu_k)_{k\geq 1}$ is the sequence of positive eigenvalues of A. It follows from ([14, part I]) that if λ is a root of (29) then $\mathrm{Re}(\lambda) < 0$ for all $k \geq 1$. Thus, we study the asymptotic behavior of these roots as $k \to \infty$. *In the sequel, we drop the subscript k.*

Let's begin by writing equation (29) in the form

$$(\lambda^2 + (\frac{1}{2}\mu^\gamma + \sqrt{x})\lambda + c_1)(\lambda^2 + (\frac{1}{2}\mu^\gamma - \sqrt{x})\lambda + c_2) = 0 \qquad (30)$$

where

$$\begin{aligned} c_1 = & \frac{(\frac{1}{2}\mu^\gamma + \sqrt{x})(\mu^{2\alpha} + \mu^\beta + \mu - \frac{1}{4}\mu^{2\gamma} + x) - \mu^{1+\gamma}}{2\sqrt{x}} \\ c_2 = & \frac{-(\frac{1}{2}\mu^\gamma - \sqrt{x})(\mu^{2\alpha} + \mu^\beta + \mu - \frac{1}{4}\mu^{2\gamma} + x) + \mu^{1+\gamma}}{2\sqrt{x}} \end{aligned} \qquad (31)$$

and x is a positive root of the equation

$$x^3 + g_2 x^2 + g_1 x - g_0 = 0 \qquad (32)$$

with

$$\begin{aligned} g_0 = & \tfrac{1}{4}\mu^{2\gamma}(\mu^{2\alpha} + \mu^\beta - \mu - \tfrac{1}{4}\mu^{2\gamma})^2 \\ g_1 = & \mu^{4\alpha} + \tfrac{3}{16}\mu^{4\gamma} + \mu^{2\beta} + \mu^2 + 2\mu^{2\alpha+\beta} + 2\mu^{2\alpha+1} - \\ & (\mu^{2(\alpha+\gamma)} + \mu^{\beta+2\gamma} + 2\mu^{\beta+1}) \\ g_2 = & 2\mu^{2\alpha} + 2\mu^\beta + 2\mu - \tfrac{3}{4}\mu^{2\gamma} \end{aligned} \qquad (33)$$

If $\beta = 1$, one has

$$\begin{aligned} g_0 = & \tfrac{1}{4}\mu^{2\gamma}(\mu^{2\alpha} - \tfrac{1}{4}\mu^{2\gamma})^2 \\ g_1 = & \mu^{4\alpha} + \tfrac{3}{16}\mu^{4\gamma} + 4\mu^{2\alpha+1} - \mu^{2(\alpha+\gamma)} + \mu^{1+2\gamma} \\ g_2 = & 2\mu^{2\alpha} + 4\mu - \tfrac{3}{4}\mu^{2\gamma} \end{aligned} \qquad (34)$$

and the solutions of equation (32) are

$$\begin{aligned} x_1 = & \tfrac{1}{4}\mu^{2\gamma} - \mu^{2\alpha} \\ x_\pm = & \tfrac{1}{4}\mu^{2\gamma} - \tfrac{1}{2}\mu^{2\alpha} - 2\mu \pm \tfrac{1}{2}\sqrt{16\mu^2 + 8\mu^{2\alpha+1} + \mu^{4\alpha} - 4\mu^{2\gamma+1}} \end{aligned} \qquad (35)$$

We then have

Lemma 9 *(a) If $\gamma > \alpha$, the roots of equation (30) are such that*

$$\lambda_1 \sim -\mu^{\gamma}; \quad \lambda_2 \sim -\mu^{1-\gamma} \quad \textit{if} \quad 2\gamma > 1$$

$$\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) \sim -\tfrac{1}{2}\mu^{\gamma} \quad \textit{if} \quad 2\gamma \leq 1$$

$$\lambda_3 \sim -\mu^{2\alpha-\gamma}; \quad \lambda_4 \sim -\mu^{1-2\alpha+\gamma} \quad \textit{if} \quad 2\alpha - \gamma > \tfrac{1}{2}$$

$$\mathrm{Re}(\lambda_3) = \mathrm{Re}(\lambda_4) \sim -\tfrac{1}{2}\mu^{2\alpha-\gamma} \quad \textit{if} \quad 2\alpha - \gamma \leq \tfrac{1}{2}$$

as $\mu \to \infty$.
(b) If $\gamma \leq \alpha$, then

$$\mathrm{Re}(\lambda_1) \;=\; \mathrm{Re}(\lambda_2) \sim -\frac{1}{2}\mu^{\gamma} \quad \textit{for all} \quad \alpha \in [0,1]$$

$$\mathrm{Re}(\lambda_3) \;=\; \mathrm{Re}(\lambda_4) \sim \begin{cases} -\tfrac{1}{2}\mu^{\gamma-2\alpha+1} & \textit{if} \;\; 2\alpha > 1 \\ -\tfrac{1}{8}\mu^{\gamma} & \textit{if} \;\; 2\alpha \leq 1 \end{cases}$$

as $\mu \to \infty$.

Proof.
(a) If $\gamma > \alpha$,by setting $x = x_1$ in (31), we obtain

$$c_1 = c_2 = \mu \tag{36}$$

and

$$\begin{aligned} \tfrac{1}{2}\mu^{\gamma} + \sqrt{x} &= \tfrac{1}{2}\mu^{\gamma} + \sqrt{\tfrac{1}{4}\mu^{2\gamma} - \mu^{2\alpha}} \quad \sim \mu^{\gamma} \\ \tfrac{1}{2}\mu^{\gamma} - \sqrt{x} &= \tfrac{\mu^{2\alpha}}{\frac{1}{2}\mu^{\gamma}+\sqrt{x}} \quad \sim \mu^{2\alpha-\gamma} \end{aligned} \tag{37}$$

as $\mu \to \infty$. It follows from (36) and $(37)_1$ that the roots λ_1 and λ_2 of the equation

$$\lambda^2 + (\frac{1}{2}\mu^{\gamma} + \sqrt{x})\lambda + c_1 = 0 \tag{38}$$

are asymptotically real if $2\gamma > 1$ and complex if $2\gamma \leq 1$ and

$$\begin{aligned} \lambda_1 \sim -\mu^{\gamma}; \quad \lambda_2 \sim -\mu^{1-\gamma} \quad &\text{if} \quad 2\gamma > 1 \\ \mathrm{Re}(\lambda_1) \sim -\tfrac{1}{2}\mu^{\gamma} \quad &\text{if} \quad 2\gamma \leq 1 \end{aligned} \tag{39}$$

while, for the roots λ_3 and λ_4 of the equation

$$\lambda^2 + (\frac{1}{2}\mu^\gamma - \sqrt{x})\lambda + c_2 = 0 \tag{40}$$

one has

$$\begin{aligned} &\lambda_3 \sim -\mu^{2\alpha-\gamma}; \quad \lambda_4 \sim -\mu^{1-2\alpha+\gamma} \quad \text{if} \quad 2\alpha-\gamma > \tfrac{1}{2} \\ &\mathrm{Re}(\lambda_3) \sim -\tfrac{1}{2}\mu^{2\alpha-\gamma} \quad \text{if} \quad 2\alpha-\gamma \leq \tfrac{1}{2} \end{aligned} \tag{41}$$

We conclude that if $\gamma > \alpha$, the eigenvalues are bounded away from the imaginary axis if and only if $\gamma \leq 2\alpha$.

If $\gamma \leq \alpha$, we set $x = x_+$. In this case, simple computations provide the following asymptotical behaviours

$$c_1 \sim \begin{cases} \mu^{2\alpha} & \text{if} \quad 2\alpha > 1 \\ a\mu & \text{if} \quad 2\alpha \leq 1 \end{cases}$$

$$c_2 \sim \begin{cases} \mu^{2(1-\alpha)} & \text{if} \quad 2\alpha > 1 \\ b\mu & \text{if} \quad 2\alpha \leq 1 \end{cases}$$

$$\frac{1}{2}\mu^\gamma + \sqrt{x} \sim \begin{cases} \mu^{\gamma} & \text{if} \quad 2\alpha > 1 \\ c\mu^\gamma & \text{if} \quad 2\alpha \leq 1 \end{cases}$$

$$\frac{1}{2}\mu^\gamma + \sqrt{x} \sim \begin{cases} \mu^{\gamma-2\alpha+1} & \text{if} \quad 2\alpha > 1 \\ d\mu^\gamma & \text{if} \quad 2\alpha \leq 1 \end{cases}$$

where a, b, c and d are various positive constants independant of μ. It appears that the roots of equation (38) are asymptotically complex with

$$\mathrm{Re}(\lambda_1) \sim -\frac{1}{2}\mu^\gamma \quad \text{for all} \quad \alpha \in [0,1]$$

while those of the equation (40) are such that

$$\mathrm{Re}(\lambda_3) \sim \begin{cases} -\frac{1}{2}\mu^{\gamma-2\alpha+1} & \text{if} \quad 2\alpha > 1 \\ -\frac{1}{8}\mu^\gamma & \text{if} \quad 2\alpha \leq 1 \end{cases}$$

All these estimates allow to conclude that if $\gamma \leq \alpha$, the eigenvalues are bounded away from the imaginary axis if and only if $\gamma \geq 2\alpha - 1$. ■

This proves the assertion of the theorem. ■

Remark 5 *The assertion in Theorem 8 is probably true even if $\beta \neq 1$ (condition (28) being replaced by (27)). It is worthwhile to note the gap between the (α, γ)-domains described by conditions (27) and (28) respectively. For example, if $\beta > 1$ then γ may be chosen equal to zero only for particular values of α.*

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