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STABILIZABILITY OF SYSTEMS OF ONE-DIMENSIONAL WAVE EQUATIONS BY ONE INTERNAL OR BOUNDARY CONTROL FORCE*

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Abstract. We study the internal and boundary stabilizability of a system of wave equations by one control force. We prove that the "classical" internal damping applied to only one of the equations never gives exponential stability if the wave speeds are different and, if the wave speeds are the same, we give explicit necessary and sufficient conditions for the stability to occur. We also study the simultaneous boundary stabilization of the same system.

Key words. stabilization, semigroups, essential spectrum

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1. Introduction.

The starting point of this work was the study of the stabilizability of two coupled abstract second order equations in Hilbert spaces using only one control force. More precisely, let us consider the system

$$\begin{cases} u''(t) = & -Au(t) + Bv'(t) \qquad \text{in } H, \\ v''(t) = & -B^*u'(t) - Cv(t) - Dv'(t) \quad \text{in } G, \end{cases}$$

where H and G are Hilbert spaces. The question then is to characterize the widest classes of operators A, B, C, and D for which the uniform stability of the semigroup associated with this system (once the conditions for its existence are ensured) holds. A general answer seems to be difficult but some results are given, with rather restrictive assumptions, by Afilal and Khodja [1] (see also [3], [5], and [7] for abstract thermoelastic systems which correspond to this system by neglecting v''). In this last paper, it was pointed out that there was a "gap" between the cases $A = C$ and $A \neq C$ in that it is easier to find a stabilizing operator D in the first case ($A = C$) than in the second ($A \neq C$); see [1] for more details.

In this paper, keeping in mind the abstract approach we have just described, we will confine ourselves to the study of one-dimensional hyperbolic systems which are close to the Timoshenko beam equations (see, for instance, [11]).

The first problem we consider is the following.

$$(1.1) \quad \begin{cases} u_{tt} = u_{xx} + b(x)v_t + f & \text{in } (0,\infty)\times(0,1), \\ v_{tt} = \eta^2 v_{xx} - b(x)u_t + g & \text{in } (0,\infty)\times(0,1), \\ u(t,0) = v(t,0) = u(t,1) = v(t,1) = 0, & t \in (0,\infty), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ v(0,x) = v_0(x),\ v_t(0,x) = v_1(x), \end{cases}$$

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where $\eta \in \mathbb{R}$, $b \in C([0,1])$ and f and g are two control forces. Our aim here is to study the exponential stability of (1.1) whenever we choose

$$f \equiv 0,\ g = -a(x)v_t, \tag{1.2}$$

where $a \in C([0,1])$. Our system then writes

$$\begin{cases} u_{tt} = u_{xx} + b(x)v_t & \text{in } (0,\infty)\times(0,1), \\ v_{tt} = \eta^2 v_{xx} - b(x)u_t \ - a(x)v_t & \text{in } (0,\infty)\times(0,1), \\ u(t,0) = v(t,0) = u(t,1) = v(t,1) = 0, & t \in (0,\infty), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ v(0,x) = v_0(x),\ v_t(0,x) = v_1(x). \end{cases} \tag{1.3}$$

The natural energy associated with (1.3) is

$$E(t) = \int_0^1 \left(|\ u_t\ |^2 + |\ u_x\ |^2 + |\ v_t\ |^2 + \eta^2\ |\ v_x\ |^2\right) dx. \tag{1.4}$$

Let us recall that (1.3) is *exponentially stable* if there exist $\omega > 0$ and $M > 0$ such that

$$E(t) \le Me^{-\omega t}E(0) \qquad \forall t > 0 \tag{1.5}$$

holds for any initial data (u_0, u_1, v_0, v_1) with finite energy. It is said to be *strongly stable* if for any initial data (u_0, u_1, v_0, v_1) with finite energy

$$\lim_{t\to\infty} E(t) = 0. \tag{1.6}$$

Our result is then the following.

THEOREM 1.1. *Assume that* $a, b \in C([0,1])$.

(i) *If* $\eta \neq 1$, *then* (1.5) *does not hold.*

(ii) *If* $\eta = 1$, *assume moreover that* a *and* b *have* disjoint supports. *Then* (1.5) *holds if and only if* (1.3) *is strongly stable and*

$$\overline{a} := \int_0^1 a(x)dx > 0,\ \overline{b} := \int_0^1 b(x)dx \notin \pi\mathbb{Z}. \tag{1.7}$$

Remark 1.2. We will see that under the condition (1.7), the exponential stability holds up in an invariant subspace of the energy space of finite codimension.

If a and b have *disjoint supports,* we are not able to prove strong stability even in the dissipative case ($a \geq 0$ on $(0,1)$). If a and b have *the same support,* the strong stability follows from Kapitonov's result [6] in the dissipative case (see the next remark).

Remark 1.3. The technique we use allows us to deal with systems with more than two wave equations. This is what is done by Kapitonov [6] in higher dimensions. His assumptions amount to taking a, b with the same support assuming that $a > 0$ on its support, and using the multiplier technique, he proves the exponential stability under additional geometrical assumptions (which are easily verified in the one-dimensional case).

With a slight modification of Kapitonov's proof, it is possible, in the one-dimensional case, to prove the exponential stability by assuming only that the supports of a and b contain a common interval.

N. Burq (lecture given at an international conference on control theory, Nancy, France, March 1999), has generalized our result to higher dimension using the microlocal defect measures of P. Gérard and L. Tartar.

The second problem we deal with is

$$\begin{cases} u_{tt} = u_{xx} + b(x)v_t & \text{in } (0,\infty)\times(0,1), \\ v_{tt} = \eta^2 v_{xx} - b(x)u_t & \text{in } (0,\infty)\times(0,1), \\ u(t,0) = v(t,0) = 0, \\ u(t,1) = f(t), \quad \eta^2 v_x(t,1) = g(t), \quad t\in(0,\infty), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ v(0,x) = v_0(x),\ v_t(0,x) = v_1(x). \end{cases}$$

Following Lions [8], we would like to stabilize *simultaneously* this system by two boundary control forces which are related by the relation

$$f' = g \qquad \text{on } (0,\infty).$$

A natural choice of the force g which makes our system dissipative (i.e., $E'(t) \le 0$ for $t > 0$) is

$$g(t) = -\alpha\left(u_x(t,1) + v_t(t,1)\right), \quad \alpha > 0.$$

We then get the system

$$\begin{cases} u_{tt} = u_{xx} + b(x)v_t & \text{in } (0,\infty)\times(0,1), \\ v_{tt} = \eta^2 v_{xx} - b(x)u_t & \text{in } (0,\infty)\times(0,1), \\ u(t,0) = v(t,0) = 0, \\ \quad \eta^2 v_x(t,1) = u_t(t,1) = -\alpha\left(u_x(t,1) + v_t(t,1)\right), \quad t\in(0,\infty), \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x),\ v(0,x) = v_0(x),\ v_t(0,x) = v_1(x). \end{cases} \tag{1.8}$$

The energy associated with this system is again given by (1.4). Our result for system (1.8) is the following.

THEOREM 1.4. *Assume that* $b \in C([0,1])$ *and* $\alpha > 0$. *Then*

(i) *if* $\eta \neq 1$, (1.5) *holds if and only if* (1.8) *is strongly stable and*

$$\eta = \frac{2p+1}{q} \ \textit{for some } (p,q) \in \mathbb{Z}\times\mathbb{Z}^*;$$

(ii) *if* $\eta = 1$, (1.5) *holds if and only if* (1.8) *is strongly stable and*

$$\bar{b} := \int_0^1 b(x)dx \neq (2k+1)\frac{\pi}{2} \quad \textit{for any} \ \ k\in\mathbb{Z}.$$

A related work which deals with simultaneous controllability of a system of one-dimensional wave equations can be found in Avodin and Tucsnak [4].

Remark 1.5. As for the previous system, we will show that exponential stability holds up in an invariant subspace of the energy space of finite codimension.

If $b \equiv 0$, it is easy to verify that strong stability holds. However, if $b \neq 0$, it seems to be difficult to find conditions on b which imply the strong stability.

The paper is organized as follows. In the second section, we begin by stating and proving a lemma extending a result of Neves, Ribeiro, and Lopes [9]. We prove Theorems 1.1 and 1.4 in the third section.

Some of the results of this paper were already announced in [2].

2. A lemma. We consider a one-dimensional hyperbolic system written in the form

$$(2.1)\quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = -M(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} - N(x)\begin{pmatrix} u \\ v \end{pmatrix} \qquad \text{on } [0,T]\times]0,l[\,, \\ \\ \dfrac{d}{dt}\left[v(t,l) - Du(t,l)\right] = Fu(t,l) + Gv(t,l), \\ \\ u(t,0) = Ev(t,0), \end{cases}$$

where

(i) $N(x)$ *is an* $n \times n$ *matrix whose entries* n_{ij} *are continuous complex valued functions of* x *in* $[0,l\,]$,

(ii) $M(x)$ *is a diagonal matrix satisfying*

$$M(x) = \text{diag}\,([M_{ii}(x)]_{i=1}^{r}, [M_{jj}(x)]_{j=r+1}^{q}),$$

where M_{ii} *(resp.,* M_{jj}*) are diagonal matrices such that*

$$\begin{aligned} M_{ii}(t,x) &= \lambda_i(x) I_{m_i}, \qquad & i &= 1,\ldots,r, \\ M_{jj}(t,x) &= \mu_j(x) I_{m_j}, \qquad & j &= r+1,\ldots,q, \end{aligned}$$

and where I_{m_i} *is the identity matrix of size* m_i *and*

$$\sum_{i=1}^{r} m_i = p; \quad \sum_{j=r+1}^{q} m_j = n - p.$$

We suppose also that the entries of $M(x)$ *are real valued* C^1 *functions in* x *with*

$$\lambda_i(x) > 0 \;\; \textit{and} \;\; \mu_j(x) < 0 \quad \forall\, i,j \quad \textit{and} \quad \forall\, x \in [0,l\,]\,,$$

(iii) $u(t,x) = (u_i(t,x))_{i=1}^{p}$ *and* $v(t,x) = (v_i(t,x))_{j=p+1}^{n}$,

(iv) D, E, F, *and* G *are matrices of appropriate sizes.*

With system (2.1), we consider the reduced system

$$(2.2)\quad \begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = -M(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} - N_0(x)\begin{pmatrix} u \\ v \end{pmatrix} \qquad \text{on } (0,T)\times]0,l[\,, \\ \\ u(t,0) = Ev(t,0) \;\;,\;\; v(t,l) = Du(t,l), \end{cases}$$

where

(v) $N_0(x) = \operatorname{diag}(N_{11}(x), N_{22}(x), \ldots, N_{qq}(x))$ *is a diagonal matrix per block whose elements* $N_{\eta\eta}(x)$ $(1 \leq \eta \leq q)$ *are* $m_\eta \times m_\eta$ *matrices* (m_η *is the algebraic multiplicity of the eigenvalue* $\lambda_\eta(x)$ (*or* $\mu_\eta(x)$)) *and each matrix* $N_{\eta\eta}(x)$ *is a block of the matrix* $N(x)$ *such that*

$$N_{\eta\eta}(x) = (n_{k,l})_{S_{\eta-1} \leq k,l \leq S_\eta} \quad \text{with } S_0 = 1, \text{ and } S_\eta = \sum_{d=1}^{\eta} m_d.$$

To illustrate this last assumption, if, for example, $M = \operatorname{diag}(1, 1, 2, -3, -3)$ and $N = (n_{ij})_{1 \leq i,j \leq 5}$, then the corresponding reduced matrix N_0 is given by

$$N_0 = \begin{pmatrix} n_{11} & n_{12} & 0 & 0 & 0 \\ n_{21} & n_{22} & 0 & 0 & 0 \\ 0 & 0 & n_{33} & 0 & 0 \\ 0 & 0 & 0 & n_{44} & n_{45} \\ 0 & 0 & 0 & n_{54} & n_{55} \end{pmatrix}.$$

In this study we prove that the two systems (2.1) and (2.2) have the same essential spectral radius.

Before giving the main result of this section, we need to recall some definitions and properties that may be found in Van Neerven [12, pp. 106–111].

DEFINITION 2.1. (i) *If* L *is a linear bounded operator in a Banach space, the essential spectral radius of* L *is*

$$r_{ess}(L) := \inf \{ r > 0 : \lambda \in \sigma(L), |\lambda| \geq r; \text{ implies } \lambda \text{ is an isolated eigenvalue of finite multiplicity} \},$$

where $\sigma(L)$ *is the spectrum of* L.

(ii) *The type (or growth bound)* $\omega(T)$ *of a* C_0*-semigroup* $(T(t))$ *generated by* A *is*

$$\omega(T) := \inf \left\{ \omega \in \mathbb{R},\ \exists M_\omega > 0,\ \|T(t)\| \leq M_\omega e^{\omega t} \quad \forall\ t \geq 0 \right\}.$$

A well-known result is that there exists a real number $\omega_{ess}(T)$ (the *essential type* of $T(t)$) such that

$$\omega_{ess}(T) := \frac{\ln [r_{ess}(T(t))]}{t}, \quad t > 0.$$

The property below will play a significant role in our study:

$$r_{ess}(L + K) = r_{ess}(L) \quad \text{for any compact operator } K. \tag{2.3}$$

We recall also (see [12, pp. 106–111]) that the type $\omega(T)$ of a semigroup $(T(t))$ is given by

$$\omega(T) = \max [s(A)\,;\, \omega_{ess}(T)],$$

where $s(A)$ is the spectral abscissa of A:

$$s(A) := \sup \{ \operatorname{Re} \lambda; \lambda \in \sigma(A) \}.$$

Returning to our problem, let us introduce the new variable:

$$z(t) = v(t, l) - Du(t, l)$$

and define, on the energy space $H = \left[L^2([0,l])\right]^n \times \mathbb{C}^{n-p}$, the following operators:

$$A_1 \begin{pmatrix} u \\ v \\ z \end{pmatrix} = \left(\left[-M(x)\frac{\partial}{\partial x} - N(x) \right] \begin{pmatrix} u \\ v \end{pmatrix}, Fu(t,l) + Gv(t,l) \right) \tag{2.4}$$

and

$$A_4 \begin{pmatrix} u \\ v \end{pmatrix} = \left[-M(x)\frac{\partial}{\partial x} - N_0(x) \right] \begin{pmatrix} u \\ v \end{pmatrix} \tag{2.5}$$

whose domains are, respectively,

$$\begin{aligned} D(A_1) = \{ (u,v,z) \in H\,; (u,v) \in \left[W^{1,2}(0,l)\right]^n\,; \\ u(0) = E\,v(0),\ z = v(l) - Du(l) \} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} D(A_4) = \{ (u,v) \in H\,; (u,v) \in \left[W^{1,2}(0,l)\right]^n\,; \\ u(0) = E\,v(0),\ v(l) = Du(l) \}. \end{aligned} \tag{2.7}$$

Equations (2.1) and (2.2) can be viewed as abstract systems

$$Y_t = A_1 Y \quad \text{in } H,$$

$$Z_t = A_4 Z \quad \text{in } \tilde{H},$$

where $\tilde{H} := \{(u,v,z) \in H;\ z \equiv 0\}$ and

$$A_1 = \begin{pmatrix} -M\dfrac{\partial}{\partial x} - N & 0 \\ 0 & R \end{pmatrix}, \qquad A_4 := -M\frac{\partial}{\partial x} - N_0$$

with $R : \mathbb{C}^{n-p} \longrightarrow \mathbb{C}$ and $Rz(t) = Fu(t,l) + Gv(t,l)$.

As in [9], under the assumptions (i)–(v), A_1 (resp., A_4) defined by (2.4) and (2.6) (resp., by (2.5) and (2.7)) generates a C_0-semigroup $T_1(t)$ on H (resp., ($T_4(t)$ on $\tilde{H}$).

Our main ingredient is the following.

LEMMA 2.2. *Suppose that the assumptions* (i)–(v) *hold; then the difference of the two semigroups $T_1(t)$ and $T_4(t)$ is a compact operator. In particular,*

$$r_{ess}(T_1(t)) = r_{ess}(T_4(t)).$$

Consequently,

$$\omega_{ess}(T_1) = \omega_{ess}(T_4).$$

Remark 2.3. This lemma has been proved by Neves, Ribeiro, and Lopes [9] in the case where the eigenvalues $\lambda_i(x)$ and $\mu_j(x)$ of the diagonal matrix $M(x)$ are all distinct (i.e., $m_i = 1$ for $i = 1, \dots, q$). In this case, these authors showed also that $\omega_{ess}(T_4) = s(A_4)$.

For the proof of Lemma 2.2, we use the same techniques as Neves, Ribeiro, and Lopes [9]. Note that the proof we propose for Lemma 2.2 works in the nonautonomous case as well.

Proof of Lemma 2.2. To simplify, we will prove Lemma 2.2 in the following situation:

- $M(x) = \operatorname{diag}(\lambda(x), \lambda(x), \mu(x))$; with $\lambda(x) > 0$ and $\mu(x) < 0$ for any x in$[0, l]$.
- $N(x) = (n_{ij}(x))_{1\le i,j\le 3}$.
- $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$, $F = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}$, $G = g$, and $E = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$, where $d_1,\ d_2,\ f_1,\ f_2,\ g,\ e_1$, and e_2 are real or complex constants.

We introduce two intermediate reduced systems:

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = -M(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u \\ v \\ w \end{pmatrix} - N_0(x)\begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad \text{on }]0, l[, \\ \\ \dfrac{d}{dt}\left[w(t,l) - d_1u(t,l) - d_2v(t,l)\right] = f_1u(t,l) + f_2v(t,l) + gw(t,l), \\ \\ u(t,0) = e_1w(t,0), \quad v(t,0) = e_2w(t,0), \end{array}\right. \tag{2.8}$$

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial t}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = -M(x)\dfrac{\partial}{\partial x}\begin{pmatrix} u \\ v \\ w \end{pmatrix} - N_0(x)\begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad \text{on }]0, l[, \\ \\ \dfrac{d}{dt}\left[w(t,l) - d_1u(t,l) - d_2v(t,l)\right] = 0, \\ \\ u(t,0) = e_1w(t,0), \ \ v(t,0) = e_2w(t,0). \end{array}\right. \tag{2.9}$$

Denote by A_2 (resp., A_3) the associated operator of system (2.8) (resp., (2.9)). These two operators are defined on the same energy space and have the same domain as A_1. A_2 and A_3 generate, respectively, two C_0-semigroups which we will denote by $T_2(t)$ and $T_3(t)$.

We introduce the space $\bar{H} := \{(u, v, w, z) \in H\ ;\ z = 0\}$ and we define the orthogonal projection P of H on $\bar{H}$:

$$\begin{aligned} P: \qquad H &\longrightarrow \bar{H}, \\ (u, v, w, z) &\longmapsto (u, v, w, 0). \end{aligned}$$

We can identify $T_4(t)$ with $T_4(t)P$ which is an operator on H. It is enough to show that : $(T_1(t) - T_4(t)P)$ is compact.

We write

$$T_1(t) - T_4(t)P = (T_1(t) - T_2(t)) + (T_2(t) - T_3(t)) + (T_3(t) - T_4(t)P)\,.$$

We will show that each term of the right-hand side member is a compact operator.

Step 1. $(T_2(t) - T_3(t))$ *is a compact operator.*

To compute explicitly the semigroups $T_2(t)$ and $T_3(t)$, we follow [9] using the characteristics method. Taking the initial data $Y_0 = ((u_0, v_0), w_0, z_0)$ in H, we put $U = (u, v)$ and $(u_0, v_0) = U_0$. Denote by $z(t) = w(t,l) - d_1u(t,l) - d_2v(t,l)$ the new variable and by $[(u, v), w, z]$ the unique solution of system (2.2) with initial data Y_0.

Given a fixed point (t, x) in $(0, T)\times]0, l[$, let $\varphi(., t, x)$ (resp., $\psi(., t, x)$) be the unique solution of

$$\left\{\begin{array}{l} \dfrac{dx}{ds} = \lambda(x(s)), \\ \quad x(t) = x, \end{array}\right. \qquad \text{resp.,} \ \left\{\begin{array}{l} \dfrac{dx}{ds} = \mu(x(s)), \\ \quad x(t) = x. \end{array}\right.$$

We define the maps

$$\begin{aligned} &\tau_i : [0,T]\times[0,l] \longrightarrow \mathbb{R},\\ &\qquad\qquad (t,x) \longrightarrow \tau_i(t,x) \quad i=1,2,\\ &\text{such that } \ \varphi(\tau_1(t,x),t,x)=0 \ \text{ and } \ \psi(\tau_2(t,x),t,x)=l. \end{aligned}$$

Denote by $R(t,y,x)$ the fundamental matrix associated with

$$\frac{d}{dt}Y(t) = N_0^{11}(x(t))Y(t),$$

where

$$N_0^{11}(x) = \begin{pmatrix} n_{11}(x) & n_{12}(x) \\ n_{21}(x) & n_{22}(x) \end{pmatrix} \text{ and } N_0(x) = \begin{pmatrix} N_0^{11}(x) & 0 \\ 0 & n_{33}(x) \end{pmatrix}.$$

Using the corresponding boundary conditions, the solution (U,w,z) is given by the following:

- If $0 \le x \le \varphi(t,0,0)$, then

$$\begin{aligned} U(t,x) = \exp\left[-\int_0^{\tau_1(t,x)} n_{33}(\psi(s,\tau_1(t,x),0))ds\right] R(t,\tau_1(t,x),x)\\ \times \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} w_0(\psi(0,\tau_1(t,x),0)). \end{aligned}$$

- If $\varphi(t,0,0) \le x \le l$, then

$$U(t,x) = R(t,0,x)\times U_0(\varphi(0,t,x)).$$

- If $0 \le x \le \psi(t,0,l)$, then

$$w(t,x) = \exp\left[-\int_0^t n_{33}(\psi(s,t,x))ds\right] w_0(\psi(0,t,x)).$$

- If $\psi(t,0,l) \le x \le l$, then

$$\begin{aligned} w(t,x) = \exp\left(-\int_{\tau_2(t,x)}^t n_{33}(\psi(s,t,x))ds\right)\\ \times\left[z(\tau_2(t,x)) + [d_1,d_2]R(\tau_2(t,x),0,l)U_0(\varphi(0,\tau_2(t,x),l))\right]. \end{aligned}$$

- For any t in the interval $J=[0,T]$, we have

$$\begin{aligned} z(t) &= \int_0^t e^{gt-s}(gd_1+f_1)u(t,l)+(gd_2+f_2)v(t,l)ds + e^{tg}z_0\\ &= \int_0^t e^{gt-s}(GD+F)U(s,l)ds + e^{tg}z_0. \end{aligned}$$

Consequently, $T_3(t)Y_0$ is obtained by cancelling the constants f_1, f_2, and g.

Thus

$$(T_2(t)-T_3(t))\left((u_0,v_0),w_0,z_0\right) = \left(0,\bar{w}(t,x),\bar{z}(t)\right),$$

where $\bar{w}$ and $\bar{z}$ are given by the following:

- If $0 \leq x \leq \psi(t,0,l)$, then

$$\bar{w}(t,x) = 0.$$

- If $\psi(t,0,l) \leq x \leq l$, then

$$\bar{w}(t,x) = \exp\left(-\int_{\tau_2(t,x)}^{t} n_{33}(\psi(s,t,x))ds \right) \bar{z}(\tau_2(t,x)).$$

- And for all $t \in J$, we have

$$\bar{z}(t) = \int_0^t e^{tg-s}(GD+F)R(s,0,l)U_0(\varphi(0,s,l))ds + (e^{tg}-1)z_0.$$

To prove the compactness of the difference $T_2(t) - T_3(t)$, it is sufficient to show that each component of $(0, \overline{w}, \overline{z}) = (T_2(t) - T_3(t))\, Y_0$ is a compact operator. Remark that the first component is null and that the operator defined by $z_0 \longrightarrow (e^{gt}-1)z_0$ is compact for any $t \in J = [0,T]$. In addition, in view of Lemma 4 in [9], the operator

$$\begin{aligned} U_0 \longrightarrow & \int_0^t \underbrace{e^{tg-s}\,[GD+F]\,R(s,0,l)}_{L(t,s)}\, U_0(\varphi(0,s,l))ds \\ = & \int_0^t L(t,s)U_0(\varphi(0,s,l))ds \end{aligned}$$

is compact since $L = (L_{ij})_{1\leq i,j\leq 2}$ is such that the L_{ij} $(1 \leq i,j \leq 2)$ are continuous functions of their arguments and

$$U_0(\varphi(0,s,l)) = \begin{pmatrix} u_0(\varphi(0,s,l)) \\ \\ v_0(\varphi(0,s,l)) \end{pmatrix}$$

$$\text{since} \quad : \frac{d}{ds}(\varphi(0,s,l)) = -\frac{\partial\varphi}{\partial x}(0,s,l)\,\lambda(l) \neq 0.$$

Consequently, $T_2(t) - T_3(t)$ is a compact operator for all t in the compact interval J.

Step 2. $(T_1(t) - T_2(t))$ *is a compact operator.*

For this we write, for a given initial data $Y_0 = ((u_0,v_0),w_0,z_0)$, the differential equation corresponding with system (2.1) in the form

$$\frac{\partial}{\partial t}Y(t) = A_1Y(t) = A_2Y(t) + BY(t),$$

where

$$B = \begin{pmatrix} N - N_0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then we have

$$Y(t) = T_1(t)Y_0 = T_2(t)Y_0 + \int_0^t T_2(t-s)BY(s)ds.$$

Consequently,

$$
\begin{aligned}
(T_1(t)-T_2(t))Y_0 &= \int_0^t T_2(t-s)BT_1(s)Y_0ds \\
&= \int_0^t T_2(t-s)BT_2(s)Y_0ds + \int_0^t T_2(t-s)B(T_1(s)-T_2(s))Y_0ds.
\end{aligned}
$$

LEMMA 2.4 (see [9, Lemma 3]). *Suppose that* $\int_0^t T_2(t-s)BT_2(s)ds$ *is a compact operator for all* $t \in J$*; then* $(T_1(t)-T_2(t))$ *is a compact operator for all* t *in* J*. In fact,* $\{(T_1(t)-T_2(t))\,Y;\ \|Y\|_H \le 1,\ t \in J\}$ *is precompact.*

To show that $\{(T_1(t)-T_2(t))\,Y; \|Y\|_H \le 1, t \in J\}$ is precompact, it is enough, according to Lemma 2.4, to prove that $\int_0^t T_2(t-s)BT_2(s)ds$ is a compact operator. However, we know that

$$
\begin{aligned}
T_2(t-s)BT_2(s) = T_3(t-s)BT_3(s) &+ [T_2(t-s)-T_3(t-s)]\,BT_3(s) \\
&+ T_2(t-s)B\,[T_2(s)-T_3(s)]
\end{aligned}
$$

As $T_2(t)-T_3(t)$ is a compact operator, $\int_0^t T_2(t-s)BT_2(s)ds$ is compact if $\int_0^t T_3(t-s)BT_3(s)ds$ is. For this, we are going to use Lemma 4 of [9]. Let $Y_0 = ((u_0,v_0),w_0,z_0) = (U_0,w_0,z_0)$ be in H. We wish to prove that each integral component of $\int_0^t T_3(t-s)BT_3(s)Y_0ds$ defines a compact operator. Note that the third component $\bar{z}$ defines an operator of finite dimensional range; thus it is compact. We can write

$$
\int_0^t T_3(t-s)BT_3(s)Y_0ds = \int_0^t \begin{pmatrix} \bar{U}(t,s,x) \\ \bar{w}(t,s,x) \\ \bar{z}(t,s,x) \end{pmatrix} ds.
$$

We deduce that $\bar{U}(t,s)$ is the first component of $\bar{Y}(t,s)$ which is the unique solution of the following Cauchy problem:

$$
\begin{cases}
\dfrac{d}{dt}Y(t) = A_3Y(t), \\
Y(s) = \tilde{Y}(s).
\end{cases}
$$

For $0 \le x \le \varphi(t,0,0)$.

We have

$$
\int_0^t \bar{U}(t,s,x)ds = \int_0^{\tau_1(t,x)} \bar{U}(t,s,x)ds + \int_{\tau_1(t,x)}^t \bar{U}(t,s,x)ds
$$

In the same way as [9], we find

$$
\begin{aligned}
\int_0^{\tau_1(t,x)} \bar{U}(t,s,x)ds = \int_0^{s_1} \xi_1(t,s,x)\, U_0(\varphi(0,s,x_\psi))ds &+ \int_{s_1}^{\tau_1(t,x)} \\
&\left[\xi_2(t,s,x)\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} w_0(\psi(0,\tau_1(s,x_\psi),0))ds\right].
\end{aligned}
$$

And

$$
\begin{aligned}
\int_{\tau_1(t,x)}^t \bar{U}(t,s,x)ds = \int_{\tau_1(t,x)}^{s_2} &\xi_3(t,s,x)\begin{pmatrix} n_{13} \\ n_{23} \end{pmatrix} w_0(\psi(0,s,x_\varphi))ds \\
&+ \int_{s_2}^t \xi_4(t,s,x)U_0(\varphi(0,\tau_2(s,x_\varphi),l))ds,
\end{aligned}
$$

where s_1 and s_2 are the two reals satisfying

$$\begin{cases} \varphi(s_1,0,0) = \psi(s_1,\tau_1(t,x),0) := x_\psi, \\ \psi(s_2,0,l) = \varphi(s_2,t,x) := x_\varphi \end{cases}$$

and $(\xi_i)_{i=1}^4$ are continuous functions of their arguments.

For $\boldsymbol{\varphi(t,0,0) \le x \le l}$.

It is easy to see that it is a particular case of that previously treated ($0 \le x \le \varphi(t,0,0)$).

To conclude, we have

$$\begin{aligned} \frac{d\varphi}{ds}(0,s,x_\psi) &= \frac{\partial\varphi}{\partial s}(0,s,x_\psi) + \frac{\partial\varphi}{\partial x}(0,s,x_\psi)\frac{\partial x_\psi}{\partial s} \\ &= \frac{\partial\varphi}{\partial x}(0,s,x_\psi)\ [\mu(x_\psi) - \lambda(x_\psi)]. \end{aligned}$$

However, $\mu(x_\psi) < 0$ and $\lambda(x_\psi) > 0$, thus

$$\frac{d\varphi}{ds}(0,s,x_\psi) \neq 0.$$

In addition we have

$$\frac{d\psi}{ds}(0,\tau_1(s,x_\psi),0) = \frac{\partial\psi}{\partial\tau_1}(0,\tau_1(s,x_\psi),0) \times \left[\frac{\partial\tau_1}{\partial s}(s,x_\psi) + \frac{\partial\tau_1}{\partial s}(s,x_\psi)\frac{\partial x_\psi}{\partial s}\right].$$

However,

$$\frac{\partial\psi}{\partial\tau_1}(0,\tau_1(s,x_\psi),0) = -\frac{\partial\psi}{\partial x}(0,\tau_1(s,x_\psi),0)\,\mu(0) \quad\text{and}\quad \frac{\partial x_\psi}{\partial s} = \mu(x_\psi).$$

Finally,

$$\begin{aligned} \frac{d\psi}{ds}(0,\tau_1(s,x_\psi),0) = &-\frac{\mu(0)}{\lambda(x_\psi)}\frac{\partial\psi}{\partial x}(0,\tau_1(s,x_\psi),0) \times \frac{\partial\varphi}{\partial x}(\tau_1(s,x_\psi),s,x_\psi) \\ &\times [\lambda(x_\psi) - \mu(x_\psi)]. \end{aligned}$$

In particular,

$$\frac{d\psi}{ds}(0,\tau_1(s,x_\psi),0) \neq 0.$$

In the same way, we get

$$\frac{d\psi}{ds}(0,s,x_\varphi) \neq 0 \quad\text{and}\quad \frac{d\varphi}{ds}(0,\tau_2(s,x_\varphi),l) \neq 0.$$

We can, according to Lemma 4 of [9], conclude that $\int_0^t \bar{U}(t,s,x)ds$ defines a compact operator.

The second integral component $\int_0^t \bar{w}(t,s,x)ds$ can be treated by similar techniques. Hence $\int_0^t T_2(t-s)BT_2(s)ds$ is compact for any $t \in J$. Thus, from Lemma 2.4, we deduce that $(T_1(t) - T_2(t))$ is a compact operator for any $t \in J$ and that $\{(T_1(t) - T_2(t))Y,\ \|Y\|_H \le 1,\ t \in J\}$ is precompact.

Finally, $(T_3(t) - T_4(t)P)$ is compact. Indeed, we have

$$T_4(t)P = T_3(t)P \quad \text{since} \quad A_3 = (A_4, 0).$$

Then $(T_3(t) - T_4(t)P) = T_3(t)(I - P)$ is an operator of finite dimensional range, thus it is compact. Moreover, $\{(T_3(t) - T_4(t)P)Y,\ \|Y\|_H \leq 1,\ t \in J\}$ is precompact.

Hence, Lemma 2.2 is proved. □

Remark 2.5. Lemma 2.2 holds also in the case where the matrices M, N, D, E, F, and G are dependent on the time.

3. Proofs. To prove our two theorems, we introduce the following new variables:

$$\begin{cases} p = u_t - u_x; & q = u_t + u_x, \\ r = v_t - \eta v_x; & s = v_t + \eta v_x. \end{cases} \tag{3.1}$$

Our system becomes

$$\frac{\partial}{\partial t}\begin{pmatrix} U \\ V \end{pmatrix} + M\frac{\partial}{\partial x}\begin{pmatrix} U \\ V \end{pmatrix} + N(x)\begin{pmatrix} U \\ V \end{pmatrix} = 0, \tag{3.2}$$

where

$$U = \begin{pmatrix} p \\ r \end{pmatrix};V = \begin{pmatrix} q \\ s \end{pmatrix} \quad \text{and} \quad M = \operatorname{diag}(1, \eta; -1, -\eta) \quad \text{with}$$

$$N(x) = \frac{1}{2}\begin{pmatrix} 0 & -b(x) & 0 & -b(x) \\ b(x) & a(x) & b(x) & a(x) \\ 0 & -b(x) & 0 & -b(x) \\ b(x) & a(x) & b(x) & a(x) \end{pmatrix}. \tag{3.3}$$

Now the boundary conditions in (1.3) transform into

$$U(t,0) = -V(t,0), \quad U(t,1) = -V(t,1), \qquad t > 0, \tag{3.4}$$

and the boundary conditions in (1.8) into

$$U(t,0) = -V(t,0),$$

$$\begin{pmatrix} 1 & \eta \\ \alpha & \eta - \alpha \end{pmatrix} U(t,1) = \begin{pmatrix} -1 & \eta \\ \alpha & \eta - \alpha \end{pmatrix} V(t,1), \tag{3.5}$$

and (3.2) will represent system (1.8) if we set $a \equiv 0$ in (3.3). The equivalence of the transformed systems with our initial systems clearly holds.

In view of the proof of Lemma 2.2, we will set

$$A_1 = -M\frac{\partial}{\partial x} - N(x)$$

with, for the proof of Theorem 1.1, the associated boundary conditions

$$U(0) = -V(0), \quad U(1) = -V(1) \tag{3.6}$$

and, for the proof of Theorem 1.4, the associated boundary conditions

$$U(0) = -V(0),$$

$$\begin{pmatrix} 1 & \eta \\ \alpha & \eta-\alpha \end{pmatrix} U(1) = \begin{pmatrix} -1 & \eta \\ \alpha & \eta-\alpha \end{pmatrix} V(1). \tag{3.7}$$

(Note that in this system, $a \equiv 0$.)

Proof of Theorem 1.1. Assume first that $\eta \neq 1$. According to Lemma 2.2 (see also [9]), the semigroup associated with (3.2), (3.4) has the same essential type as the semigroup associated with the system

$$\begin{aligned} &\frac{\partial}{\partial t}\begin{pmatrix} U \\ V \end{pmatrix} + M\frac{\partial}{\partial x}\begin{pmatrix} U \\ V \end{pmatrix} + \widetilde{N}(x)\begin{pmatrix} U \\ V \end{pmatrix} = 0, \\ &U(t,0) = -V(t,0), \quad U(t,1) = -V(t,1), \qquad t > 0, \end{aligned} \tag{3.8}$$

with $\widetilde{N}(x) = \operatorname{diag}(0, a(x), 0, a(x))$. We set

$$A_4 = -M\frac{\partial}{\partial x} - \widetilde{N}(x)$$

with the associated boundary conditions (3.6). It is sufficient (see Remark 2.3) to prove that the spectral abscissa $s(A_4)$ is 0. So, let us consider the system

$$\begin{aligned} &(\lambda - A_4)\begin{pmatrix} U \\ V \end{pmatrix} = 0, \\ &U(0) = -V(0), \qquad U(1) = -V(1). \end{aligned}$$

Computing the solutions of this last system, it is easy to prove that λ is an eigenvalue if and only if it satisfies

$$\left(e^{2\lambda} - 1\right)\left(e^{\frac{2}{\eta}(\lambda+\overline{a})} - 1\right) = 0.$$

Thus, the eigenvalues are

$$\begin{aligned} &\lambda = ik\pi, \ k \in \mathbb{Z}^*, \\ &\lambda = -\overline{a} + ik\eta\pi, \qquad k \in \mathbb{Z}, \end{aligned}$$

where $\overline{a} = \int_0^1 a(x)dx$. This proves that $s(A_4) = 0$ and concludes the proof of (i).

Let us now consider the case $\eta = 1$. We set for simplicity $\{x, a(x) \neq 0\} = [\alpha, \beta]$ and $\{x, b(x) \neq 0\} = [\delta, \gamma]$ with $\beta < \delta$ and $(\alpha, \beta, \delta, \gamma) \in ([0,1])^4$.

According to Lemma 2.2, let $A_4 = -M\frac{\partial}{\partial x} - N_0(x)$ be the reduced operator associated with A_1, where

$$N_0(x) = \frac{1}{2}\begin{pmatrix} 0 & -b(x) & 0 & 0 \\ b(x) & a(x) & 0 & 0 \\ 0 & 0 & 0 & -b(x) \\ 0 & 0 & b(x) & a(x) \end{pmatrix}. \tag{3.9}$$

We denote by $T_4(t)$ the C_0-semigroup generated by A_4.

Lemma 2.2 implies that

$$\omega_{ess}(T_1) = \omega_{ess}(T_4).$$

Now we compute $s(A_4)$. Given $Y = (U, V) \in D(A_4)$,

$$(\lambda I - A_4) \begin{pmatrix} U \\ V \end{pmatrix} = 0$$

or

$$\begin{cases} \dfrac{dU}{dx}(x) = -(\lambda I + N_{11}(x))U(x), \\ \dfrac{dV}{dx}(x) = (\lambda I + N_{11}(x))V(x), \end{cases}$$

where

$$N_{11}(x) = \frac{1}{2} \begin{pmatrix} 0 & -b(x \\ b(x) & a(x) \end{pmatrix}.$$

Solving the differential system above and taking into account the boundary conditions leads to

$$P(\lambda)V(0) := \begin{pmatrix} (e^{-\lambda} - e^{\lambda}) \cos \frac{\bar{b}}{2} & (e^{-\lambda - \frac{\bar{a}}{2}} + e^{\lambda + \frac{\bar{a}}{2}}) \sin \frac{\bar{b}}{2} \\ -(e^{-\lambda} + e^{\lambda}) \sin \frac{\bar{b}}{2} & (e^{-\lambda - \frac{\bar{a}}{2}} + e^{\lambda + \frac{\bar{a}}{2}}) \cos \frac{\bar{b}}{2} \end{pmatrix} V(0) = 0.$$

Consequently, it is clear that

$$\begin{aligned} \lambda \in \sigma(A_4) &\Leftrightarrow \det P(\lambda) = 0, \\ &\Leftrightarrow e^{4\lambda} - (e^{-\bar{a}} + 1) \cos\left(\bar{b}\right) e^{2\lambda} + e^{-\bar{a}} = 0. \end{aligned}$$

Let $\delta = (e^{-\bar{a}} + 1)^2 \cos^2\left(\bar{b}\right) - 4e^{-\bar{a}}$. Thus, if we put

$$\begin{cases} x_1 = \frac{1}{2}(e^{-\bar{a}} + 1) \cos\left(\bar{b}\right) + \dfrac{\sqrt{\delta}}{2}, \\ x_2 = \frac{1}{2}(e^{-\bar{a}} + 1) \cos\left(\bar{b}\right) - \dfrac{\sqrt{\delta}}{2}, \end{cases}$$

then we have

$$s(A_4) = \begin{cases} \left.\begin{array}{ll} \frac{1}{2} \ln x_1 & \text{if } \cos\left(\bar{b}\right) \geq 0, \\ \frac{1}{2} \ln(-x_2) & \text{if } \cos\left(\bar{b}\right) \leq 0 \end{array}\right\} \text{if } \delta \geq 0, \\ -\frac{1}{4}\bar{a} = -\frac{1}{4} \int_0^1 a(t)dt \qquad \text{if } \delta < 0. \end{cases}$$

Remark that if $\bar{a} \leq 0$, then $s(A_4) \geq 0$ and we deduce the following.

If $\bar{a} > 0$, then $s(A_4) \leq 0$ and $s(A_4) = 0$ if and only if $\cos\left(\bar{b}\right) = \pm 1$, that is, if and only if $\bar{b} \in \pi\mathbb{Z}$.

A simple computation shows that the eigenvalues of A_4 are distributed on at most two vertical axes.

We conclude with the help of the following result.

LEMMA 3.1 (Renardy [10, Theorem 1, p. 1300]). *Let H be a Hilbert space and let $L = L_0 + B$ be the infinitesimal generator of a C_0-semigroup of operators in H. Assume that L_0 is normal and B is bounded. Assume that there exists a number $M > 0$ and an integer n such that the following hold:*

(a) *If $\lambda \in \sigma(L_0)$ and $|\lambda| > M - 1$, then λ is an isolated eigenvalue of finite multiplicity.*

(b) *If $|z| > M$, then the number of eigenvalues of L_0 in the unit disk centered at z (counted by multiplicity) does not exceed n.*

Then $\omega_{ess}(e^{Lt}) \leq s(L)$.

We apply this lemma with $L = A_4$ (defined by (2.5) and (2.7)), $L_0 = -M\frac{\partial}{\partial x}$ with $D(L_0) = D(A_4)$ and $B = N_0$. Since M is a constant matrix, it is easy to see that L_0 is normal. Its eigenvalues are $\lambda_k = ik\pi, k \in \mathbb{Z}$, and their (algebraic and geometric) multiplicities are equal to 2. We can then take $n = 2$ and since $|\lambda_{k+1} - \lambda_k| = \pi$, assertion (b) in Lemma 3.1 is satisfied. Thus, $\omega_{ess}(T_4) \leq s(A_4) < 0$. Note that, according to the definition of the essential spectral radius and the previous inequality, one deduces that $s(A_4) = \omega_{ess}(T_4)$. □

Proof of Theorem 1.4. (i) Assume first that $\eta = \frac{2p+1}{q}$ for some $(p, q) \in \mathbb{Z} \times \mathbb{Z}^*$ with $\eta \neq 1$.

As in the previous proof, computing the essential type amounts to computing the eigenvalues λ of the reduced system, namely,

$$\lambda \begin{pmatrix} U \\ V \end{pmatrix} + M\frac{\partial}{\partial x}\begin{pmatrix} U \\ V \end{pmatrix} = 0$$

with the boundary conditions

$$U(0) = -V(0),$$

$$\begin{pmatrix} 1 & \eta \\ \alpha & \eta - \alpha \end{pmatrix} U(1) = \begin{pmatrix} -1 & \eta \\ \alpha & \eta - \alpha \end{pmatrix} V(1).$$

It is then easy to see that λ is an eigenvalue if and only if it satisfies the equation

$$\alpha \sinh\lambda \sinh\frac{\lambda}{\eta} + \eta \sinh\lambda \cosh\frac{\lambda}{\eta} + \alpha\eta \cosh\lambda \cosh\frac{\lambda}{\eta} = 0. \tag{3.10}$$

To prove the assertion of the theorem, we proceed by contradiction. Assume that there exists a sequence (λ_n) of eigenvalues such that

$$\lim_{n\to\infty} \operatorname{Re}\lambda_n = 0. \tag{3.11}$$

Let us set $\lambda_n = x_n + iy_n$. Using the relations

$$\sinh(a + ib) = \sinh a \cos b + i \cosh a \sin b,$$
$$\cosh(a + ib) = \cosh a \cos b + i \sinh a \sin b,$$

(3.10), and (3.11), we get

$$-\alpha \sin y_n \sin\frac{y_n}{\eta} + i\eta \sin y_n \cos\frac{y_n}{\eta} + \alpha\eta \cos y_n \cos\frac{y_n}{\eta} \underset{n\to\infty}{\to} 0.$$

This last condition is equivalent to

$$\sin y_n \cos \frac{y_n}{\eta} \underset{n\to\infty}{\to} 0 \tag{3.12}$$

and

$$\eta \cos y_n \cos \frac{y_n}{\eta} - \sin y_n \sin \frac{y_n}{\eta} \underset{n\to\infty}{\to} 0. \tag{3.13}$$

From (3.12), it follows that either $\underset{n\to\infty}{\to} 0$ or $\cos \frac{y_n}{\eta} \underset{n\to\infty}{\to} 0$. If one of the alternatives holds, (3.13) will imply the second one. We deduce that (3.12)–(3.13) are equivalent to

$$\sin y_n \underset{n\to\infty}{\to} 0 \text{ and } \cos \frac{y_n}{\eta} \underset{n\to\infty}{\to} 0. \tag{3.14}$$

Now, if (y_n) is bounded, there exists a subsequence that converges to y such that

$$\sin y = 0 \text{ and } \cos \frac{y}{\eta} = 0.$$

But this is possible if and only if there exists $(k,j) \in \mathbb{Z}^2$ such that $\eta = \frac{2k}{2j+1}$. This contradicts our assumption on η. It follows that (y_n) is unbounded and we may assume that $\mid y_n \mid \underset{n\to\infty}{\to} \infty$. In this case, (3.14) is equivalent to the existence of a sequence $(k_n, j_n)_n \subset \mathbb{Z} \times \mathbb{Z}$ such that

$$k_n - \eta \left(j_n + \frac{1}{2} \right) \to 0 \text{ as } n \to \infty$$

or equivalently

$$k_n - \eta j_n \to \frac{\eta}{2} \text{ as } n \to \infty. \tag{3.15}$$

Let us consider the set

$$G = \left\{ k - \eta j, (k,j) \in \mathbb{Z}^2 \right\}. \tag{3.16}$$

It is an additive subgroup of $\mathbb{R}$. A well-known result of algebra asserts that either there exists a real number $a > 0$ such that $G = a\mathbb{Z}$ or $\overline{G} = \mathbb{R}$.

The first alternative holds if and only if $a \in \mathbb{Q}$ since $\mathbb{Z} \subset G$. On the other hand, G is closed in $\mathbb{R}$ and (3.15) holds if and only if $\frac{\eta}{2} \in G$. This means that there exists $j \in \mathbb{Z}$ such that $\eta = 2aj = \frac{2p}{q}$ for some $(p,q) \in \mathbb{Z} \times \mathbb{Z}^*$. But this contradicts the form of η.

The second alternative holds if and only if $\eta \in \mathbb{R} \backslash \mathbb{Q}$. So the sufficiency part is proved.

Assume now that $\eta \neq \frac{2p+1}{q}$ for all $(p,q) \in \mathbb{Z} \times \mathbb{Z}^*$. Equation (3.10) rewrites

$$(\alpha + \eta + \alpha\eta) e^{2\lambda(1+\frac{1}{\eta})} - (\alpha + \eta - \alpha\eta) e^{2\frac{\lambda}{\eta}} - (\alpha - \eta - \alpha\eta) e^{2\lambda} + \alpha - \eta + \alpha\eta = 0.$$

We set

$$\begin{aligned} f(\lambda) &= a_0 e^{2\lambda(1+\frac{1}{\eta})} - a_1 e^{2\frac{\lambda}{\eta}} - a_2 e^{2\lambda} + a_3, \quad \lambda \in \mathbb{C}, \\ a_0 &= \alpha + \eta + \alpha\eta, \quad a_1 = \alpha + \eta - \alpha\eta, \\ a_2 &= \alpha - \eta - \alpha\eta, \quad a_3 = \alpha - \eta + \alpha\eta. \end{aligned}$$

We will apply Rouché's theorem to f to prove that there exists a sequence of eigenvalues such that (3.11) holds true. According to the previous computations, there exists a sequence $(k_n, j_n) \in \mathbb{Z}^2$ such that

$$\varepsilon_n := (2k_n - (2j_n + 1)\eta)\,\frac{\pi}{2} \underset{n\to\infty}{\to} 0. \tag{3.17}$$

Let us set $\lambda_n = ik_n\pi$. To achieve our goal, it suffices to prove that there exists a sequence of positive numbers r_n such that $\lim_{n\to\infty} r_n = 0$ and

$$|f(\lambda) - f'(\lambda_n)(\lambda - \lambda_n)| < |f'(\lambda_n)(\lambda - \lambda_n)| \text{ if } |\lambda - \lambda_n| = r_n. \tag{3.18}$$

To estimate the left-hand term in the previous inequality, we have from Taylor's formula

$$|f(\lambda) - f'(\lambda_n)(\lambda - \lambda_n)| \le |f(\lambda_n)| + \sum_{p\ge 2} \frac{\left|f^{(p)}(\lambda_n)\right|}{p!} \left|\lambda - \lambda_n\right|^p. \tag{3.19}$$

We first have, using (3.17) and noting that $a_0 - a_1 = -\, a_2 + a_3 = 2\alpha\eta$

$$\begin{aligned} |f(\lambda_n)| &= \left|(a_0 - a_1)e^{2i\frac{\varepsilon_n}{\eta}} + a_2 - a_3\right| \\ &= 2\alpha\eta\left|e^{2i\frac{\varepsilon_n}{\eta}} - 1\right| \le 4\alpha\left|\varepsilon_n\right|. \end{aligned} \tag{3.20}$$

Moreover, for $p \ge 1$

$$f^{(p)}(\lambda_n) = -2^p\left[\left(a_0\left(1 + \frac{1}{\eta}\right)^p - \frac{a_1}{\eta^p}\right)e^{2i\frac{\varepsilon_n}{\eta}\pi} + a_2\right] \tag{3.21}$$

and for any $p \ge 2$

$$\begin{aligned} \left|f^{(p)}(\lambda_n)\right| &= 2^p\left|\left(a_0\left(1 + \frac{1}{\eta}\right)^p - \frac{a_1}{\eta^p}\right)e^{2i\frac{\varepsilon_n}{\eta}\pi} + a_2\right| \\ &\le 2^p\left(|a_0|\left(1 + \frac{1}{\eta}\right)^p + \frac{|a_1|}{\eta^p} + |a_2|\right). \end{aligned} \tag{3.22}$$

Thus (3.19), (3.20), and (3.22) imply

$$\begin{aligned} |f(\lambda) - f'(\lambda_n)(\lambda - \lambda_n)| &\le |a_0|\left(e^{2r_n\left(1+\frac{1}{\eta}\right)} - 2r_n\left(1 + \frac{1}{\eta}\right) - 1\right) \\ &\quad + |a_1|\left(e^{2\frac{r_n}{\eta}} - 2\frac{r_n}{\eta} - 1\right) + |a_2|\left(e^{2r_n} - 2r_n - 1\right) \\ &\le 4\left(\left(1 + \frac{1}{\eta}\right)^2|a_0| + \frac{|a_1|}{\eta^2} + |a_2|\right)r_n^2 \quad \text{for } n \ge n_0. \end{aligned}$$

On the other hand, it is easy to see, using the definition of the a_i, that

$$\begin{aligned} |f'(\lambda_n)(\lambda - \lambda_n)| &= 2\Bigg[\left(\frac{a_1 - a_0}{\eta} - a_0\right)^2 - 2a_2\left(\frac{a_1 - a_0}{\eta} - a_0\right)\cos\left(2\frac{\varepsilon_n}{\eta}\right) \\ &\qquad + a_2^2\Bigg]^{1/2} r_n \\ &\ge 2\alpha r_n. \end{aligned}$$

We then need, in order to satisfy (3.18), to find r_n such that, for n sufficiently large

$$4\alpha\,|\varepsilon_n| + 4\left(\left(1+\frac{1}{\eta}\right)^2 |a_0| + \frac{|a_1|}{\eta^2} + |a_2|\right) r_n^2 < 2\alpha r_n.$$

Choosing $r_n = \sqrt{|\varepsilon_n|}$, there will exist $N \in \mathbb{N}$ such that for all $n \geq N$

$$|f(\lambda) - f'(\lambda_n)(\lambda-\lambda_n)| < |f'(\lambda_n)(\lambda-\lambda_n)| \quad \text{for} \quad |\,\lambda - \lambda_n| = \sqrt{|\varepsilon_n|}\,.$$

This ends the proof of the point (i) in Theorem 1.4.

(ii) Assume that $\eta = 1$.

As in the previous proof, computing the essential type amounts to computing the eigenvalues λ of the reduced system, namely,

$$\lambda\begin{pmatrix} U \\ V \end{pmatrix} + M\frac{\partial}{\partial x}\begin{pmatrix} U \\ V \end{pmatrix} + N_0\begin{pmatrix} U \\ V \end{pmatrix} = 0$$

with the boundary conditions

$$U(0) = -V(0),$$
$$\begin{pmatrix} 1 & 1 \\ \alpha & 1-\alpha \end{pmatrix} U(1) = \begin{pmatrix} -1 & 1 \\ \alpha & 1-\alpha \end{pmatrix} V(1).$$

Here

$$N_0(x) = \frac{1}{2}\begin{pmatrix} 0 & -b(x) & 0 & 0 \\ b(x) & 0 & 0 & 0 \\ 0 & 0 & 0 & -b(x) \\ 0 & 0 & b(x) & 0 \end{pmatrix}.$$

The eigenvalues satisfy the equation

$$e^{4\lambda} + \frac{4\alpha}{2\alpha+1}\sin(\overline{b})e^{2\lambda} + \frac{2\alpha-1}{2\alpha+1} = 0.$$

Let $\delta = \frac{4\alpha^2}{(1+2\alpha)^2}\sin^2(\overline{b}) - \frac{2\alpha-1}{2\alpha+1}$. Then we have the following.

- If $\delta < 0$, then all solutions satisfy $e^{2\,\mathrm{Re}\,\lambda} = \left|e^{2\lambda}\right| = \frac{2\alpha-1}{2\alpha+1}$. It follows that $\mathrm{Re}\,\lambda = \frac{1}{2}\ln\frac{2\alpha-1}{2\alpha+1} < 0$.
- If $\delta \geq 0$, then $e^{2\,\mathrm{Re}\,\lambda} = |e^{2\lambda}| = |-\frac{2\alpha}{2\alpha+1}\sin(\overline{b}) \pm \sqrt{\delta}|$. Thus

$$\mathrm{Re}\,\lambda = \frac{1}{2}\ln\left|-\frac{2\alpha}{2\alpha+1}\sin(\overline{b}) \pm \sqrt{\delta}\right| \leq 0.$$

Consequently,

$$s(A_4) = \begin{cases} \frac{1}{2}\ln(-\frac{2\alpha}{2\alpha+1}\sin(\overline{b}) + \sqrt{\delta}) \quad \text{if } \sin(\overline{b}) \leq 0 & \\ & \text{if } \delta \geq 0, \\ \frac{1}{2}\ln(\frac{2\alpha}{2\alpha+1}\sin(\overline{b}) + \sqrt{\delta}) \quad \text{if } \sin(\overline{b}) \geq 0 & \\ \frac{1}{2}\ln\frac{2\alpha-1}{2\alpha+1} & \text{if } \delta < 0. \end{cases}$$

A simple computation shows that

$$s(A_4) = 0 \text{ if and only if } \sin(\bar{b}) = \pm 1.$$

We deduce that

$$s(A_4) < 0 \text{ if and only if } \bar{b} \neq (2k+1)\frac{\pi}{2} \quad \forall\ k \in \mathbb{Z}.$$

Next we use Lemma 3.1 and the definition of essential spectral radius; we get (in the same way as in the proof of Theorem 1.1)

$$\omega_{ess}(T_1) = \omega_{ess}(T_4) = s(A_4).$$

We conclude that the semigroup $T_1(t)$ is exponentially stable in an invariant subspace of finite codimension.

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