On "quasi-Richards" equation and finite volume approximation of two-phase flow with unlimited air mobility

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   - Two-phase model in porous medium.
     High/infinite mobility μ of the “air” phase?
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   - Asymptotics of the scheme as $\mu \to 0$
   - Numerical illustrations

Models

Assumptions about groundwater flow

Water and air	incompressible phases
Porous medium	homogeneous and isotropic
Gravity	neglected
Source term	of a special form

Gravity neglected and source term of a special form } lower bound on saturation

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Porous medium — homogeneous and isotropic

Gravity — neglected
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} lower bound on saturation

Richards model

$$\begin{cases} u_t - \mathrm{div}(k_w(u)\nabla p) = s_w, \\ u = p_c{}^{-1}(p_{atm} - p), \end{cases}$$

Two-phase model

$$\begin{cases} u_t - \mathrm{div}(k_w(u)\nabla p) = s_w \\ (1-u)_t - \mathrm{div}(\mu k_a(u)\nabla(p + p_c(u))) = s_a \end{cases}$$

where $\mu :=$ Ratio between the phase mobilities (we want $\mu \to \infty$)

Assumptions

1. Ω is a polygonal subset of $\mathbb{R}^d$, $d = 2$ or 3,
2. $T > 0$ is given,
3. $u_m \in (0, 1)$,
4. $u_0 \in L^\infty(\Omega)$ and $u_m \leq u_0(x) \leq 1$ for a.e $x \in \Omega$,
5. $c \in L^\infty(\Omega \times (0, T))$, $u_m \leq c(t, x) \leq 1$ for a.e. $(x, t) \in \Omega \times (0, T)$,
6. $\overline{s} \in L^2(\Omega \times (0, T))$, $\overline{s} \geq 0$, $\underline{s} \in L^2(\Omega \times (0, T))$, $\underline{s} \geq 0$ and $\displaystyle\int_\Omega (\overline{s}(x, t) - \underline{s}(x, t))dx = 0$ a.e.,
7. $k_w \in \mathcal{C}^0([0, 1], \mathbb{R})$, k_w is non-decreasing with $k_w(0) = 0$, $k_w(1) = 1$ and $k_w(u_m) > 0$,
8. $k_a \in \mathcal{C}^0([0, 1], \mathbb{R})$, k_a is non-increasing with $k_a(1) = 0$, $k_a(0) = 1$ and $k_a(s) > 0$ for all $s \in [0, 1)$,
9. $p_c \in \mathcal{C}^0([u_m, 1], \mathbb{R})$, $p_c \in \mathrm{Lip}_{loc}([u_m, 1), \mathbb{R})$, p_c is strictly decreasing

Mathematical setting of the problem

Set

$$f_\mu(u) := \frac{k_w(u)}{k_w(u) + \mu k_a(u)} \longrightarrow_{\mu\to\infty} 1\!\!1_{[u=1]}$$

Two-phase problem: find (u, p) such that:

$$\begin{cases} u_t - \mathrm{div}(k_w(u)\nabla p) = f_\mu(c)\,\overline{s} - f_\mu(u)\,\underline{s} \quad \text{on } \Omega\times(0,T), \end{cases}$$

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Tools: vanishing viscosity regularization or finite volume scheme

The "quasi-Richards" equation

Theorem (Eymard, Henry, Hilhorst'09, DCDS-S'12.)

There exist solutions (u^μ, p^μ) for the two-phase flow problem that obey uniform estimates: lower bound u_m on the saturations u^μ, $L^2(0,T;H^1)$ bound on the pressures p^μ and on the 1/2-Kirchoff transform $\zeta(u^\mu)$,

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$$\begin{cases} u_t - \mathrm{div}(k_w(u)\nabla p) = s_w \\ \nabla(p + p_c(u)) = 0 \ \ \textit{a.e. on the set } [u < 1], \end{cases}$$

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$$\begin{cases} u_t - \mathrm{div}(k_w(u)\nabla p) = s_w = \overline{s}\, 1\!\!1_{[c=1]} - \theta\, \underline{s}\, 1\!\!1_{[u=1]} \\ \nabla(p + p_c(u)) = 0 \ \ a.e. \ on \ the \ set \ [u<1], \end{cases}$$

where θ is an unknown $[0,1]$-valued function .

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Thus: solution of the quasi-Richards eqn. is a triple (u,p,θ) with $\nabla p = -\nabla p_c(u)$ on $[u<1]$ and with θ defined on $[u=1]$.
Regularity:
u is $[u_m, 1]$-valued with $\zeta(u) \in L^2(0,T;H^1)$, $p \in L^2(0,T;H^1)$.

Is quasi-Richards well-posed? Is it different from Richards?

- Richards is well-posed: Alt, Luckhaus'83 .
  L^1 contraction inequality holds. ($\Rightarrow$ uniqueness, stability)
- Existence of sols to quasi-Richards: Eymard, Henry, Hilhorst .
  Uniqueness ? Relation to the unique solution of Richards ?

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Theorem (A., Eymard, Ghilani, Marhraoui'12)

Assume $u, \hat{u}$ are weak solutions of the quasi-Richards equation corresponding to data $(u_0, \overline{s})$ and $(\hat{u}_0, \widehat{\overline{s}})$. Then we have the following incomplete contraction inequality : for a.e. t,

$$\int_\Omega (u-\hat{u})^+(t,\cdot) \leq \int_\Omega (u_0-\hat{u}_0)^+ + \int_0^t\int_\Omega (\overline{s}-\widehat{\overline{s}})^+ + \int_0^t\int_{[u=1=\hat{u}]} \overline{s}. \quad (1)$$

Proof: use renormalized solutions of Plouvier-Debaigt, Gagneux .

Is quasi-Richards well-posed? Is it different from Richards?

Theorem (A., Eymard, Ghilani, Marhraoui'12, dedicated to M.Madaune-Tort)

Assume there is no water injection: $\bar{s}1\!\!1_{[c=1]} = 0$ *a.e. on* $(0,T)\times\Omega$ *(with* $c = c(t,x)$ *the saturation in water of the injected fluid).*
Then for every datum u_0 *there exists a unique* u *such that* (u,p,θ) *is a solution of the quasi-Richards equation.*

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Then for every datum u_0 *there exists a unique* u *such that* (u,p,θ) *is a solution of the quasi-Richards equation.*
Moreover, in absence of water injection we have $\theta\, \underline{s} = 0$ *a.e. (no water production!); and the saturation* u *given by quasi-Richards eqn coincides with the unique solution of the Richards eqn.*

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In general, we do not expect that quasi-Richards and Richards coincide:

- Physical reasons: p_{atm} is not the good pressure for air when air is captured by saturated water phase
- While uniqueness of u in the triple (u,p,θ) can be hoped for, we do not expect uniqueness of (p,θ) in the saturated set $[u=1]$.

More work needed to understand quasi-Richards !

Finite volume scheme

Write $k_w(u) = f_\mu(u) M_\mu(u)$, $M_\mu = k_w + \mu k_a$. Set $\delta_{K,L}^{n+1}(Z_{\mathcal{D}}) = Z_L^{n+1} - Z_K^{n+1}$.

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The scheme is: find $U_\mathcal{D}=(U_K^n)_{n,K}$, $P_\mathcal{D}=(P_K^n)_{n,K}$ satisfying

$$\frac{U_K^{n+1}-U_K^n}{\delta t^n}m_K=\sum_{L\in\mathcal{N}_K}\tau_{K|L}f_\mu(U_{K|L}^{n+1})M_\mu(\bar{U}_{K|L}^{n+1})\delta_{K,L}^{n+1}(P_\mathcal{D})-\text{source}$$

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$$\frac{(1 - U_K^{n+1}) - (1 - U_K^n)}{\delta t^n} m_K = \text{air source}$$

Kirchoff transform: $\downarrow$ $g' = k_a {p_c}'$

$$+ \sum_{L \in \mathcal{N}_K} \tau_{K|L} (1 - f_\mu(U_{K|L}^{n+1})) M_\mu(\bar{U}_{K|L}^{n+1}) \delta_{K,L}^{n+1}(P_\mathcal{D}) - \mu \sum_{L \in \mathcal{N}_K} \tau_{K|L} \delta_{K,L}^{n+1}(g(U_\mathcal{D}))$$

(+ discretization of IC u_0, + normalization of $P_\mathcal{D}$ due to Neumann BC)

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where

- $U^{n+1}_{K|L}$ is the upwind value : $U^{n+1}_{K|L} = \begin{cases} U^{n+1}_L & \text{if } \delta^{n+1}_{K,L}(P_\mathcal{D}) \geq 0, \\ U^{n+1}_K & \text{otherwise,} \end{cases}$

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$$\frac{(1 - U_K^{n+1}) - (1 - U_K^n)}{\delta t^n} m_K = \text{air source}$$

Kirchoff transform: $\downarrow$ $g' = k_a p_c'$

$$+ \sum_{L \in \mathcal{N}_K} \tau_{K|L} (1 - f_\mu(U_{K|L}^{n+1})) M_\mu(\bar{U}_{K|L}^{n+1}) \delta_{K,L}^{n+1}(P_\mathcal{D}) - \mu \sum_{L \in \mathcal{N}_K} \tau_{K|L} \delta_{K,L}^{n+1}(g(U_\mathcal{D}))$$

(+ discretization of IC u_0, + normalization of $P_\mathcal{D}$ due to Neumann BC)

where

- $U_{K|L}^{n+1}$ is the upwind value : $U_{K|L}^{n+1} = \begin{cases} U_L^{n+1} & \text{if } \delta_{K,L}^{n+1}(P_\mathcal{D}) \geq 0, \\ U_K^{n+1} & \text{otherwise,} \end{cases}$
- $\bar{U}_{K|L}^{n+1} \in [\min(U_K^{n+1}, U_L^{n+1}), \max(U_K^{n+1}, U_L^{n+1})]$ is the auxiliary value:

$$k_a(\bar{U}_{K|L}^{n+1})\, \delta_{K,L}^{n+1}(p_c(U_\mathcal{D})) = \delta_{K,L}^{n+1}(g(U_\mathcal{D})) \text{ i.e., } k_a(\bar{U}_{K|L}^{n+1}) = \frac{g(U_L^{n+1}) - g(U_K^{n+1})}{p_c(U_L^{n+1}) - p_c(U_K^{n+1})}.$$

Properties of the scheme

- the choice $\bar{U}_{K|L}^{n+1}$ makes appear $\mu k_a(\bar{U}_{K|L}^{n+1})\delta_{K,L}^{n+1}(P_{\mathcal{D}} - p_c(U_{\mathcal{D}})) \Rightarrow$ uniform in μ, h (discrete) estimates as for Eymard, Henry, Hilhorst
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- numerical tests: $\|U_{\mathcal D}^{\mu}-U_{\mathcal D}^{\text{Rich}}\|_{1\text{ or }\infty}\sim \dfrac{const}{\mu}+residual(h)\Rightarrow$ Asymptotics of the scheme as $\mu\to\infty$: a scheme for Richards ? In the gradually saturated regime $(u\le u_M<1)$ we find

$$\frac{U_K^{n+1}-U_K^n}{\delta t^n}m_K-\sum_{L\in\mathcal N_K}\tau_{K|L}k_w(U_{K|L}^{n+1})\frac{k_a(\bar U_{K|L}^{n+1})}{k_a(U_{K|L}^{n+1})}\delta_{K,L}^{n+1}(P_{\mathcal D})=0,$$

while the straightforward discretization of Richards equation yields $k_w(U_{K|L}^{n+1})\delta_{K,L}^{n+1}(P_{\mathcal D})$. One can see that $\frac{k_a(\bar U_{K|L}^{n+1})}{k_a(U_{K|L}^{n+1})}\to 1$ as $h\to 0$, so we have an "almost asymptotic preserving" scheme: (limit $\mu\to\infty$ of the two-phase scheme is a "strange" scheme for Richards eqn).

Merci pour votre attention !