On “quasi-Richards” equation and finite volume approximation of two-phase flow with unlimited air mobility

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Introduction and motivation
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- Two-phase model in porous medium.
  High/infinite mobility $\mu$ of the “air” phase?
  Is the classical Richards model appropriate?
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   - Solutions of two-phase flow equations; estimates
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   - Asymptotics of the scheme as $\mu \to 0$
   - Numerical illustrations
Models

Assumptions about groundwater flow

- Water and air: incompressible phases
- Porous medium: homogeneous and isotropic
- Gravity: neglected
- Source term: of a special form

\[ \text{lower bound on saturation} \]
Assumptions about groundwater flow

- Water and air incompressible phases
- Porous medium homogeneous and isotropic
- Gravity neglected
- Source term of a special form

\[
\left\{ \begin{array}{l}
\frac{du}{dt} - \text{div}(k_w(u)\nabla p) = s_w, \\
u = p_c^{-1}(p_{atm} - p),
\end{array} \right.
\]

Richards model

Two-phase model

\[
\left\{ \begin{array}{l}
\frac{du}{dt} - \text{div}(k_w(u)\nabla p) = s_w, \\
(1 - u)\frac{du}{dt} - \text{div}(\mu k_a(u)\nabla (p + p_c(u))) = s_a
\end{array} \right.
\]

where \(\mu := \text{Ratio between the phase mobilities (we want } \mu \rightarrow \infty)\)
Assumptions

1. \( \Omega \) is a polygonal subset of \( \mathbb{R}^d \), \( d = 2 \) or \( 3 \),
2. \( T > 0 \) is given,
3. \( u_m \in (0, 1) \),
4. \( u_0 \in L^\infty(\Omega) \) and \( u_m \leq u_0(x) \leq 1 \) for a.e. \( x \in \Omega \),
5. \( c \in L^\infty(\Omega \times (0, T)) \), \( u_m \leq c(t, x) \leq 1 \) for a.e. \( (x, t) \in \Omega \times (0, T) \),
6. \( \bar{s} \in L^2(\Omega \times (0, T)) \), \( \bar{s} \geq 0 \), \( s \in L^2(\Omega \times (0, T)) \), \( s \geq 0 \) and \( \int_{\Omega} (\bar{s}(x, t) - s(x, t)) dx = 0 \) a.e.,
7. \( k_w \in C^0([0, 1], \mathbb{R}) \), \( k_w \) is non-decreasing with \( k_w(0) = 0 \), \( k_w(1) = 1 \) and \( k_w(u_m) > 0 \),
8. \( k_a \in C^0([0, 1], \mathbb{R}) \), \( k_a \) is non-increasing with \( k_a(1) = 0 \), \( k_a(0) = 1 \) and \( k_a(s) > 0 \) for all \( s \in [0, 1) \),
9. \( p_c \in C^0([u_m, 1], \mathbb{R}) \), \( p_c \in \text{Lip}_{loc}([u_m, 1], \mathbb{R}) \), \( p_c \) is strictly decreasing
Mathematical setting of the problem

Set

\[ f_\mu(u) := \frac{k_w(u)}{k_w(u) + \mu k_a(u)} \quad \xrightarrow{\mu \to \infty} \quad 1_{[u=1]} \]

Two-phase problem: find \((u, \rho)\) such that:

\[
\begin{cases}
  u_t - \text{div}(k_w(u)\nabla \rho) = f_\mu(c) \bar{s} - f_\mu(u) s & \text{on } \Omega \times (0, T), \\
  \nabla \rho \cdot n = 0 & \text{on } \partial \Omega \times (0, T), \\
  \int_\Omega \rho(x, t) \, dx = 0 & \text{on } (0, T), \\
  u(\cdot, 0) = u_0 \geq u_m > 0 & \text{on } \Omega 
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\begin{aligned}
&u_t - \text{div}(k_w(u)\nabla p) = f_\mu(c) \overline{s} - f_\mu(u) s \\
&(1 - u)_t - \text{div}(\mu k_a(u)\nabla (p + p_c(u))) = (1 - f_\mu(c)) \overline{s} - (1 - f_\mu(u)) s
\end{aligned}
\]

on \(\Omega \times (0, T)\),
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    \nabla p \cdot \mathbf{n} &= 0 & \text{on } \partial\Omega \times (0, T), \\
    \nabla (p + p_c(u)) \cdot \mathbf{n} &= 0 & \text{on } \partial\Omega \times (0, T), \\
    \int_{\Omega} p(x, t) \, dx &= 0 & \text{on } (0, T).
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  \nabla p \cdot n = 0 & \text{on } \partial\Omega \times (0, T), \\
  \nabla(p + p_c(u)) \cdot n = 0 & \text{on } \partial\Omega \times (0, T), \\
  \int_{\Omega} p(x, t) \, dx = 0 & \text{on } (0, T), \\
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Tools: vanishing viscosity regularization or finite volume scheme
The “quasi-Richards” equation

**Theorem (Eymard, Henry, Hilhorst'09, DCDS-S'12.)**

There exist solutions \((u^\mu, p^\mu)\) for the two-phase flow problem that obey uniform estimates: lower bound \(u_m\) on the saturations \(u^\mu\), \(L^2(0, T; H^1)\) bound on the pressures \(p^\mu\) and on the \(1/2\)-Kirchoff transform \(\zeta(u^\mu)\),
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\begin{cases}
  u_t - \text{div}(k_w(u) \nabla p) = s_w \\
  \nabla (p + p_c(u)) = 0 \quad \text{a.e. on the set } [u < 1],
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\[
\begin{cases}
  u_t - \text{div}(k_w(u)\nabla p) = s_w = \bar{s} \mathbb{1}_{[c=1]} - \theta \bar{s} \mathbb{1}_{[u=1]} \\
  \nabla(p + p_c(u)) = 0 \text{ a.e. on the set } [u < 1],
\end{cases}
\]

where \(\theta\) is an unknown \([0, 1]\)-valued function.
The “quasi-Richards” equation

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There exist solutions \((u^\mu, p^\mu)\) for the two-phase flow problem that obey **uniform estimates**: lower bound \(u_m\) on the saturations \(u^\mu\), \(L^2(0, T; H^1)\) bound on the pressures \(p^\mu\) and on the 1/2-Kirchhoff transform \(\zeta(u^\mu)\), estimate on \(\mu \int \int k_a(u^\mu) \left| \nabla (p^\mu + p_c(u^\mu)) \right|^2\); any accumulation point of \((u^\mu, p^\mu)\) as \(\mu \to \infty\) satisfies

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\]

where \(\theta\) is an unknown \([0, 1]\)-valued function.

Thus: solution of the quasi-Richards eqn. is a **triple** \((u, p, \theta)\) with \(\nabla p = -\nabla p_c(u)\) on \([u < 1]\) and with \(\theta\) defined on \([u = 1]\).

**Regularity:**

\(u\) is \([u_m, 1]\)-valued with \(\zeta(u) \in L^2(0, T; H^1)\), \(p \in L^2(0, T; H^1)\).
Is quasi-Richards well-posed? Is it different from Richards?

- Richards is well-posed: Alt, Luckhaus’83.  
  $L^1$ contraction inequality holds. ($\Rightarrow$ uniqueness, stability)
- Existence of sols to quasi-Richards: Eymard, Henry, Hilhorst.
  Uniqueness? Relation to the unique solution of Richards?
Is quasi-Richards well-posed? Is it different from Richards?

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**Theorem (A., Eymard, Ghilani, Marhraoui’12)**

Assume \(u, \hat{u}\) are weak solutions of the quasi-Richards equation corresponding to data \((u_0, \bar{s})\) and \((\hat{u}_0, \hat{s})\). Then we have the following *incomplete* contraction inequality: for a.e. \(t\),

\[
\int_{\Omega} (u - \hat{u})^+(t, \cdot) \leq \int_{\Omega} (u_0 - \hat{u}_0)^+ + \int_0^t \int_{\Omega} (\bar{s} - \hat{s})^+ + \int_0^t \int_{\{u=1=\hat{u}\}} \bar{s}. \quad (1)
\]

Proof: use renormalized solutions of Plouvier-Debaigt, Gagneux.
Is quasi-Richards well-posed? Is it different from Richards?

**Theorem (A., Eymard, Ghilani, Marhraoui’12, dedicated to M.Madaune-Tort)**

*Assume there is no water injection: \( \bar{s}1_{[c=1]} = 0 \) a.e. on \((0, T) \times \Omega\) (with \(c = c(t,x)\) the saturation in water of the injected fluid).

Then for every datum \(u_0\) there exists a unique \(u\) such that \((u,p,\theta)\) is a solution of the quasi-Richards equation.*
Theorem (A., Eymard, Ghilani, Marhraoui'12, dedicated to M. Madaune-Tort)

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Then for every datum \( u_0 \) there exists a unique \( u \) such that \((u, p, \theta)\) is a solution of the quasi-Richards equation.

Moreover, in absence of water injection we have \( \theta s = 0 \) a.e. (no water production!); and the saturation \( u \) given by quasi-Richards eqn coincides with the unique solution of the Richards eqn.
Is quasi-Richards well-posed? Is it different from Richards?

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Moreover, in absence of water injection we have \( \theta s = 0 \) a.e. (no water production!); and the saturation \( u \) given by quasi-Richards eqn coincides with the unique solution of the Richards eqn.

In general, we do not expect that quasi-Richards and Richards coincide:

- Physical reasons: \( p_{atm} \) is not the good pressure for air when air is captured by saturated water phase

- While uniqueness of \( u \) in the triple \((u, p, \theta)\) can be hoped for, we do not expect uniqueness of \((p, \theta)\) in the saturated set \([u = 1]\).

More work needed to understand quasi-Richards!
Write $k_w(u) = f_{\mu}(u)M_\mu(u)$, $M_\mu = k_w + \mu k_a$. Set $\delta_{K,L}^{n+1}(Z_D) = Z_L^{n+1} - Z_K^{n+1}$.
Write \( k_w(u) = f_\mu(u) M_\mu(u) \), \( M_\mu = k_w + \mu k_a \). Set \( \delta^{n+1}_{K,L} (Z_D) = Z^{n+1}_L - Z^{n+1}_K \).

The scheme is: find \( U_D = (U^K_n)_n, K, P_D = (P^K_n)_n, K \) satisfying

\[
\frac{U^{n+1}_K - U^n_K}{\delta t^n} m_K = \sum_{L \in \mathcal{N}_K} \tau_{K|L} f_\mu(U^{n+1}_{K|L}) M_\mu (\bar{U}^{n+1}_{K|L}) \delta^{n+1}_{K,L} (P_D) - \text{source}
\]
Finite volume scheme

Write $k_w(u) = f_\mu(u) M_\mu(u)$, $M_\mu = k_w + \mu k_a$. Set $\delta_{K,L}^{n+1}(Z_D) = Z_L^{n+1} - Z_K^{n+1}$.

The scheme is: find $U_D = (U^n_K)_{n,K}$, $P_D = (P^n_K)_{n,K}$ satisfying

$$
\frac{U_{K}^{n+1} - U_{K}^{n}}{\delta t^n} m_K = \sum_{L \in \mathcal{N}_K} \tau_{K|L} f_\mu(U_{K|L}^{n+1}) M_\mu(\bar{U}_{K|L}^{n+1}) \delta_{K,L}^{n+1}(P_D) - \text{source}
$$

$$
\frac{(1 - U_{K}^{n+1}) - (1 - U_{K}^{n})}{\delta t^n} m_K = \text{air source} \quad \text{Kirchoff transform: } \downarrow \ g' = k_a p_c'
$$

$$
+ \sum_{L \in \mathcal{N}_K} \tau_{K|L} (1 - f_\mu(U_{K|L}^{n+1})) M_\mu(\bar{U}_{K|L}^{n+1}) \delta_{K,L}^{n+1}(P_D) - \mu \sum_{L \in \mathcal{N}_K} \tau_{K|L} \delta_{K,L}^{n+1}(g(U_D))
$$

(+ discretization of IC $u_0$, + normalization of $P_D$ due to Neumann BC)
Finite volume scheme

Write \( k_w(u) = f_\mu(u) M_\mu(u) \), \( M_\mu = k_w + \mu k_a \). Set \( \delta^{n+1}_{K,L}(Z_D) = Z^{n+1}_L - Z^{n+1}_K \).

The scheme is: find \( U_D = (U^n_K)_{n,K} \), \( P_D = (P^n_K)_{n,K} \) satisfying

\[
\frac{U^{n+1}_K - U^n_K}{\delta t^n} m_K = \sum_{L \in \mathcal{N}_K} \tau_{K\mid L} f_\mu(U^{n+1}_K|L) M_\mu(\bar{U}^{n+1}_{K\mid L}) \delta^{n+1}_{K,L}(P_D) - \text{source}
\]

\[
\frac{(1 - U^{n+1}_K) - (1 - U^n_K)}{\delta t^n} m_K = \text{air source} \quad \text{Kirchoff transform: } \downarrow \quad g' = k_a p_c'
\]

\[
+ \sum_{L \in \mathcal{N}_K} \tau_{K\mid L} (1 - f_\mu(U^{n+1}_K|L)) M_\mu(\bar{U}^{n+1}_{K\mid L}) \delta^{n+1}_{K,L}(P_D) - \mu \sum_{L \in \mathcal{N}_K} \tau_{K\mid L} \delta^{n+1}_{K,L}(g(U_D))
\]

(+ discretization of IC \( u_0 \), + normalization of \( P_D \) due to Neumann BC)

where

- \( U^{n+1}_{K\mid L} \) is the upwind value: \( U^{n+1}_{K\mid L} = \begin{cases} U^{n+1}_L & \text{if } \delta^{n+1}_{K,L}(P_D) \geq 0, \\ U^n_K & \text{otherwise,} \end{cases} \)
Finite volume scheme

Write $k_w(u) = f_\mu(u)M_\mu(u)$, $M_\mu = k_w + \mu k_a$. Set $\delta^{n+1}_{K,L}(Z_D) = Z_L^{n+1} - Z_K^{n+1}$.

The scheme is: find $U_D = (U_K^n)_{n,K}$, $P_D = (P_K^n)_{n,K}$ satisfying

$$\frac{U_K^{n+1} - U_K^n}{\delta t^n} m_K = \sum_{L \in \mathcal{N}_K} \tau_{K|L} f_\mu(U_K^{n+1}) M_\mu(\bar{U}_K^{n+1}) \delta^{n+1}_{K,L}(P_D) - \text{source}$$

$$(1 - U_K^{n+1}) - (1 - U_K^n) m_K = \text{air source} \quad \text{Kirchhoff transform: } \downarrow \quad g' = k_a p_c'$$

$$+ \sum_{L \in \mathcal{N}_K} \tau_{K|L} (1 - f_\mu(U_K^{n+1})) M_\mu(\bar{U}_K^{n+1}) \delta^{n+1}_{K,L}(P_D) - \mu \sum_{L \in \mathcal{N}_K} \tau_{K|L} \delta^{n+1}_{K,L}(g(U_D))$$

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where

- $U_K^{n+1}$ is the upwind value: $U_K^{n+1} = \begin{cases} U_L^{n+1} & \text{if } \delta^{n+1}_{K,L}(P_D) \geq 0, \\ U_K^{n+1} & \text{otherwise}, \end{cases}$

- $\bar{U}_K^{n+1} \in [\min(U_K^{n+1}, U_L^{n+1}), \max(U_K^{n+1}, U_L^{n+1})]$ is the auxiliary value:

$$k_a(\bar{U}_K^{n+1}) \delta^{n+1}_{K,L}(p_c(U_D)) = \delta^{n+1}_{K,L}(g(U_D)) \quad \text{i.e., } k_a(\bar{U}_K^{n+1}) = \frac{g(U_L^{n+1}) - g(U_K^{n+1})}{p_c(U_L^{n+1}) - p_c(U_K^{n+1})}.$$
Properties of the scheme

- The choice $\bar{U}_{k|l}^{n+1}$ makes appear $\mu k_a(\bar{U}_{k|l}^{n+1}) \delta_{k,l}^{n+1}(P_D - p_c(U_D)) \Rightarrow$ uniform in $\mu, h$ (discrete) estimates as for Eymard, Henry, Hilhorst

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Asymptotics of the scheme as $\mu \to \infty$: a scheme for Richards?

In the gradually saturated regime ($u \leq u_M < 1$) we find

$$U_{K|L}^{n+1} - U_{K|L}^{n} \frac{m_K}{\delta t^n} - \sum_{L \in \mathcal{N}_K} \tau_{K|L} k_w(U_{K|L}^{n+1}) \frac{k_a(\bar{U}_{K|L}^{n+1})}{k_a(U_{K|L}^{n+1})} \delta_{K,L}^{n+1}(P_D) = 0,$$

while the straightforward discretization of Richards equation yields

$k_w(U_{K|L}^{n+1})\delta_{K,L}^{n+1}(P_D)$. One can see that $\frac{k_a(\bar{U}_{K|L}^{n+1})}{k_a(U_{K|L}^{n+1})} \to 1$ as $h \to 0$, so we have an “almost asymptotic preserving” scheme: (limit $\mu \to \infty$ of the two-phase scheme is a “strange” scheme for Richards eqn).
Merci pour votre attention !