# On "quasi-Richards" equation and finite volume approximation of two-phase flow with unlimited air mobility

B. Andreianov [1], R. Eymard[2],
M. Ghilani[3,4] and N. Marhraoui[4]

[1]Université de Franche-Comte Besançon, and LJLL, Paris 6
[2]Université de Marne-la-Vallée, Paris,
[3]Ecole Nationale Supérieure des Arts et Metiers, Meknès (Maroc),
[4]Université Moulay Ismail, Faculté des Sciences, Meknès (Maroc)

February 25-27, 2014
9th DFG-CNRS Workshop, LJLL, Paris

# Outline

1. Introduction and motivation

## Outline

1. Introduction and motivation
   - Two-phase model in porous medium.
     High/infinite mobility $\mu$ of the “air” phase?
     Is the classical Richards model appropriate?

# Outline

1. Introduction and motivation
   - Two-phase model in porous medium.
     High/infinite mobility $\mu$ of the "air" phase?
     Is the classical Richards model appropriate?
   - "Quasi-Richards" equation as the singular limit as $\mu \to \infty$
     Comparison with the Richards equation?
     Existence, uniqueness of solutions?

# Outline

1. Introduction and motivation
   - Two-phase model in porous medium.
     High/infinite mobility $\mu$ of the "air" phase?
     Is the classical Richards model appropriate?
   - "Quasi-Richards" equation as the singular limit as $\mu \to \infty$
     Comparison with the Richards equation?
     Existence, uniqueness of solutions?
   - Robust with respect to $\mu$ finite volume approximation ?
     Asymptotic-preserving scheme ?

# Outline

1. Introduction and motivation
   - Two-phase model in porous medium.
     High/infinite mobility $\mu$ of the "air" phase?
     Is the classical Richards model appropriate?
   - "Quasi-Richards" equation as the singular limit as $\mu \to \infty$
     Comparison with the Richards equation?
     Existence, uniqueness of solutions?
   - Robust with respect to $\mu$ finite volume approximation ?
     Asymptotic-preserving scheme ?
2. Some theory of the "quasi-Richards" equation

# Outline

1. Introduction and motivation
   - Two-phase model in porous medium.
     High/infinite mobility $\mu$ of the "air" phase?
     Is the classical Richards model appropriate?
   - "Quasi-Richards" equation as the singular limit as $\mu \to \infty$
     Comparison with the Richards equation?
     Existence, uniqueness of solutions?
   - Robust with respect to $\mu$ finite volume approximation ?
     Asymptotic-preserving scheme ?
2. Some theory of the "quasi-Richards" equation
   Solutions of two-phase flow equations; estimates
   Passage to the limit (singular) and "quasi-Richards"
   Renormalization and incomplete contraction inequality

# Outline

1. Introduction and motivation
   - Two-phase model in porous medium.
     High/infinite mobility $\mu$ of the "air" phase?
     Is the classical Richards model appropriate?
   - "Quasi-Richards" equation as the singular limit as $\mu \to \infty$
     Comparison with the Richards equation?
     Existence, uniqueness of solutions?
   - Robust with respect to $\mu$ finite volume approximation ?
     Asymptotic-preserving scheme ?
2. Some theory of the "quasi-Richards" equation
   Solutions of two-phase flow equations; estimates
   Passage to the limit (singular) and "quasi-Richards"
   Renormalization and incomplete contraction inequality
3. Finite volumes for two-phase flow with unlimited mobility

# Outline

1. Introduction and motivation
   - Two-phase model in porous medium.
     High/infinite mobility $\mu$ of the “air” phase?
     Is the classical Richards model appropriate?
   - “Quasi-Richards” equation as the singular limit as $\mu \to \infty$
     Comparison with the Richards equation?
     Existence, uniqueness of solutions?
   - Robust with respect to $\mu$ finite volume approximation ?
     Asymptotic-preserving scheme ?
2. Some theory of the “quasi-Richards” equation
   Solutions of two-phase flow equations; estimates
   Passage to the limit (singular) and “quasi-Richards”
   Renormalization and incomplete contraction inequality
3. Finite volumes for two-phase flow with unlimited mobility
   The idea of the scheme
   A priori estimates, existence, convergence at fixed $\mu$
   Asymptotics of the scheme as $\mu \to 0$
   Numerical illustrations

# Models

## Assumptions about groundwater flow

| | |
|---|---|
| Water and air | incompressible phases |
| Porous medium | homogeneous and isotropic |
| Gravity | neglected |
| Source term | of a special form |

Gravity neglected and source term of a special form $\Big\}$ lower bound on saturation

## Models

### Assumptions about groundwater flow

| | |
|---|---|
| Water and air | incompressible phases |
| Porous medium | homogeneous and isotropic |
| Gravity | neglected |
| Source term | of a special form |

(Gravity neglected and source term of a special form) } lower bound on saturation

### Richards model

$$\begin{cases} u_t - \operatorname{div}(k_w(u)\nabla p) = s_w, \\ u = {p_c}^{-1}(p_{atm} - p), \end{cases}$$

### Two-phase model

$$\begin{cases} u_t - \operatorname{div}(k_w(u)\nabla p) = s_w \\ (1-u)_t - \operatorname{div}(\mu k_a(u)\nabla(p + p_c(u))) = s_a \end{cases}$$

where $\mu :=$ Ratio between the phase mobilities (we want $\mu \to \infty$)

## Assumptions

1. $\Omega$ is a polygonal subset of $\mathbb{R}^d$, $d = 2$ or $3$; $T > 0$ is given,
2. $u_m \in (0,1)$; initial saturation $u_0$, source saturation $c$ with $u_m \leq u_0(x) \leq 1$ a.e on $\Omega$, and $u_m \leq c(t,x) \leq 1$ a.e. on $\Omega \times (0,T)$,
3. source $\overline{s} \in L^2$, sink $\underline{s} \in L^2$, $\quad \overline{s}, \underline{s} \geq 0$, global conservation: $\displaystyle\int_\Omega (\overline{s}(x,t) - \underline{s}(x,t))dx = 0$ on $(0,T)$,
4. $k_w \in \mathcal{C}^0([0,1])$, $k_w$ non-decreasing with $k_w(0) = 0$, $k_w(1) = 1$ and $k_w(u_m) > 0$,
   $k_a \in \mathcal{C}^0([0,1])$, $k_a$ non-increasing with $k_a(1) = 0$, $k_a(0) = 1$ and $k_a(s) > 0$ for all $s \in [0,1)$,
   $p_c \in \mathcal{C}^0([u_m,1]) \cap \mathrm{Lip}_{loc}([u_m,1))$, $p_c$ strictly decreasing
5. $\mu \in [1, +\infty)$

## Mathematical setting of the problem

Set $f_\mu(u) := \dfrac{k_w(u)}{k_w(u) + \mu k_a(u)}$; one has $f_\mu \longrightarrow_{\mu\to\infty} 1\!\!1_{[u=1]}$

**Two-phase problem: find $(u, p)$ such that:**

$$\begin{cases} u_t - \operatorname{div}(k_w(u)\nabla p) = f_\mu(c)\,\overline{s} - f_\mu(u)\,\underline{s} \quad \text{on } \Omega \times (0, T), \end{cases}$$

## Mathematical setting of the problem

Set $f_\mu(u) := \dfrac{k_w(u)}{k_w(u) + \mu k_a(u)}$; one has $f_\mu \longrightarrow_{\mu\to\infty} 1\!\!1_{[u=1]}$

**Two-phase problem: find $(u, p)$ such that:**

$$\begin{cases} u_t - \mathrm{div}(k_w(u)\nabla p) = f_\mu(c)\,\overline{s} - f_\mu(u)\,\underline{s} & \text{on } \Omega \times (0,T), \\ (1-u)_t - \mathrm{div}(\mu k_a(u)\nabla(p + p_c(u))) & \\ \qquad = (1 - f_\mu(c))\,\overline{s} - (1 - f_\mu(u))\,\underline{s} & \text{on } \Omega \times (0,T), \end{cases}$$

## Mathematical setting of the problem

Set $f_\mu(u) := \dfrac{k_w(u)}{k_w(u) + \mu k_a(u)}$; one has $f_\mu \longrightarrow_{\mu\to\infty} 1\!\!1_{[u=1]}$

**Two-phase problem: find $(u, p)$ such that:**

$$
\begin{cases}
u_t - \mathrm{div}(k_w(u)\nabla p) = f_\mu(c)\,\overline{s} - f_\mu(u)\,\underline{s} & \text{on } \Omega\times(0,T),\\
(1-u)_t - \mathrm{div}(\mu k_a(u)\nabla(p + p_c(u))) & \\
\qquad = (1-f_\mu(c))\,\overline{s} - (1-f_\mu(u))\,\underline{s} & \text{on } \Omega\times(0,T),\\
\nabla p\cdot\mathbf{n} = 0 & \text{on } \partial\Omega\times(0,T),\\
\nabla(p + p_c(u))\cdot\mathbf{n} = 0 & \text{on } \partial\Omega\times(0,T),\\
\int_\Omega p(x,t)\,dx = 0 & \text{on } (0,T)
\end{cases}
$$

## Mathematical setting of the problem

Set $f_\mu(u) := \dfrac{k_w(u)}{k_w(u) + \mu k_a(u)}$; one has $f_\mu \longrightarrow_{\mu\to\infty} 1\!\!1_{[u=1]}$

**Two-phase problem: find $(u, p)$ such that:**

$$\begin{cases} u_t - \mathrm{div}(k_w(u)\nabla p) = f_\mu(c)\,\overline{s} - f_\mu(u)\,\underline{s} & \text{on } \Omega \times (0,T), \\ (1-u)_t - \mathrm{div}(\mu k_a(u)\nabla(p + p_c(u))) & \\ \qquad\qquad = (1 - f_\mu(c))\,\overline{s} - (1 - f_\mu(u))\,\underline{s} & \text{on } \Omega \times (0,T), \\ \nabla p \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ \nabla(p + p_c(u)) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ \int_\Omega p(x,t)\,dx = 0 & \text{on } (0,T) \\ u(\cdot,0) = u_0 \geq u_m > 0 & \text{on } \Omega \\ c(\cdot) \geq u_m > 0 & \text{on } \Omega \times (0,T). \end{cases}$$

# Mathematical setting of the problem

Set $f_\mu(u) := \dfrac{k_w(u)}{k_w(u) + \mu k_a(u)}$; one has $f_\mu \longrightarrow_{\mu\to\infty} 1\!\!1_{[u=1]}$

## Two-phase problem: find $(u, p)$ such that:

$$\begin{cases} u_t - \mathrm{div}(k_w(u)\nabla p) = f_\mu(c)\,\overline{s} - f_\mu(u)\,\underline{s} & \text{on } \Omega \times (0,T), \\ (1-u)_t - \mathrm{div}(\mu k_a(u)\nabla(p + p_c(u))) & \\ \qquad\qquad = (1 - f_\mu(c))\,\overline{s} - (1 - f_\mu(u))\,\underline{s} & \text{on } \Omega \times (0,T), \\ \nabla p \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ \nabla(p + p_c(u)) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ \int_\Omega p(x,t)\,dx = 0 & \text{on } (0,T) \\ u(\cdot,0) = u_0 \geq u_m > 0 & \text{on } \Omega \\ c(\cdot) \geq u_m > 0 & \text{on } \Omega \times (0,T). \end{cases}$$

To let $\mu \to \infty$: uniform estimates for discrete/regularized problem

## The "quasi-Richards" equation

**Theorem (Eymard, Henry, Hilhorst'09, DCDS-S'12.)**

*There exist solutions $(u^\mu, p^\mu)$ for the two-phase flow problem that obey uniform estimates: lower bound $u_m$ on the saturations $u^\mu$, $L^2(0,T;H^1)$ bound on the pressures $p^\mu$ and on the $1/2$-**Kirchoff transform** $\zeta(u^\mu)$ ,*

## The "quasi-Richards" equation

**Theorem (Eymard, Henry, Hilhorst'09, DCDS-S'12.)**

*There exist solutions $(u^\mu, p^\mu)$ for the two-phase flow problem that obey uniform estimates: lower bound $u_m$ on the saturations $u^\mu$, $L^2(0,T;H^1)$ bound on the pressures $p^\mu$ and on the $1/2$-***Kirchoff transform** $\zeta(u^\mu)$ , *estimate on* $\mu \iint k_a(u^\mu)|\nabla(p^\mu + p_c(u^\mu))|^2$.

## The "quasi-Richards" equation

**Theorem (Eymard, Henry, Hilhorst'09, DCDS-S'12.)**

*There exist solutions* $(u^\mu, p^\mu)$ *for the two-phase flow problem that obey uniform estimates: lower bound* $u_m$ *on the saturations* $u^\mu$, $L^2(0,T;H^1)$ *bound on the pressures* $p^\mu$ *and on the* $1/2$-**Kirchoff transform** $\zeta(u^\mu)$ , *estimate on* $\mu \iint k_a(u^\mu)|\nabla(p^\mu + p_c(u^\mu))|^2$.
*Any accumulation point of* $(u^\mu, p^\mu)_\mu$ *as* $\mu \to \infty$ *satisfies*

$$\begin{cases} u_t - \operatorname{div}(k_w(u)\nabla p) = s_w \\ \nabla(p + p_c(u)) = 0 \ \ \textit{a.e. on the set } [u < 1], \end{cases}$$

## The "quasi-Richards" equation

**Theorem (Eymard, Henry, Hilhorst'09, DCDS-S'12.)**

*There exist solutions $(u^\mu, p^\mu)$ for the two-phase flow problem that obey uniform estimates: lower bound $u_m$ on the saturations $u^\mu$, $L^2(0,T;H^1)$ bound on the pressures $p^\mu$ and on the $1/2$-***Kirchoff transform** $\zeta(u^\mu)$ , *estimate on* $\mu \iint k_a(u^\mu)|\nabla(p^\mu + p_c(u^\mu))|^2$.
*Any accumulation point of $(u^\mu, p^\mu)_\mu$ as $\mu \to \infty$ satisfies*

$$\begin{cases} u_t - \mathrm{div}(k_w(u)\nabla p) = s_w = \overline{s}\,\mathbb{1}_{[c=1]} - \theta\,\underline{s}\,\mathbb{1}_{[u=1]} \\ \nabla(p + p_c(u)) = 0 \ \ a.e.\ \ on\ the\ set\ [u<1], \end{cases}$$

*where $\theta$ is an unknown $[0,1]$-valued function .*

## The "quasi-Richards" equation

**Theorem (Eymard, Henry, Hilhorst'09, DCDS-S'12.)**

*There exist solutions $(u^\mu, p^\mu)$ for the two-phase flow problem that obey uniform estimates: lower bound $u_m$ on the saturations $u^\mu$, $L^2(0,T;H^1)$ bound on the pressures $p^\mu$ and on the 1/2-**Kirchoff transform** $\zeta(u^\mu)$, estimate on $\mu \iint k_a(u^\mu)|\nabla(p^\mu + p_c(u^\mu))|^2$. Any accumulation point of $(u^\mu, p^\mu)_\mu$ as $\mu \to \infty$ satisfies*

$$\begin{cases} u_t - \operatorname{div}(k_w(u)\nabla p) = s_w = \bar{s}\, 1\!\!1_{[c=1]} - \theta\, \underline{s}\, 1\!\!1_{[u=1]} \\ \nabla(p + p_c(u)) = 0 \ \textit{a.e. on the set } [u<1], \end{cases}$$

*where $\theta$ is an unknown $[0,1]$-valued function .*

Thus: solution of the quasi-Richards eqn. is a triple $(u, p, \theta)$ with $\nabla p = -\nabla p_c(u)$ on $[u<1]$ and with $\theta$ defined on $[u=1]$.
Regularity:
$u$ is $[u_m, 1]$-valued with $\zeta(u) \in L^2(0,T;H^1)$, $p \in L^2(0,T;H^1)$.

## Formal justification of the estimates:

• Multiply first equation by $p$, second equation by $(p + p_c(u))$, sum up, use chain rule on $p_c(u)u_t$, use $k_w(u) \geq k_w(u_m) > 0$
$\Rightarrow$ $L^2$ bounds on $|\nabla p|^2$, on $k_a(u)\,\mu|\nabla(p + p_c(u))|^2$ .

## Formal justification of the estimates:

- Multiply first equation by $p$, second equation by $(p + p_c(u))$, sum up, use chain rule on $p_c(u)u_t$, use $k_w(u) \ge k_w(u_m) > 0$ $\Rightarrow$ $L^2$ bounds on $|\nabla p|^2$, on $k_a(u)\,\mu|\nabla(p + p_c(u))|^2$ .
- Write a "global flux" formulation:

$$-\operatorname{div} q = \overline{s} - \underline{s}, \quad q \cdot n = 0,$$

$$u_t - \operatorname{div}[\, f_\mu(u) q - k_w(u) \nabla Q(u)\,] = s_w, \quad \nabla Q(u) \cdot n = 0$$

with $q = M_\mu(u)\,\nabla(p + Q(u))$ and $Q(u) := \int_0^u (1 - f_\mu(s)) p_c'(s)\, ds$.

## Formal justification of the estimates:

- Multiply first equation by $p$, second equation by $(p + p_c(u))$, sum up, use chain rule on $p_c(u)u_t$, use $k_w(u) \geq k_w(u_m) > 0$ $\Rightarrow$ $L^2$ bounds on $|\nabla p|^2$, on $k_a(u)\,\mu|\nabla(p + p_c(u))|^2$ .
- Write a "global flux" formulation:

$$-\operatorname{div} q = \bar{s} - \underline{s}, \quad q \cdot n = 0,$$

$$u_t - \operatorname{div}[\, f_\mu(u)q - k_w(u)\nabla Q(u)\,] \;=\; s_w, \quad \nabla Q(u) \cdot n = 0$$

with $q = M_\mu(u)\,\nabla(p + Q(u))$ and $Q(u) := \int_0^u (1 - f_\mu(s))p_c'(s)\,ds$.

Eliminate $q$ from the resulting system: test fct $-p_c(u)$ in the 2nd, test function $F_\mu(u) := \int_0^u f_\mu(s)p_c'(s)\,ds$ in the 1st equation $\Rightarrow$

$$k_w(u)\nabla Q(u) \cdot \nabla p_c(u) = \frac{k_a(u)\,\mu k_w(u)}{k_w(u) + \mu k_a(u)} |p_c'(u)|^2 |\nabla u|^2 \text{ bounded in } L^1$$

## Formal justification of the estimates:

- Multiply first equation by $p$, second equation by $(p + p_c(u))$, sum up, use chain rule on $p_c(u)u_t$, use $k_w(u) \geq k_w(u_m) > 0$
$\Rightarrow$ $L^2$ bounds on $|\nabla p|^2$, on $k_a(u)\,\mu|\nabla(p + p_c(u))|^2$ .
- Write a "global flux" formulation:

$$-\operatorname{div} q = \bar{s} - \underline{s}, \quad q \cdot n = 0,$$

$$u_t - \operatorname{div}[\, f_\mu(u)q - k_w(u)\nabla Q(u)\,] = s_w, \quad \nabla Q(u) \cdot n = 0$$

with $q = M_\mu(u)\,\nabla(p + Q(u))$ and $Q(u) := \int_0^u (1 - f_\mu(s))p_c'(s)\,ds$.

Eliminate $q$ from the resulting system: test fct $-p_c(u)$ in the 2nd, test function $F_\mu(u) := \int_0^u f_\mu(s)p_c'(s)\,ds$ in the 1st equation $\Rightarrow$

$$k_w(u)\nabla Q(u) \cdot \nabla p_c(u) = \frac{k_a(u)\,\mu k_w(u)}{k_w(u) + \mu k_a(u)}|p_c'(u)|^2|\nabla u|^2 \text{ bounded in } L^1$$

$\Rightarrow$ using $k_w(u) \geq k_w(u_m) > 0$, we get $L^1$ bound on

$$|\nabla\zeta(u)|^2 = k_w(u)|p_c'(u)||\nabla u|^2 \geq \frac{k_a(u)\,\mu k_w(u)}{k_w(u) + \mu k_a(u)}|p_c'(u)|^2|\nabla u|^2.$$

## Justification of passage to the limit; multiplier $\theta$

• In addition to estimate of $\nabla\zeta(u^\mu)$, use translation estimates in time to get strong compactness of $\zeta(u^\mu)$; use invertibility of $\zeta(\cdot)$ $\Rightarrow$ create a strong limit $u$ of $u^\mu$ .

## Justification of passage to the limit; multiplier $\theta$

- In addition to estimate of $\nabla\zeta(u^\mu)$, use translation estimates in time to get strong compactness of $\zeta(u^\mu)$; use invertibility of $\zeta(\cdot)$ $\Rightarrow$ create a strong limit $u$ of $u^\mu$ .
- Pass to strong limit in nonlinearities $k_a(u^\mu)$, $k_w(u^\mu)$, to weak limit in $\nabla p^\mu$ $\Rightarrow$ create a weak limit $p$ of $p^\mu$ .

## Justification of passage to the limit; multiplier $\theta$

• In addition to estimate of $\nabla\zeta(u^\mu)$, use translation estimates in time to get strong compactness of $\zeta(u^\mu)$; use invertibility of $\zeta(\cdot)$ $\Rightarrow$ create a strong limit $u$ of $u^\mu$ .

• Pass to strong limit in nonlinearities $k_a(u^\mu)$, $k_w(u^\mu)$, to weak limit in $\nabla p^\mu$ $\Rightarrow$ create a weak limit $p$ of $p^\mu$ .

• Pass to weak-$*$ $L^\infty$ limit in $[0,1]$-valued multipliers $f_\mu(u^\mu)$ $\Rightarrow$ create a weak limit $\theta$ of $f_\mu(u^\mu)$ .

Identify the limit $\theta$ to 0 on the set $[u=1]$

## Justification of passage to the limit; multiplier $\theta$

• In addition to estimate of $\nabla\zeta(u^\mu)$, use translation estimates in time to get strong compactness of $\zeta(u^\mu)$; use invertibility of $\zeta(\cdot)$
$\Rightarrow$ create a strong limit $u$ of $u^\mu$ .

• Pass to strong limit in nonlinearities $k_a(u^\mu)$, $k_w(u^\mu)$, to weak limit in $\nabla p^\mu$ $\Rightarrow$ create a weak limit $p$ of $p^\mu$ .

• Pass to weak-$*$ $L^\infty$ limit in $[0,1]$-valued multipliers $f_\mu(u^\mu)$
$\Rightarrow$ create a weak limit $\theta$ of $f_\mu(u^\mu)$ .
Identify the limit $\theta$ to 0 on the set $[u=1]$

• Combine everything
$\Rightarrow$ get 1st line (equation) of weak quasi-Richards formulation .

# Justification of passage to the limit; multiplier $\theta$

- In addition to estimate of $\nabla\zeta(u^\mu)$, use translation estimates in time to get strong compactness of $\zeta(u^\mu)$; use invertibility of $\zeta(\cdot)$ $\Rightarrow$ create a strong limit $u$ of $u^\mu$ .
- Pass to strong limit in nonlinearities $k_a(u^\mu)$, $k_w(u^\mu)$, to weak limit in $\nabla p^\mu$ $\Rightarrow$ create a weak limit $p$ of $p^\mu$ .
- Pass to weak-$*$ $L^\infty$ limit in $[0,1]$-valued multipliers $f_\mu(u^\mu)$ $\Rightarrow$ create a weak limit $\theta$ of $f_\mu(u^\mu)$ .
Identify the limit $\theta$ to 0 on the set $[u=1]$
- Combine everything $\Rightarrow$ get 1st line (equation) of weak quasi-Richards formulation .
- On the set $[u<1]$ where $k_a(u)>0$,
pass to the limit $\mu\to\infty$ in $L^1$ bound $k_a(u^\mu)\mu|\nabla(p+p_c(u^\mu))|^2$
$\Rightarrow$ get 2nd line (constraint) of weak quasi-Richards formulation .

# Is quasi-Richards well-posed? Is it different from Richards?

- Richards is well-posed: Alt, Luckhaus'83 .
  $L^1$ contraction inequality holds. ($\Rightarrow$ uniqueness, stability)
- Existence of sols to quasi-Richards: Eymard, Henry, Hilhorst .
  Uniqueness ? Relation to the unique solution of Richards ?

## Is quasi-Richards well-posed? Is it different from Richards?

- Richards is well-posed: Alt, Luckhaus'83 .
  $L^1$ contraction inequality holds. ($\Rightarrow$ uniqueness, stability)
- Existence of sols to quasi-Richards: Eymard, Henry, Hilhorst .
  Uniqueness ? Relation to the unique solution of Richards ?

**Theorem (A., Eymard, Ghilani, Marhraoui'12)**

*Assume $u, \hat{u}$ are weak solutions of the quasi-Richards equation corresponding to data $(u_0, \bar{s})$ and $(\hat{u}_0, \widehat{\bar{s}})$. Then we have the following incomplete contraction inequality : for a.e. $t$,*

$$\int_\Omega (u-\hat{u})^+(t,\cdot) \le \int_\Omega (u_0-\hat{u}_0)^+ + \int_0^t\int_\Omega (\bar{s}-\widehat{\bar{s}})^+ + \int_0^t\int_{[u=1=\hat{u}]} \bar{s}. \quad (1)$$

Proof: use renormalized solutions of Plouvier-Debaigt, Gagneux .

## Is quasi-Richards well-posed? Is it different from Richards?

**Theorem (A., Eymard, Ghilani, Marhraoui'12 )**

*Assume there is no water injection:* $\bar{s}1\!\!1_{[c=1]} = 0$ *a.e. on* $(0, T) \times \Omega$ *(with* $c = c(t, x)$ *the saturation in water of the injected fluid).*
*Then for every datum* $u_0$ *there exists a unique* $u$ *such that* $(u, p, \theta)$ *is a solution of the quasi-Richards equation.*

# Is quasi-Richards well-posed? Is it different from Richards?

**Theorem (A., Eymard, Ghilani, Marhraoui'12 )**

*Assume there is no water injection:* $\bar{s}1\!\!1_{[c=1]} = 0$ *a.e. on* $(0,T)\times\Omega$ *(with* $c = c(t,x)$ *the saturation in water of the injected fluid).*
*Then for every datum* $u_0$ *there exists a unique* $u$ *such that* $(u,p,\theta)$ *is a solution of the quasi-Richards equation.*
*Moreover, in absence of water injection we have* $\theta\,\underline{s} = 0$ *a.e. (no water production!); and the saturation* $u$ *given by quasi-Richards eqn coincides with the unique solution of the Richards eqn.*

## Is quasi-Richards well-posed? Is it different from Richards?

**Theorem (A., Eymard, Ghilani, Marhraoui'12 )**

*Assume there is no water injection: $\bar{s}1\!\!1_{[c=1]} = 0$ a.e. on $(0,T)\times\Omega$ (with $c = c(t,x)$ the saturation in water of the injected fluid).*
*Then for every datum $u_0$ there exists a unique $u$ such that $(u,p,\theta)$ is a solution of the quasi-Richards equation.*
*Moreover, in absence of water injection we have $\theta\,\underline{s} = 0$ a.e. (no water production!); and the saturation $u$ given by quasi-Richards eqn coincides with the unique solution of the Richards eqn.*

In general, we do not expect that quasi-Richards and Richards coincide:

- Physical reasons: $p_{atm}$ is not the good pressure for air when air is captured by saturated water phase
- While uniqueness of $u$ in the triple $(u,p,\theta)$ can be hoped for, we do not expect uniqueness of $(p,\theta)$ in the saturated set $[u=1]$.

More work needed to understand quasi-Richards !

## Renormalized solutions...

- Idea: multiply quasi-Richards by a nonlinear truncation $T_n(u)$,

$$T_n \equiv 1 \text{ on } [0, 1 - \tfrac{1}{n}], \quad T_n(1) = 0.$$

Use chain rules, obtain family of evolution equations

$$(RenEq_n) \qquad b_n(u)_t - \Delta\varphi_n(u) = \bar{s}\, T_n(u) + |\nabla\psi_n(u)|^2$$

with *ad hoc* nonlinearities $b_n, \varphi_n, \psi_n$.
The information from $[u < 1]$ is recovered as $n \to \infty$ but the information from $[u = 1]$ is lost.

## Renormalized solutions...

- Idea: multiply quasi-Richards by a nonlinear truncation $T_n(u)$,

$$T_n \equiv 1 \ \text{ on } [0, 1 - \tfrac{1}{n}], \quad T_n(1) = 0.$$

Use chain rules, obtain family of evolution equations

$$(RenEq_n) \qquad b_n(u)_t - \Delta\varphi_n(u) = \bar{s}\, T_n(u) + |\nabla\psi_n(u)|^2$$

with *ad hoc* nonlinearities $b_n, \varphi_n, \psi_n$.
The information from $[u < 1]$ is recovered as $n \to \infty$ but the information from $[u = 1]$ is lost.

- To recover some information from $[u = 1]$, compute

$$(Cstr) \qquad \lim_{n\to\infty} \int_0^t\!\int_\Omega |\nabla\psi_n(u)|^2 \ = \int_0^t\!\int_{[u=1]} (\bar{s} - \theta\ \underline{s}).$$

## Renormalized solutions...

- Idea: multiply quasi-Richards by a nonlinear truncation $T_n(u)$,

$$T_n \equiv 1 \text{ on } [0, 1-\tfrac{1}{n}], \quad T_n(1) = 0.$$

Use chain rules, obtain family of evolution equations

$$(RenEq_n) \qquad b_n(u)_t - \Delta\varphi_n(u) = \bar{s}\, T_n(u) + |\nabla\psi_n(u)|^2$$

with *ad hoc* nonlinearities $b_n, \varphi_n, \psi_n$.
The information from $[u<1]$ is recovered as $n \to \infty$ but the information from $[u=1]$ is lost.

- To recover some information from $[u=1]$, compute

$$(Cstr) \qquad \lim_{n\to\infty} \int_0^t\!\int_\Omega |\nabla\psi_n(u)|^2 = \int_0^t\!\int_{[u=1]} (\bar{s} - \theta\, \underline{s}).$$

Def. Combination of equations $(RenEq_n)$, $n \to \infty$ and of constraint $(Cstr)$ is called renormalized formulation of quasi-Richards.

Prop. A weak solution is also a renormalized solution.

## Use of renormalized solutions.

- $(RenEq_n)$ is a "standard parabolic-elliptic problem" $\Rightarrow$ $L^1$-contraction ok. Given two solutions $u$, $\hat{u}$, we find

$$\|(b_n(u)-b_n(\hat{u}))^+\|_{L^1}(t) \le \|(b(u_0)-b(\hat{u}_0))^+\|_{L^1}$$
$$+\int_0^t\int_\Omega \mathrm{sign}^+(b_n(u)-b_n(\hat{u}))\,\big(\,\overline{s}\,T_n(u)-\widehat{\overline{s}}\,T_n(\hat{u})+|\nabla\psi_n(u)|^2-|\nabla\psi_n(\hat{u})|^2\big).$$

- Let $n\to\infty$ : using $b_n\to \mathrm{Id}$, we get

$$\|(u-\hat{u})^+\|_{L^1}(t) \le \|(u_0-\hat{u}_0)^+\|_{L^1}$$
$$+\int_0^t\int_{[u>\hat{u}]}\big(\,\overline{s}\,1\!\!1_{[u<1]}-\widehat{\overline{s}}\,1\!\!1_{[\hat{u}<1]}\big)+\lim_{n\to\infty}\int_0^t\int_{u>\hat{u}}\big(|\nabla\psi_n(u)|^2-|\nabla\psi_n(\hat{u})|^2\big).$$

## Use of renormalized solutions.

- $(RenEq_n)$ is a "standard parabolic-elliptic problem" $\Rightarrow$ $L^1$-contraction ok. Given two solutions $u$, $\hat{u}$, we find

$$\|(b_n(u)-b_n(\hat{u}))^+\|_{L^1}(t) \le \|(b(u_0)-b(\hat{u}_0))^+\|_{L^1}$$
$$+\int_0^t\int_\Omega \mathrm{sign}^+(b_n(u)-b_n(\hat{u}))\left(\bar{s}\,T_n(u)-\widehat{\bar{s}}\,T_n(\hat{u})+|\nabla\psi_n(u)|^2-|\nabla\psi_n(\hat{u})|^2\right).$$

- Let $n\to\infty$ : using $b_n \to \mathrm{Id}$, we get

$$\|(u-\hat{u})^+\|_{L^1}(t) \le \|(u_0-\hat{u}_0)^+\|_{L^1}$$
$$+\int_0^t\int_{[u>\hat{u}]}\left(\bar{s}\,1\!\!1_{[u<1]}-\widehat{\bar{s}}\,1\!\!1_{[\hat{u}<1]}\right)+\lim_{n\to\infty}\int_0^t\int_{u>\hat{u}}\left(|\nabla\psi_n(u)|^2-|\nabla\psi_n(\hat{u})|^2\right).$$

- Use $(Cstr)$ trying to simplify the right-hand side...

...aie aie... the term $\int_0^t\int_{[u=1=\hat{u}]}\bar{s}$ survives.

$\Rightarrow$ incomplete contraction inequality follows.

## Finite volume scheme

Write $k_w(u) = f_\mu(u) M_\mu(u)$, $M_\mu = k_w + \mu k_a$. Set $\delta_{K,L}^{n+1}(Z_{\mathcal{D}}) = Z_L^{n+1} - Z_K^{n+1}$.

# Finite volume scheme

Write $k_w(u) = f_\mu(u) M_\mu(u)$, $M_\mu = k_w + \mu k_a$. Set $\delta_{K,L}^{n+1}(Z_\mathcal{D}) = Z_L^{n+1} - Z_K^{n+1}$.
The scheme is: find $U_\mathcal{D} = (U_K^n)_{n,K}$, $P_\mathcal{D} = (P_K^n)_{n,K}$ satisfying

$$\frac{U_K^{n+1} - U_K^n}{\delta t^n} m_K = \sum_{L \in \mathcal{N}_K} \tau_{K|L} f_\mu(U_{K|L}^{n+1}) M_\mu(\bar{U}_{K|L}^{n+1}) \delta_{K,L}^{n+1}(P_\mathcal{D}) + \text{water sources}$$

# Finite volume scheme

Write $k_w(u)=f_\mu(u)M_\mu(u)$, $M_\mu=k_w+\mu k_a$. Set $\delta_{K,L}^{n+1}(Z_\mathcal{D})=Z_L^{n+1}-Z_K^{n+1}$.
The scheme is: find $U_\mathcal{D}=(U_K^n)_{n,K}$, $P_\mathcal{D}=(P_K^n)_{n,K}$ satisfying

$$\frac{U_K^{n+1}-U_K^n}{\delta t^n}m_K=\sum_{L\in\mathcal{N}_K}\tau_{K|L}f_\mu(U_{K|L}^{n+1})M_\mu(\bar{U}_{K|L}^{n+1})\delta_{K,L}^{n+1}(P_\mathcal{D})+\text{water sources}$$

$$\frac{(1-U_K^{n+1})-(1-U_K^n)}{\delta t^n}m_K=\text{air sources}$$

Kirchoff transform ↓ $g'=k_a{p_c}'$

$$+\sum_{L\in\mathcal{N}_K}\tau_{K|L}(1-f_\mu(U_{K|L}^{n+1}))M_\mu(\bar{U}_{K|L}^{n+1})\delta_{K,L}^{n+1}(P_\mathcal{D})-\mu\sum_{L\in\mathcal{N}_K}\tau_{K|L}\delta_{K,L}^{n+1}(g(U_\mathcal{D}))$$

(+ discretization of IC $u_0$, + normalization of $P_\mathcal{D}$ due to Neumann BC)

## Finite volume scheme

Write $k_w(u) = f_\mu(u) M_\mu(u)$, $M_\mu = k_w + \mu k_a$. Set $\delta_{K,L}^{n+1}(Z_\mathcal{D}) = Z_L^{n+1} - Z_K^{n+1}$.
The scheme is: find $U_\mathcal{D} = (U_K^n)_{n,K}$, $P_\mathcal{D} = (P_K^n)_{n,K}$ satisfying

$$\frac{U_K^{n+1} - U_K^n}{\delta t^n} m_K = \sum_{L \in \mathcal{N}_K} \tau_{K|L} f_\mu(U_{K|L}^{n+1}) M_\mu(\bar{U}_{K|L}^{n+1}) \delta_{K,L}^{n+1}(P_\mathcal{D}) + \text{water sources}$$

$$\frac{(1 - U_K^{n+1}) - (1 - U_K^n)}{\delta t^n} m_K = \text{air sources}$$

Kirchoff transform $\downarrow$ $g' = k_a {p_c}'$

$$+ \sum_{L \in \mathcal{N}_K} \tau_{K|L} (1 - f_\mu(U_{K|L}^{n+1})) M_\mu(\bar{U}_{K|L}^{n+1}) \delta_{K,L}^{n+1}(P_\mathcal{D}) - \mu \sum_{L \in \mathcal{N}_K} \tau_{K|L} \delta_{K,L}^{n+1}(g(U_\mathcal{D}))$$

(+ discretization of IC $u_0$, + normalization of $P_\mathcal{D}$ due to Neumann BC)

where

- $U_{K|L}^{n+1}$ is the upwind value : $U_{K|L}^{n+1} = \begin{cases} U_L^{n+1} & \text{if } \delta_{K,L}^{n+1}(P_\mathcal{D}) \geq 0, \\ U_K^{n+1} & \text{otherwise,} \end{cases}$

## Finite volume scheme

Write $k_w(u)=f_\mu(u)M_\mu(u)$, $M_\mu=k_w+\mu k_a$. Set $\delta_{K,L}^{n+1}(Z_\mathcal{D})=Z_L^{n+1}-Z_K^{n+1}$.
The scheme is: find $U_\mathcal{D}=(U_K^n)_{n,K}$, $P_\mathcal{D}=(P_K^n)_{n,K}$ satisfying

$$\frac{U_K^{n+1}-U_K^n}{\delta t^n}m_K=\sum_{L\in\mathcal{N}_K}\tau_{K|L}f_\mu(U_{K|L}^{n+1})M_\mu(\bar{U}_{K|L}^{n+1})\delta_{K,L}^{n+1}(P_\mathcal{D})+\text{water sources}$$

$$\frac{(1-U_K^{n+1})-(1-U_K^n)}{\delta t^n}m_K=\text{air sources}$$

Kirchoff transform $\downarrow$ $g'=k_a p_c{}'$

$$+\sum_{L\in\mathcal{N}_K}\tau_{K|L}(1-f_\mu(U_{K|L}^{n+1}))M_\mu(\bar{U}_{K|L}^{n+1})\delta_{K,L}^{n+1}(P_\mathcal{D})-\mu\sum_{L\in\mathcal{N}_K}\tau_{K|L}\delta_{K,L}^{n+1}(g(U_\mathcal{D}))$$

(+ discretization of IC $u_0$, + normalization of $P_\mathcal{D}$ due to Neumann BC)

where

- $U_{K|L}^{n+1}$ is the upwind value : $U_{K|L}^{n+1}=\begin{cases}U_L^{n+1} & \text{if } \delta_{K,L}^{n+1}(P_\mathcal{D})\geq 0,\\ U_K^{n+1} & \text{otherwise,}\end{cases}$
- $\bar{U}_{K|L}^{n+1}$ between $U_K^{n+1}$ and $U_L^{n+1}$ is the auxiliary value:

$$k_a(\bar{U}_{K|L}^{n+1})\,\delta_{K,L}^{n+1}(p_c(U_\mathcal{D}))=\delta_{K,L}^{n+1}(g(U_\mathcal{D}))\text{ i.e., }k_a(\bar{U}_{K|L}^{n+1})=\frac{g(U_L^{n+1})-g(U_K^{n+1})}{p_c(U_L^{n+1})-p_c(U_K^{n+1})}.$$

## Properties of the scheme

- the choice $\bar{U}_{K|L}^{n+1}$ makes appear $\mu k_a(\bar{U}_{K|L}^{n+1})\delta_{K,L}^{n+1}(P_{\mathcal{D}} - p_c(U_{\mathcal{D}})) \Rightarrow$ uniform in $\mu, h$ (discrete) estimates as for Eymard, Henry, Hilhorst
  … except for time translation estimate on $U_{\mathcal{D}}$ (not uniform in $\mu$)

## Properties of the scheme

- the choice $\bar{U}_{K|L}^{n+1}$ makes appear $\mu k_a(\bar{U}_{K|L}^{n+1})\delta_{K,L}^{n+1}(P_{\mathcal{D}} - p_c(U_{\mathcal{D}})) \Rightarrow$ uniform in $\mu, h$ (discrete) estimates as for Eymard, Henry, Hilhorst ... except for time translation estimate on $U_{\mathcal{D}}$ (not uniform in $\mu$)
- robustness even for $\mu \sim 10^7$ (with $10^2..10^3$ needed in practice) empirical convergence orders: 0.9 in natural norms (for simple tests)

## Properties of the scheme

- the choice $\bar{U}^{n+1}_{K|L}$ makes appear $\mu k_a(\bar{U}^{n+1}_{K|L})\delta^{n+1}_{K,L}(P_{\mathcal{D}} - p_c(U_{\mathcal{D}})) \Rightarrow$ uniform in $\mu, h$ (discrete) estimates as for Eymard, Henry, Hilhorst ... except for time translation estimate on $U_{\mathcal{D}}$ (not uniform in $\mu$)
- robustness even for $\mu \sim 10^7$ (with $10^2..10^3$ needed in practice) empirical convergence orders: 0.9 in natural norms (for simple tests)
- existence, compactness, convergence as $h \to 0$: "as usual"

## Properties of the scheme

- the choice $\bar{U}^{n+1}_{K|L}$ makes appear $\mu k_a(\bar{U}^{n+1}_{K|L})\delta^{n+1}_{K,L}(P_{\mathcal{D}} - p_c(U_{\mathcal{D}})) \Rightarrow$ uniform in $\mu, h$ (discrete) estimates as for Eymard, Henry, Hilhorst ... except for time translation estimate on $U_{\mathcal{D}}$ (not uniform in $\mu$)
- robustness even for $\mu \sim 10^7$ (with $10^2..10^3$ needed in practice) empirical convergence orders: 0.9 in natural norms (for simple tests)
- existence, compactness, convergence as $h \to 0$: "as usual"
- numerical tests: $\|U^{\mu}_{\mathcal{D}} - U^{\text{Rich}}_{\mathcal{D}}\|_{1 \text{ or } \infty} \sim \dfrac{const}{\mu} + residual(h) \Rightarrow$ Asymptotics of the scheme as $\mu \to \infty$: a scheme for Richards ?

## Properties of the scheme

- the choice $\bar U^{n+1}_{K|L}$ makes appear $\mu k_a(\bar U^{n+1}_{K|L})\delta^{n+1}_{K,L}(P_{\mathcal D}-p_c(U_{\mathcal D}))\Rightarrow$ uniform in $\mu, h$ (discrete) estimates as for Eymard, Henry, Hilhorst ... except for time translation estimate on $U_{\mathcal D}$ (not uniform in $\mu$)
- robustness even for $\mu\sim 10^7$ (with $10^2..10^3$ needed in practice) empirical convergence orders: 0.9 in natural norms (for simple tests)
- existence, compactness, convergence as $h\to 0$: "as usual"
- numerical tests: $\|U^{\mu}_{\mathcal D}-U^{\mathrm{Rich}}_{\mathcal D}\|_{1\text{ or }\infty}\sim \dfrac{const}{\mu}+residual(h)\Rightarrow$
  Asymptotics of the scheme as $\mu\to\infty$: a scheme for Richards ?
  In the gradually saturated regime ($u\le u_M<1$) we find

$$\frac{U^{n+1}_K-U^n_K}{\delta t^n}m_K-\sum\nolimits_{L\in\mathcal N_K}\tau_{K|L}k_w(U^{n+1}_{K|L})\frac{k_a(\bar U^{n+1}_{K|L})}{k_a(U^{n+1}_{K|L})}\delta^{n+1}_{K,L}(P_{\mathcal D})=0,$$

  while the straightforward discretization of Richards equation yields $k_w(U^{n+1}_{K|L})\delta^{n+1}_{K,L}(P_{\mathcal D})$. One can see that $\frac{k_a(\bar U^{n+1}_{K|L})}{k_a(U^{n+1}_{K|L})}\to 1$ as $h\to 0$, so we have an "almost asymptotic preserving" scheme: (limit $\mu\to\infty$ of the two-phase scheme is a "strange" scheme for Richards eqn).

# Merci pour votre attention !