Hidden regular variations of point processes with an application to risk theory

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Séminaire Besançon-Dijon, 25 Juin 2018.

Work (in progress !) with R.Biard, C.Tillier and O.Wintenberger.

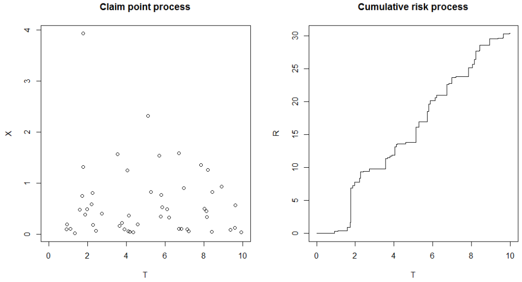
Structure of the talk

Motivating example

- Simple risk process based on claims $(T_i, X_i)_{i \geq 1}$ with :
  - arrival times $0 \leq T_1 \leq T_2 \leq \cdots$ given by a homogeneous Poisson process with intensity $\lambda > 0$,
  - independent claims $X_1, X_2, \ldots \geq 0$ with common distribution F.
- The corresponding risk process

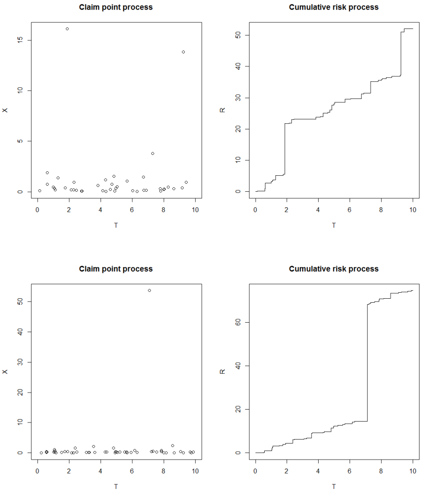
$$R(t) = \sum_{i \geq 1} X_i 1_{\{T_i \leq t\}}, \quad t \geq 0$$

represents the cumulative claim and is a compound Poisson process.
- Simulation on $[0, 10]$ with $\lambda = 5$, $F(x) = (1 + x)^{-\alpha}$, $\alpha = 1/3$:



Motivating example

- In reinsurance application, the claims can often be modeled by a heavy tailed distribution and the risk is dominated by some major extreme event.
- Worst case scenario over 100 (top) and 1000 replications (bottom) :



Motivating example

- This phenomenon is known as the *single big jump heuristic* for regularly varying Lévy process (Hult and Lindskog 2006).
- In our particular example, we have

Proposition

The risk process $R = (R(t))_{0 \leq t \leq T}$ is regularly varying in the Skohorod space $D([0, T])$: in polar coordinates, the radius has a power tail

$$\mathbb{P}(\|R\|_\infty > r) \sim r^{-\alpha}, \quad \text{as } r \to \infty,$$

and the angular component satifies, conditionally on $\|R\|_\infty > r$,

$$(R(t)/\|R\|_\infty)_{0 \leq t \leq T} \xrightarrow{d} (1_{t \geq U})_{0 \leq t \leq T} \quad \text{with } U \rightsquigarrow \mathrm{Unif}([0, T]).$$

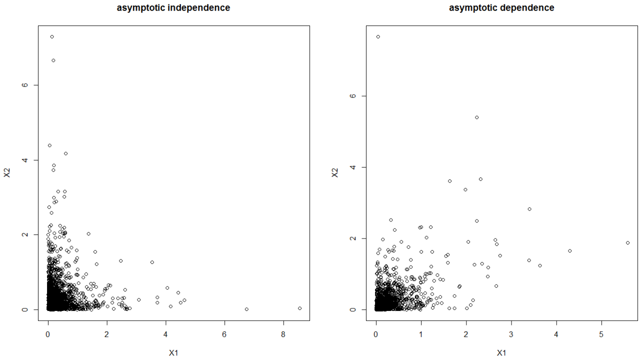
Motivating example

- Reinsurance treaties are used by insurance companies to limit their risk (Embrechts et al. Chapter 8)
- The most common treaty is the *stop loss reinsurance* where the insurance covers the total claims up to some fix value K and the excess of risk $(R(T) - K)_+$ is transferred to the reinsurer.
- We consider here the natural but less common *largest claim reinsurance* where the k largest claims are covered by the reinsurer, $k \geq 1$ fixed.
- Notations :
  - $N = N(T)$: (random) number of claims in the period $[0, T]$,
  - $X_{1:N} \leq X_{2:N} \leq \cdots \leq X_{N:N}$: order statistics of the claims,
  - $R_k^+(T) = \sum_{i=1}^{k} X_{N+1-i:N}$: risk covered by the re-insurer,
  - $R_k^-(T) = \sum_{i=1}^{N-k} X_{i:N}$: residual risk for the insurer,
- Objectif :
  - asymptotic of the residual risk $R_-^k(T)$,
  - keep in mind the temporal aspect so that the company can monitor the risk over time.

Structure of the talk

Multivariate regular variations

- For a multivariate random vector $\mathbf{X}$, $\mathbf{X}/u \to \mathbf{0}$ as $u \to \infty$.
- What is the behaviour of $\mathbb{P}(X/u \in A)$?
- Simulation with same (shifted) α-Pareto margins but different dependence



Multivariate regular variations

Trois définitions équivalentes

- Polar coordinates :

$$n\mathbb{P}(\|X\| > a_n x, X/\|X\| \in B) \to x^{-\alpha}\sigma(B)$$

  with $\mathbb{P}(\|X\| > a_n) \sim n^{-1}$ and $\sigma(\partial B) = 0$.
- Vague convergence on $[-\infty,\infty]^d \setminus \{0\}$:

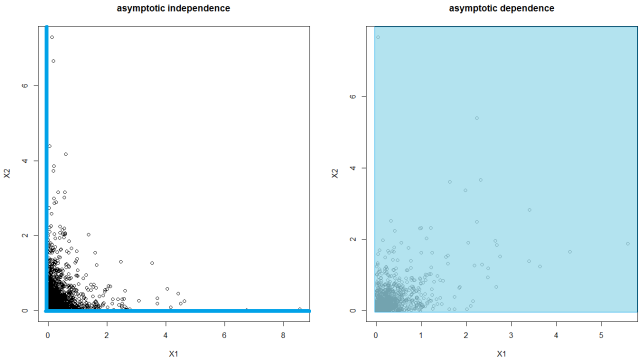
$$n\mathbb{P}(X/a_n \in \cdot) \xrightarrow{v} \mu_{\alpha,\sigma} \simeq \alpha x^{-\alpha-1}\mathrm{d}x \otimes \sigma$$

  test functions : continuous with compact support.
- M_0-convergence on $\mathbb{R}^d$:

$$n\mathbb{P}(X/a_n \in \cdot) \xrightarrow{M_0} \mu_{\alpha,\sigma} \simeq \alpha x^{-\alpha-1}\mathrm{d}x \otimes \sigma$$

  test functions : bounded continuous with support bounded away from 0.

Multivariate regular variations



$$\sigma(d\theta) = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{\pi/2}$$

$$\mu(d\mathbf{x}) = x_1^{-\alpha-1}dx_1\delta_0(dx_2)+\delta_0(dx_1)x_2^{-\alpha-1}dx_2$$

$$\sigma(d\theta) = \frac{\pi}{2}1_{(0,\pi/2)}(\theta)d\theta$$

$$\mu(d\mathbf{x}) = (x_1^2+x_2^2)^{-\alpha/2-1}dx_1dx_2$$

M_0-convergence on a metric space

Abstract framework by Hult and Lindskog (2006)

- metric space (E, d) with an origin 0_E.
- $M_0(F)$: the space of Borel measures μ that are finite on $B(0_F, r)^c$, $r > 0$.
- M_0-convergence $\mu_n \xrightarrow{M_0} \mu$ if and only if

$$\int f d\mu_n \to \int f d\mu \quad \text{for all continuous } f \geq 0 \text{ with support separated from } 0_F.$$

- M_0 convergence metrized by the following distance on $M_0(F)$:

$$\rho(\mu, \mu') = \int_0^\infty e^{-r}(\rho_r(\mu, \mu') \wedge 1)dr,$$

where ρ_r the Prohorov distance on the set of finite measures on $B(0_F, r)^c$, that is

$$\rho_r(\mu, \mu') = \inf(\varepsilon > 0 : \ \mu(A) \leq \mu'(A^\varepsilon) + \varepsilon \text{ and } \mu'(A) \leq \mu(A^\varepsilon) + \varepsilon.$$

M_0-convergence on on a metric space

Theorem (Hult and Lindskog)

- If (E, d) is complete separable, then so is $(M_0(F), \rho)$.
- Portmanteau theorem : equivalence of
  - i) $\mu_n \xrightarrow{M_0} \mu$;
  - ii) $\mu_n(A) \to \mu(A)$ for all Borel set A bounded away from 0_E and such that $\mu(\partial A) = 0$;
  - iii) $\underline{\lim}\, \mu_n(\overset{\circ}{A}) \le \mu(A) \le \overline{\lim}\, \mu_n(\bar{A})$ for all Borel set A bounded away from 0_E.
- Continuous mapping theorem : if $\mu_n \to \mu$ in $M_0(E)$ and $h : E \to F$ is continuous and such that $h(0_E) = 0_F$, then $\mu_n \circ h^{-1} \to \mu \circ h^{-1}$ in $M_0(F)$.

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Regular variations on a metric space

(E,d) equipped with a multiplication by scalars

$$(u, x) \in \mathbb{R}_+ \times E \mapsto u.x \in E$$

such that, for $0 \leq u \leq v$ and $x \in E$,

$$0.x = 0_E \quad \text{and} \quad d(ux, 0_E) \leq d(vx, 0_E).$$

Regular variations

Let $a_n \to \infty$.
We say that $X \in \mathrm{RV}(E, \{a_n\}, \mu)$ if

$$n\,\mathbb{P}(a_n^{-1} X \in \cdot) \xrightarrow{M_0} \mu.$$

Necessarily, there exists $\alpha > 0$ such that a_n is regularly varying of order $1/\alpha$ and μ is α-homogeneous.

RV and conditional limit theorem

Regular variations provides not only the asymptotic behaviour of

$$\mathbb{P}(X \in uA) \quad \text{as } u \to \infty,$$

but also the typical behaviour given this rare event.

Conditional limit theorem

Assume $X \in \mathrm{RV}(E, \{a_n\}, \mu)$.
Let A bounded away from 0_E such that $\mu(\partial A) = 0$ and $\mu(A) > 0$.
Then,

$$\mathbb{P}(u^{-1}X \in \cdot \mid X \in uA) \xrightarrow{d} \frac{\mu(\cdot \cap A)}{\mu(A)}.$$

Structure of the talk

Regular variations of point processes

- $\mathcal{N}_0(E) \subset \mathcal{M}_0(E)$: subspace of point measures (points may accumulate to 0_E).
- Complete separable metric space with the induced metric ρ.
- Scaling : for $u > 0$ and $\pi = \sum_{i\geq 0} \delta_{x_i}$, $u\pi = \sum_{i\geq 0} \delta_{ux_i}$.
- Good control of the distance to the origin (= null measure) :

$$\frac{1}{2}(\|\pi\| \wedge 1) \leq \rho(0,\pi) \leq \|\pi\| \quad \text{with } \|\pi\| = \max_{x\in\pi} d(0_F, x).$$

Laplace criterion (D., Hashorva, Soulier (AoAP 18+))

Let $\mu, \mu_1, \mu_2 \ldots \in \mathcal{M}_0(\mathcal{N}_0(F))$. The following are equivalent :

i) $\mu_n \to \mu$ in $\mathcal{M}_0(\mathcal{N}_0(E))$.

ii) $\int_{\mathcal{N}_0(E)} \left(1 - e^{-\pi(f)}\right) \mu_n(\mathrm{d}\pi) \to \int_{\mathcal{N}_0(E)} \left(1 - e^{-\pi(f)}\right) \mu(\mathrm{d}\pi)$ for all bounded continuous f with support bounded away from 0_F.

This extends Zhao (2016) where weak convergence of probability distribution on $\mathcal{N}_0(F)$ is considered.

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Regular variations of Poisson point process

Theorem (D., Hashorva, Soulier (AoAP 18+))

Let $\mu \in \mathcal{M}_0(F)$ such that $n\mu(a_n^{-1}\cdot) \xrightarrow{M_0} \nu$.
Consider $\Pi \sim \mathrm{PRM}(E,\mu)$ as a random element of $\mathcal{N}_0(E)$. Then,

$$\Pi \in \mathrm{RV}_\alpha\left(\mathcal{N}_0(F), \{a_n\}, \nu^*\right) \quad \text{with} \quad \nu^*(\cdot) = \int \mathbf{1}_{\{\delta_x \in \cdot\}}\, \nu(\mathrm{d}x).$$

Proof : Laplace functional of PRM is explicit and

$$n\mathbb{E}\left[1 - e^{-\int_F f(x/a_n)\Pi(dx)}\right] = n\left(1 - \exp\left[\int_F (e^{-f(x/a_n)} - 1)\mu(dx)\right]\right)$$
$$= n\left(1 - \exp\left[n^{-1}\int_F (e^{-f(x)} - 1)n\mu(a_n dx)\right]\right)$$
$$\longrightarrow \int_F (1 - e^{-f(x)})\nu(dx).$$

Comment : single large point heuristic similar to the single big jump heuristic for RV Lévy processes.

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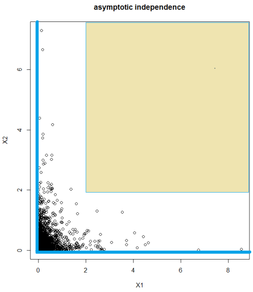


- For $\mathbf{X} \in \mathrm{RV}(\mathbb{R}_+^2, \mu)$ with asymptotically independent components, μ concentrates on the axes.
  What is the behaviour of $\mathbb{P}(X_1 > ux, X_2 > uy)$ as $u \to \infty$?
- In this simple independent case

$$\mathbb{P}(X_1 > ux, X_2 > uy) \sim u^{-2\alpha} x^{-\alpha} y^{-\alpha}.$$

More generally, for A bounded away from the axes,

$$n^{-2}\mathbb{P}(a_n^{-1}\mathbf{X} \in A) \to \int_A \alpha x_1^{-\alpha-1} \alpha x_2^{-\alpha-1}\, dx_1\, dx_2.$$

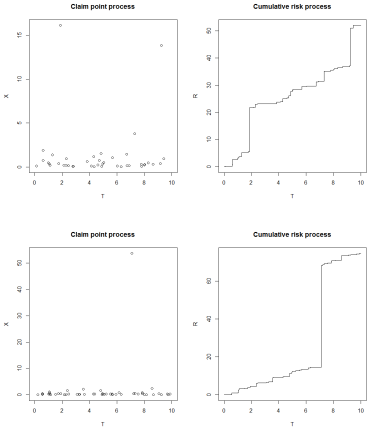
Hidden regular variations

- Notion of regular variation when a cone C is removed from E (Lindskog, Resnick, Roy 2014).
- One can write, in the preceding example :

$$n^{-2}\mathbb{P}(a_n^{-1}\mathbf{X} \in \cdot) \to \alpha^2 x_1^{-\alpha-1} \alpha x_2^{-\alpha-1}\, dx_1\, dx_2 \quad \in M(\mathbb{R}_+^2 \setminus C)$$

  with $C = \{0\} \times \mathbb{R}_+ \cup \mathbb{R}_+ \times \{0\}$ the cone of axes.
- Most of the theory goes through.

Back to our motivating example



- space of finite point measures :

$$\mathcal{N}([0,T]\times[0,\infty))$$

- scaling on component x only :

$$u.\sum_{i=1}^{n}\delta_{(t_i,x_i)}=\sum_{i=1}^{n}\delta_{(t_i,ux_i)}$$

- cone to be removed :

$$C=\mathcal{N}([0,T]\times\{0\})$$

A regular variation result

(Expected !) Theorem

Assume $F \in \mathrm{RV}_\alpha([0,+\infty),\{a_n\},\alpha dx^{-\alpha-1})$ and let $\Pi \sim \mathrm{PRM}(dtF(dx))$. Then,

$$n\mathrm{P}(a_n^{-1}\Pi \in \cdot) \xrightarrow{M} \int_0^T \int_0^\infty 1_{\{\delta_{(t,x)}\in\,\cdot\,\}} dt\alpha dx^{-\alpha-1},$$

M-convergence in $\mathcal{N}([0,T]\times[0,\infty))$ with the cone $\mathcal{N}([0,T]\times\{0\})$ removed.

Problem : the functional $\pi \mapsto r_k^-(\pi)$ corresponding to the residual risk after reinsurance of the k largest claim is identically zero a.e. for the limit measure.

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A hidden regular variation result

The support of the functional r_k^- is the cone of point measures with at most k points in $[0, T] \times (0, \infty)$ denoted by $\mathcal{N}_{\leq k}$.

(Expected !) Theorem

Assume $F \in \mathrm{RV}_\alpha([0, +\infty), \{a_n\}, \alpha dx^{-\alpha-1})$ and let $\Pi \sim \mathrm{PRM}(dtF(dx))$. Then,

$$n^{-(k+1)} \mathrm{P}(a_n^{-1}\Pi \in \cdot) \overset{M(\mathcal{N} \setminus \mathcal{N}_{\leq k})}{\longrightarrow} \int_0^T \int_0^\infty 1_{\sum_{i=1}^{k+1} \delta_{(t_i, x_i)} \in \cdot} \otimes_{i=1}^{k+1} dt_i \alpha dx_i^{-\alpha-1}.$$

Strategy : using this, we expect to

- quantify the probability that the residual risk is large ;
- obtain a conditional limit theorem given the residual risk is large (typical scenario in a rare event situation) ;
- monitor the risk given what has happens at time $T_0 < T$.

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