Dentability Indices and Locally Uniformly Convex Renormings

by

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Abstract : We prove that if the dentability index $\delta(X)$ of a Banach space X is less than ω_1 (first uncountable ordinal), then X admits an equivalent locally uniformly convex norm. We prove also that if its weak* dentability index $\delta^*(X)$ is less than ω_1, then X admits an equivalent norm whose dual norm is locally uniformly convex.

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1. Introduction - Notations

Two important questions in Banach space theory are : does a Banach space with the Radon-Nikodym Property (RNP) have an equivalent locally uniformly convex (LUC) norm ? Does an Asplund space (or equivalently a Banach space whose dual space has the RNP) admit an equivalent Fréchet - differentiable norm ? A complete reference on Asplund spaces and spaces with the RNP is the book of R.D. Bourgin [B].After M. Talagrand [T] proved that $C([0,\omega_1])$, where ω_1 is the first uncountable ordinal, has an equivalent Fréchet differentiable norm, but does not admit any equivalent norm with a strictly convex dual norm, R. Haydon answered negatively the second question in [H1] by constructing a scattered compact space K such that $C(K)$ does not admit a Gateaux-differentiable renorming, nor a strictly convex renorming. On the other hand R. Deville [D] proved that if K is a scattered compact space and if its ω_1^{th} Cantor derived set $K^{(\omega_1)}$ is empty, then $C(K)$ admits a Fréchet - differentiable renorming. Moreover R. Haydon and C.A. Rogers [H-R] proved that, under the same assumptions, $C(K)$ admits an equivalent LUC norm.

In this paper we prove that if the unit ball B_X of a Banach space X is "quickly" dentable, then X admits a LUC renorming and that if the unit ball of its dual space X^* is "quickly" weak*-dentable, then X^* admits a dual LUC renorming. To be more precise, we shall introduce two ordinal indices related to these notions.

Dentability index of $X, \delta(X)$:

Let C be a closed bounded subset of X, we call a slice of C a set of the form : $S(y,a) = \{x \in C : \ y(x) > a\}$ where $y \in X^*$ and $a \in \mathbb{R}$. For $\epsilon > 0$, $C'_\epsilon = \{x \in C$ such that any slice of C containing x is of diameter $> \epsilon\}$. For α ordinal we construct F_ϵ^α inductively in the following way :

$$F_\epsilon^0 = F = B_X$$

$$F_\epsilon^{\alpha+1} = (F_\epsilon^\alpha)'_\epsilon$$

$$F_\epsilon^\alpha = \bigcap_{\beta<\alpha} F_\epsilon^\beta \text{if } \alpha \text{ is a limit ordinal.}$$

$(F_\epsilon^\alpha)_\alpha$ is a decreasing family of closed convex symmetric subsets of B_X.
We define $\delta(X,\epsilon) = \begin{cases} \inf\{\alpha < \omega_1 : \ F_\epsilon^\alpha = \emptyset\} & \text{if it exists} \\ \omega_1 & \text{otherwise.} \end{cases}$
And $\delta(X) = \sup_{\epsilon>0} \delta(X,\epsilon)$.

Weak* - dentability index, $\delta^*(X)$:

Let C be a closed bounded subset of X^*, we call a weak* - slice of C a set of the form : $S(x,a) = \{y \in C : \ y(x) > a\}$ where $x \in X$ and $a \in \mathbb{R}$. For $\epsilon > 0$ $\quad C_\epsilon^{(1)} = \{y \in C$ such that any weak* slice of C containing y is of diameter $> \epsilon\}$. We denote

$$\begin{aligned} K_\epsilon^{(0)} &= K = B_{X^*} \\ K_\epsilon^{(\alpha+1)} &= \left(K_\epsilon^{(\alpha)}\right)_\epsilon^{(1)} \\ K_\epsilon^{(\alpha)} &= \bigcap_{\beta<\alpha} K_\epsilon^{(\beta)} \text{ if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

The $K_\epsilon^{(\alpha)}$'s are weak* compact convex and symmetric.
Then $\delta^*(X,\epsilon) = \begin{cases} \inf\{\alpha < \omega_1 : \ K_\epsilon^{(\alpha)} = \emptyset\} & \text{if it exists} \\ \omega_1 & \text{otherwise} \end{cases}$
And $\delta^*(X) = \sup_{\epsilon>0} \delta^*(X,\epsilon)$.

Let us recall that a norm $|| \quad ||$ on a real vector space is locally uniformly convex (LUC) if, for a sequence (x_n) in X and for $x \in X$, the two hypotheses $||x_n|| = ||x|| = 1$ and $||\frac{x+x_n}{2}|| \longrightarrow 1$ imply $||x - x_n|| \longrightarrow 0$.

2. Main Results

Theorem 2.1 : Let X be a Banach space. If $\delta(X) < \omega_1$ then X admits an equivalent locally uniformly convex norm.

Proof : For n positive integer and $\alpha < \delta(X, 2^{-n})$ we choose $a_{\alpha,n} > 0$ in such a way that :

$$\text{for any } n \geq 1, \sum_{\alpha<\delta(X,2^{-n})} a_{\alpha,n}^2 = 2^{-n}$$

$$\text{so} \sum_{n=1}^{\infty} \sum_{\alpha<\delta(X,2^{-n})} a_{\alpha,n}^2 = 1.$$

Then we denote $\psi_{\alpha,n}(x) = a_{\alpha,n} d(x, F_{2^{-n}}^{\alpha})$, where $d(x, F_{2^{-n}}^{\alpha})$ is the distance from x to $F_{2^{-n}}^{\alpha}$ for the original norm $|| \quad ||$ of X.

And we define $f(x) = \left(||x||^2 + \sum_{n=1}^{\infty} \sum_{\alpha<\delta(X,2^{-n})} \psi_{\alpha,n}^2(x) \right)^{1/2}$.

Clearly $f(x) \geq ||x||$.

On the other hand, since for any $n \geq 1$ and any $\alpha < \delta(X, 2^{-n}), 0 \in F_{2^{-n}}^{\alpha} : \ d\left(x, F_{2^{-n}}^{\alpha}\right) \leq ||x||$.

So $f(x) \leq \sqrt{2}||x||$.

Let $C = \{x \in X, \ f(x) \leq 1\}$, C is $|| \quad ||$-closed, convex, symmetric and $\frac{1}{\sqrt{2}} B_X \subseteq C \subseteq B_X$.

Let us denote by $| \quad |$ the gauge of C. $| \quad |$ is equivalent to $|| \quad ||$.

Lemma 2.2 : Let x be in X and $\{x_k\}$ be a sequence in X. If $f(x) = f(x_k) = 1$, for any k, and $f\left(\frac{x+x_k}{2}\right) \longrightarrow 1$, then $||x - x_k|| \longrightarrow 0$.

Since f is uniformly continuous in norm on B_X, the conclusion of theorem 2.1 follows immediately.

Proof of Lemma 2.2 : Let x and $\{x_k\}$ be as in the hypotheses.

For any k in $\mathbb{N}$:

$$\begin{aligned} f^2\left(\frac{x+x_k}{2}\right) &= ||\frac{x+x_k}{2}||^2 + \sum_{n=1}^{\infty} \sum_{\alpha<\delta(X,2^{-n})} \psi_{\alpha,n}^2\left(\frac{x+x_k}{2}\right) \\ &\leq \left(\frac{||x_k|| + ||x||}{2}\right)^2 + \sum_{n=1}^{\infty} \sum_{\alpha<\delta(X,2^{-n})} \left(\frac{\psi_{\alpha,n}(x_k) + \psi_{\alpha,n}(x)}{2}\right)^2 \\ &\leq \frac{1}{2}\left(f^2(x_k) + f^2(x)\right) = 1. \end{aligned}$$

because of the convexity of the functions $\psi_{\alpha,n}$, $||.||$ and $t \mapsto t^2$.

But $f^2\left(\frac{x+x_k}{2}\right) \longrightarrow 1$.

$$\text{So} \left(\frac{||x_k|| + ||x||}{2}\right)^2 + \sum_{n=1}^{\infty} \sum_{\alpha<\delta(X,2^{-n})} \left(\frac{\psi_{\alpha,n}(x_k) + \psi_{\alpha,n}(x)}{2}\right)^2 \longrightarrow 1.$$

Since $\ell_2(\mathbb{N})$ is uniformly convex, this implies that

$$(||x_k|| - ||x||)^2 + \sum_{n=1}^{\infty} \sum_{\alpha<\delta(X,2^{-n})} (\psi_{\alpha,n}(x_k) - \psi_{\alpha,n}(x))^2 \longrightarrow 0.$$

So, in particular, for any $n \geq 1$ and any $\alpha < \delta(X, 2^{-n})$, $d\left(x_k, F^{\alpha}_{2^{-n}}\right) \longrightarrow d\left(x, F^{\alpha}_{2^{-n}}\right)$.

Now let $\epsilon > 0$, we want to show that for k large enough, $||x - x_k|| < \epsilon$. Take $n_0 \geq 1$ such that $2^{1-n_0} < \epsilon$. Since $f(x) = 1, x \in B_X$. So there exists $\alpha_0 < \delta\left(X, 2^{-n_0}\right)$ such that $x \in F^{\alpha_0}_{2^{-n_0}} \backslash F^{\alpha_0+1}_{2^{-n_0}}$. We know that $d\left(x_k, F^{\alpha_0}_{2^{-n_0}}\right) \longrightarrow 0$. Thus there is a sequence $\{x'_k\} \subseteq F^{\alpha_0}_{2^{-n_0}}$ such that $||x_k - x'_k|| \longrightarrow 0$. If $\frac{x+x'_k}{2} \in F^{\alpha_0+1}_{2^{-n_0}}$, we call $\delta = \frac{1}{2}\psi^2_{\alpha_0+1,n_0}(x) > 0$. Then

$$\psi^2_{\alpha_0+1,n_0}\left(\frac{x+x'_k}{2}\right) = 0 \leq \frac{1}{2}\psi^2_{\alpha_0+1,n_0}(x) + \frac{1}{2}\psi^2_{\alpha_0+1,n_0}(x'_k) - \delta$$

Thus

$$f^2\left(\frac{x+x'_k}{2}\right) \leq \frac{1}{2} + \frac{1}{2}f^2(x'_k) - \delta, \tag{1}$$

because the $\psi_{\alpha,n}$'s and $||\ \ ||$ are convex. But since f is uniformly continuous in norm on B_X, we have that $f(x'_k) \longrightarrow 1$ and $f\left(\frac{x+x'_k}{2}\right) \longrightarrow 1$. So, for k large enough, (1) cannot hold. Therefore, there is $k_0 \in \mathbb{N}$ such that, for any $k \geq k_0$, $\frac{x+x'_k}{2} \notin F^{\alpha_0+1}_{2^{-n_0}}$. But $\frac{x+x'_k}{2} \in F^{\alpha_0}_{2^{-n_0}}$ because x and x'_k are in $F^{\alpha_0}_{2^{-n_0}}$. Then there is a slice T of $F^{\alpha_0}_{2^{-n_0}}$ containing $\frac{x+x'_k}{2}$ and with diameter $\leq 2^{-n_0}$. This slice must contain either x or x'_k. Therefore $||x - x'_k|| \leq 2\mathrm{diam}T \leq 2^{-n_0}$, for any $k \geq k_0$. Now, since $||x_k - x'_k|| \to 0$, there exists $k_1 \geq k_0$ such that : for any $k \geq k_1$, $\quad ||x - x_k|| < \epsilon$. $\square$

Remarks : 1) If X is a separable Banach space with the RNP then $\delta(X) < \omega_1$ (the converse being false), which in turn implies X has the RNP.

2) Let us mention the following simple fact : $\delta(X) \leq \omega_0$ if and only if X admits an equivalent uniformly convex norm (or equivalently X super-reflexive). Where ω_0 denotes the first infinite ordinal.

Proof : From the existence of an equivalent uniformly convex norm, it follows easily that for any $\epsilon > 0$, $\delta(X, \epsilon) < \omega_0$.

Let us now assume that X is not super-reflexive. Then X has the finite tree property (see R.C. James [J1]). So there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$ there is a dyadic tree $(x_s)_{s \in 2^{\leq n}} \subseteq B_X$ (where $2^{\leq n}$ denotes the set of sequences of 0 and 1 with length $\leq n$) satisfying : for any $s \in 2^{\leq n-1}$, $||x_{s\frown 0} - x_{s\frown 1}|| \geq 2\epsilon$ and $x_s = \frac{1}{2}(x_{s\frown 0} + x_{s\frown 1})$. It is now easy to see that $(x_s)_{s \in 2^{\leq n-1}} \subseteq F'_\epsilon$. Indeed for $s \in 2^{\leq n-1}$, any slice containing x_s must contain either $x_{s\frown 0}$ or $x_{s\frown 1}$. Therefore, this slice is of diameter $> \epsilon$. Proceeding inductively we obtain that $F^n_\epsilon \neq \emptyset$. Thus, for any n, $0 \in F^n_\epsilon$, because F^n_ϵ is convex and symmetric. Therefore $0 \in F^{\omega_0}_\epsilon$. So $\delta(X) > \omega_0$. $\square$

Theorem 2.3 : Let X be a Banach space. If $\delta^*(X) < \omega_1$, then X^* admits an equivalent dual norm that is locally uniformly convex. Consequently, X admits an equivalent Fréchet-differentiable norm.

Proof : We consider the function $f(y) = \left(||y||^2 + \sum_{n=1}^{\infty} \sum_{\alpha < \delta^*(X, 2^{-n})} a^2_{\alpha,n} d^2(y, K^{(\alpha)}_{2^{-n}}) \right)^{1/2}$ defined on X^*, where the $a_{\alpha,n}$'s are chosen as in the proof of Theorem 2.1. Then the proof is identical. We only have to show that the norm defined this way is a dual norm, or equivalently that $\{y : \ f(y) \leq 1\}$ is weak* closed. This follows from the weak* lower semi continuity of the functions $d(., K^{(\alpha)}_{2^{-n}})$ and $||\,.\,||$. $\square$

It is well known that when X is separable, $\delta^*(X) < \omega_1$ holds if and only if X^* is separable. The next proposition gives an example of a non separable Asplund space, other than a super-reflexive space and satisfying $\delta^*(X) < \omega_1$.

Proposition 2.4 : For any set Γ :

$$\delta\left(\ell_1(\Gamma)\right) \leq \delta^*\left(c_0(\Gamma)\right) < \omega_1.$$

Proof : The first inequality is clear.

We will need the following lemma :

Lemma 2.5 : If j is a bijective isometry on a Banach space X, then for any ordinal α and any $\epsilon > 0$: $j^*\left(K_\epsilon^{(\alpha)}\right) = K_\epsilon^{(\alpha)}$.

The proof of this lemma is a straightforward transfinite induction.

Let us denote $K = B_{\ell_1(\Gamma)}$. Γ being infinite, we fix a countable subset D of Γ. For any $y \in \ell_1(\Gamma)$ there is a bijective isometry j_y on $c_0(\Gamma)$ such that the support of $j_y^*(y)$ is included in D. This, combined with Lemma 2.5, implies that for any $\epsilon > 0$ and any ordinal α : $K_\epsilon^{(\alpha)} = \bigcup\left\{(j^*)^{-1}\left(K_\epsilon^{(\alpha)} \cap B_{\ell_1(D)}\right)/j \text{ bijective isometry on } c_0(\Gamma)\right\}$. But, since $c_0(\Gamma)$ is an Asplund space, $\left(K_\epsilon^{(\alpha)}\right)_\alpha$ is strictly decreasing (as long as $K_\epsilon^{(\alpha)} \neq \emptyset$). Therefore $\left(K_\epsilon^{(\alpha)} \cap B_{\ell_1(D)}\right)_\alpha$ is strictly decreasing (as long as $K_\epsilon^{(\alpha)} \neq \emptyset$) and $K_\epsilon^{(\alpha)} = \emptyset$ if and only if $K_\epsilon^{(\alpha)} \cap B_{\ell_1(D)} = \emptyset$. Since $\ell_1(D)$ is separable, there exists an ordinal $\alpha < \omega_1$ such that $K_\epsilon^{(\alpha)} \cap B_{\ell_1(D)} = \emptyset$. So $\delta^*\left(c_0(\Gamma)\right) < \omega_1$. $\square$

Remark : Another consequence of Lemma 2.5 is that the renormings of Theorem 2.1 and Theorem 2.3 preserve the bijective isometries on X.

3. Szlenk indices

Let X be a Banach space. We shall now introduce two ordinal indices related to X that have been essentially defined by W. Szlenk [S].

Szlenk index of $X, Sz(X)$:

Let C be a closed bounded subset of X^*. For $\epsilon > 0$ we define $C_\epsilon^{[1]} = \{y \in C$ such that for any weak*-neighborhood V of y, $\operatorname{diam}(V \cap C) > \epsilon\}$.

We construct

$$\begin{aligned} K_\epsilon^{[0]} &= K = B_{X^*} \\ K_\epsilon^{[\alpha+1]} &= \left(K_\epsilon^{[\alpha]}\right)_\epsilon^{[1]} \\ K_\epsilon^{[\alpha]} &= \bigcap_{\beta<\alpha} K_\epsilon^{[\beta]} \text{ if } \alpha \text{ is a limit ordinal} \end{aligned}$$

$$Sz(X,\epsilon) = \begin{cases} \inf\{\alpha < \omega_1 : \ K_\epsilon^{[\alpha]} = \emptyset\} & \text{if it exists} \\ \omega_1 & \text{otherwise} \end{cases}$$

$$Sz(X) = \sup_{\epsilon>0} Sz(X,\epsilon).$$

Weak - Szlenk index of $X, Sz_w(X)$:

For C closed bounded subset of X and $\epsilon > 0$, $C_\epsilon^{\langle 1\rangle} = \{x \in C$ such that for any weak-neighborhood V of $x, \operatorname{diam}(V \cap C) > \epsilon\}$. Then we define $F_\epsilon^{\langle\alpha\rangle}$, $Sz_w(X,\epsilon)$ and $Sz_w(X)$ in the usual way. It is clear that $Sz_w(X) \leq \delta(X)$, but $Sz_w(X) < \omega_1$ does not imply $\delta(X) < \omega_1$. Indeed the predual B of the James tree space has the Point of Continuity Property and is separable, so $Sz_w(X) < \omega_1$; but B does not have the RNP, so $\delta(X) = \omega_1$ (see R.C. James [J2], J. Lindenstrauss and C. Stegall [L-S], C.A. Edgar and R.F. Wheeler [E-W]).

In the dual case we also have $Sz(X) \leq \delta^*(X)$. On the other hand, if X is separable, the following are equivalent : i) $\delta^*(X) < \omega_1$, ii) $Sz(X) < \omega_1$, iii) X^* is separable. However we do not know if $\delta^*(X) < \omega_1$ and $Sz(X) < \omega_1$ are still equivalent when X is non-separable.

The question is now : What kind of renormings can we find on X, under the weaker assumptions $Sz(X) < \omega_1$ and $Sz_w(X) < \omega_1$? In this section we present the partial results allowed by the methods of Section 2. The main obstacle is the non convexity of the derived sets $K_\epsilon^{[\alpha]}$ or $F_\epsilon^{\langle\alpha\rangle}$.

Proposition 3.1 : Let X be a Banach space. If $Sz(X) < \omega_1$ there is a weak* lower semi continuous function f defined on X^* satisfying :

i) $||y|| < \frac{1}{2} \Rightarrow f(y) < 1$ and $f(y) \leq 1 \Rightarrow ||y|| \leq 1$.

ii) the weak* topology and the norm topology coincide on the sets $S_a = \{y \in X^* : f(y) = a\}$, for any $0 < a \leq 1$.

Proof : We choose a sequence $\{a_{1,n}\}_{n=1}^{\infty}$ of positive real numbers such that $\sum\limits_{n=1}^{\infty} a_{1,n} = \frac{3}{4}$. Then, for any $n \geq 1$ and any $1 < \alpha < Sz(X, 2^{-n})$ we choose $a_{\alpha,n} > 0$ in such a way that : $\sum\limits_{1<\alpha<Sz(X,2^{-n})} a_{\alpha,n} = \frac{2^{-n}}{4}$. Therefore $\sum\limits_{n=1}^{\infty} \sum\limits_{1<\alpha<Sz(X,2^{-n})} a_{\alpha,n} = \frac{1}{4}$ and $\sum\limits_{n=1}^{\infty} \sum\limits_{\alpha<Sz(X,2^{-n})} a_{\alpha,n} = 1$. Now we consider $f(y) = ||y|| + \sum\limits_{n=1}^{\infty} \sum\limits_{\alpha<Sz(X,2^{-n})} \psi_{\alpha,n}(y)$, defined on X^*, where $\psi_{\alpha,n}(y) = a_{\alpha,n} d\left(y, K_{2^{-n}}^{[\alpha]}\right)$. Since the $K_{2^{-n}}^{[\alpha]}$'s are weak* compact, the $\psi_{\alpha,n}$'s are weak* lower semi continuous. Thus f is weak* lower semi continuous. The inclusion $\{y : \ f(y) \leq 1\} \subseteq B_{X^*}$ is clear. If $||y|| < \frac{1}{2}$, for any $n, y \in K_{2^{-n}}^{[1]}$ (we may assume that X is infinite dimensional). Thus $f(y) = ||y|| + \sum\limits_{n=1}^{\infty} \sum\limits_{1<\alpha<Sz(X,2^{-n})} a_{\alpha,n} d(y, K_{2^{-n}}^{[\alpha]}) < 1$. Only the assertion ii) remains to be shown.

Claim : Let $0 < a \leq 1$ and $y \in S_a$.
$\forall \gamma > 0, \ \forall n \geq 1, \ \forall \alpha < Sz(X, 2^{-n}), \ \exists W$ weak* neighborhood of y such that :

$$\forall y' \in W \cap S_a, |d(y', K_{2^{-n}}^{[\alpha]}) - d(y, K_{2^{-n}}^{[\alpha]})| < \gamma$$

It is enough to show that for any $\gamma' > 0$ there is a weak* neighborhood U of y such that :

$$\forall n \geq 1, \ \forall \alpha < Sz(X, 2^{-n}), \ \forall y' \in U \cap S_a, \ |\psi_{\alpha,n}(y') - \psi_{\alpha,n}(y)| < \gamma'.$$

Take $(\alpha_1, n_1), \ldots, (\alpha_r, n_r)$ such that $||y|| + \sum\limits_{i=1}^{r} \psi_{\alpha_i,n_i}(y) > a - \frac{\gamma'}{2}$. Since $||\ .\ ||$ and the $\psi_{\alpha,n}$'s are weak* lower semi-continuous, there is a weak* neighborhood U of y such that :

$$\forall y' \in U : \quad ||y'|| > ||y|| - \frac{\gamma'}{2(r+1)} \text{ and } \forall \quad 1 \leq i \leq r, \ \psi_{\alpha_i,n_i}(y') > \psi_{\alpha_i,n_i}(y) - \frac{\gamma'}{2(r+1)}.$$

If $y' \in U \cap S_a$, $||y'|| + \sum\limits_{i=1}^{r} \psi_{\alpha_i,n_i}(y') > a - \gamma'$. So if $(\alpha, n) \notin \{(\alpha_1, n_1), \ldots, (\alpha_r, n_r)\}$, $\psi_{\alpha,n}(y') < \gamma'$ while $\psi_{\alpha,n}(y) < \frac{\gamma'}{2}$. Therefore $|\psi_{\alpha,n}(y') - \psi_{\alpha,n}(y)| < \gamma'$. On the other hand if $y' \in U$

and $\psi_{\alpha_i,n_i}(y') \geq \psi_{\alpha_i,n_i}(y) + \gamma'$ then $||y'|| + \sum_{i=1}^{r} \psi_{\alpha_i,n_i}(y') > a$, so $y' \notin S_a$. This finishes the proof of the claim.

Now, let $\epsilon > 0,\ 0 < a \leq 1$ and y in S_a. We need to find a weak* neighborhood V of y such that, for any $y' \in V \cap S_a,\ ||y - y'|| < \epsilon$. Take n such that $2^{-n} < \epsilon$. Since $y \in B_{X^*}$, there exists $\alpha < Sz(X, 2^{-n})$ such that $y \in K_{2^{-n}}^{[\alpha]} \backslash K_{2^{-n}}^{[\alpha+1]}$. So there is a weak* open neighborhood W_0 of y such that diam $(K_{2^{-n}}^{[\alpha]} \cap W_0) \leq 2^{-n}$. Unless $K_{2^{-n}}^{[\alpha]} = \{y\}$ which is a trivial case (the claim gives directly the weak* neighborhood we need), we may assume that $K_{2^{-n}}^{[\alpha]} \backslash W_0 \neq \emptyset$. We denote $\beta = d(y,\ K_{2^{-n}}^{[\alpha]} \backslash W_0) > 0$. Since the function $d(., K_{2^{-n}}^{[\alpha]} \backslash W_0)$ is weak* lower semicontinuous, there is a weak* neighborhood W_1 of y so that :

$$\forall y' \in W_1 \quad d(y',\ K_{2^{-n}}^{[\alpha]} \backslash W_0) > \frac{\beta}{2}.$$

Moreover, from the claim above, it follows that there is a weak* neighborhood W_2 of y such that :

$$\forall y' \in W_2 \cap S_a \quad d(y', K_{2^{-n}}^{[\alpha]}) < \mathrm{Min}\{\frac{\beta}{2}, \epsilon - 2^{-n}\}.$$

Since $d(., K_{2^{-n}}^{[\alpha]}) = \inf \Big\{ d(., K_{2^{-n}}^{[\alpha]} \backslash W_0),\ d(., K_{2^{-n}}^{[\alpha]} \cap W_0) \Big\}$, we have :

$$\forall y' \in W_1 \cap W_2 \cap S_a \quad d(y', K_{2^{-n}}^{[\alpha]} \cap W_0) < \epsilon - 2^{-n}$$

So $\forall y' \in W_1 \cap W_2 \cap S_a \qquad ||y - y'|| < \epsilon$. This concludes the proof of the proposition. $\square$

In the non dual case, although the distance functions to the derived sets $F_{2^{-n}}^{\langle\alpha\rangle}$ are not necessarily weakly lower semi-continuous, we obtain a similar result.

Proposition 3.2 : Let X be a Banach space. If $Sz_w(X) < \omega_1$ there is a weakly lower semi-continuous function f defined on X and satisfying

i) $||x|| < \frac{1}{2} \Rightarrow f(x) < 1$ and $f(x) \leq 1 \Rightarrow ||x|| \leq 1$

ii) the weak topology and the norm topology coincide on the sets $S_a = \{x \in X \quad f(x) = a\}$, for any $0 < a \leq 1$.

Proof : For a function $\varphi:\ X\to \mathbb{R}^+$, we denote by $\check{\varphi}$ the weakly lower semi-continuous regularization of $\varphi:\ \check{\varphi}(x)=\sup\left\{\inf\limits_{x'\in V} f(x');\ V \text{ weak neighborhood of } x\right\}$. We choose the coefficients $a_{\alpha,n}$, for $n\geq 1$ and $\alpha<Sz_w(X,2^{-n})$ as in the proof of proposition 3.1. Then we call : $\varphi_{\alpha,n}(x)=d(x,\ F_{2^{-n}}^{\langle\alpha\rangle})$

$g(x)=||x||+\sum\limits_{n=1}^{\infty}\sum\limits_{\alpha<Sz_w(X,2^{-n})} a_{\alpha,n}\varphi_{\alpha,n}(x)$

and $f(x)=\check{g}(x)=||x||+\sum\limits_{n=1}^{\infty}\sum\limits_{\alpha<Sz_w(X,2^{-n})} a_{\alpha,n}\check{\varphi}_{\alpha,n}(x)$. It is easy to check that the condition i) holds for g and for $f=\check{g}$.

Let $0<a\leq 1$ and $x\in S_a$. Like in the dual case we have : $\forall\gamma>0,\ \forall n\geq 1,\ \forall\alpha< Sz_w(X,2^{-n})$, there is a weak neighborhood W of x such that :

$$\forall x'\in W\cap S_a \qquad |\check{\varphi}_{\alpha,n}(x')-\check{\varphi}_{\alpha,n}(x)|<\gamma.$$

Now, let $\epsilon>0$. We want to find a weak neighborhood V of x such that, for any $x'\in V\cap S_a,\ ||x-x'||\leq\epsilon$. We take $n\geq 1$ and $\alpha<S_{z_w}(X,2^{-n})$ such that $2^{-n}<\frac{\epsilon}{2}$ and $x\in F_{2^{-n}}^{\langle\alpha\rangle}\backslash F_{2^{-n}}^{\langle\alpha+1\rangle}$. Then there is a weak neighborhood U of x with diam $(F_{2^{-n}}^{\langle\alpha\rangle}\cap U)\leq 2^{-n}$. As in the dual case we may assume $F_{2^{-n}}^{\langle\alpha\rangle}\backslash U\neq\emptyset$. We call $h_1=d(.,\ F_{2^{-n}}^{\langle\alpha\rangle}\backslash U)$ and $h_2=d(.,F_{2^{-n}}^{\langle\alpha\rangle}\cap U)$. It is clear that $\check{\varphi}_{\alpha,n}=\inf\left\{\check{h}_1,\check{h}_2\right\}$. We also may assume $U=\{x'\in X:\ |y_i(x')-y_i(x)|<\lambda,\ \forall_i\in\{1,\dots,r\}\}$ where $y_i\in X^*,\ ||y_i||=1$ and $\lambda>0$.

$$\text{Let} U'=\left\{x'\in X:\ |y_i(x')-y_i(x)|<\frac{\lambda}{2},\ \forall i\in\{1,\dots,r\}\right\}$$

For any $x'\in U', h_1(x')>\frac{\lambda}{2}$, so $\check{h}_1(x)\geq\frac{\lambda}{2}$. But we know that there is a weak neighborhood W of x such that for any $x'\in W\cap S_a\ \check{\varphi}_{\alpha,n}(x')<\text{Min}\left\{\frac{\lambda}{2},\frac{\epsilon}{2}\right\}$

Therefore : $\forall x'\in W\cap U'\cap S_a,\ \check{h}_2(x')<\epsilon/2$. But $\check{h}_2(x')<\epsilon/2$ implies $||x-x'||\leq\epsilon$. Indeed if $||x-x'||>\epsilon$, by Hahn Banach, there is a weak neighborhood W' of x' such that for any $x''\in W',\ ||x-x''||>\epsilon$. So for any $x''\in W',\ h_2(x'')>\frac{\epsilon}{2}$, because diam $(F_{2^{-n}}^{\langle\alpha\rangle}\cap U)<\frac{\epsilon}{2}$. Therefore $\check{h}_2(x')\geq\frac{\epsilon}{2}$. $\square$

Let us mention that R. Haydon obtained recently in [H2] two results connected with this section : 1) The scattered compact space K constructed in [H1] is such that $C(K)$ does not admit a non zero real valued Fréchet-differentiable function with bounded support. 2) There is a scattered compact space K so that $C(K)$ is not strictly convexifiable although it admits an equivalent norm such that the norm and weak topologies coincide on its unit sphere.

We want to mention that the definition we use for $Sz(X)$ is not the definition originally introduced by Szlenk in [S]. The derivation he considered is the following : Let X be a separable Banach space, C be a closed bounded subset of X^* and $\epsilon > 0$:

$$C'_\epsilon = \Big\{ y \in C : \exists \{y_n\} \subseteq C,\ \exists \{x_n\} \subseteq B_X \text{ such that } \\ y_n \xrightarrow{w^*} y,\ x_n \xrightarrow{w} 0,\ y_n(x_n) \geq \epsilon\ \forall_n \in \mathbb{N} \Big\}.$$

Let us call $\sigma(X,\epsilon)$ and $\sigma(X)$ the ordinal indices associated in the usual way to this operation. If X is a separable Banach space, it is clear that $\sigma(X) \leq Sz(X)$. The equality is not true in general. Indeed we have that $\sigma(\ell_1) = 1$ because ℓ_1 enjoys the Schur property while $Sz(\ell_1) = \omega_1$ since $\ell_1^* = \ell_\infty$ is not separable. However this counterexample is essentially the only one.

Proposition 3.3 : Let X be a separable Banach space. If X does not contain any isomorphic copy of ℓ_1, then $Sz(X) = \sigma(X)$.

Proof : For $\epsilon > 0$, $(K_\epsilon^\alpha)_\alpha$ will denote the family of derived sets obtained with the original Szlenk-derivation. For the index $Sz(X)$ we will use the derivation $C_\epsilon^{[1]} = \Big\{ y \in C :\ \exists \{y_n\} \subseteq C \text{ s. t. } y_n \xrightarrow{w^*} y \text{ and } ||y_n - y|| \geq \epsilon \text{ for all } n \geq 0 \Big\}$ which is equivalent to the derivation defined in Section 2 when X is separable. We will show by transfinite induction that $K_\epsilon^{[\alpha]} \subseteq K_{\epsilon/16}^\alpha$. The conclusion of the proposition follows clearly. This property is true for $\alpha = 0$ and passes easily to the limit ordinals. Let us now assume that $K_\epsilon^{[\alpha]} \subseteq K_{\epsilon/16}^\alpha$. Let $y \in K_\epsilon^{[\alpha+1]}$, there is a sequence $\{y_n\} \subseteq K_\epsilon^{[\alpha]}$ such that $y_n \xrightarrow{w^*} y$ and $||y_n - y|| \geq \epsilon$ for any $n \geq 0$.

Lemma 3.4 : There exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ $||y_n\Big|_{\mathrm{Ker}y}|| > \frac{\epsilon}{4}$.

Proof of Lemma 3.4 : Of course, we may assume $y \neq 0$.

Observe that $||y_n\Big|_{\mathrm{Ker}y}|| = d(y_n, \mathbb{R}y)$.

Suppose $d(y_n, \mathbb{R}y) \leq \frac{\epsilon}{4}$. Let $y'_n \in \mathbb{R}y$ such that $||y_n - y'_n|| = d(y_n, \mathbb{R}y)$. Since $||y_n - y|| \geq \epsilon$, $||y'_n - y|| \geq \frac{3\epsilon}{4}$. Let $x \in B_X$ such that $y(x_0) > \frac{2}{3}||y||$ then $|(y'_n - y)(x_0)| > \frac{\epsilon}{2}$.

Thus $|(y_n - y)(x_0)| > \frac{\epsilon}{4}$. But this contradicts $y_n \xrightarrow{w^*} y$. □

End of the proof of proposition 3.3

Consequently we may assume that there is a sequence $\{x_n\}$ in $B_X \cap \mathrm{Ker}y$ such that, for any n in $\mathbb{N}$, $y_n(x_n) > \epsilon/4$. Since $X \not\supset \ell_1$, we may also assume that x_n is weak-Cauchy (see [O-R]). On the oter hand $\forall p \in \mathbb{N}$, $x_p \in \mathrm{Ker}y$. So$\forall p \in \mathbb{N}$, $y_n(x_p) \longrightarrow 0$. Therefore we can construct an increasing sequence of integers $\{n_k\}$ such that $|y_{n_{k+1}}(x_{n_k})| \leq \frac{\epsilon}{8}$. Then we call $x'_k = \frac{x_{n_k} - x_{n_{k-1}}}{2}$ and $y'_k = y_{n_k}$. $\{x'_n\} \subseteq B_X$, $x'_k \xrightarrow{w} 0$ because $\{x_n\}$ is weak Cauchy and $y'_k(x'_k) > \frac{\epsilon}{16}$. So $y \in K^{\alpha+1}_{\frac{\epsilon}{16}}$. □

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