A PRIORI BOUNDS AND MULTIPLICITY OF SOLUTIONS FOR AN INDEFINITE ELLIPTIC PROBLEM WITH CRITICAL GROWTH IN THE GRADIENT

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Abstract. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a smooth bounded domain. We consider a boundary value problem of the form

$$-\Delta u = c_\lambda(x)u + \mu(x)|\nabla u|^2 + h(x)\,, \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

where c_λ depends on a parameter $\lambda \in \mathbb{R}$, the coefficients c_λ and h belong to $L^q(\Omega)$ with $q > N/2$ and $\mu \in L^\infty(\Omega)$. Under suitable assumptions, but without imposing a sign condition on any of these coefficients, we obtain an a priori upper bound on the solutions. Our proof relies on a new boundary weak Harnack inequality. This inequality, which is of independent interest, is established in the general framework of the p-Laplacian. With this a priori bound at hand, we show the existence and multiplicity of solutions.

Résumé. Soit $\Omega \subset \mathbb{R}^N$, $N \geq 2$, un domaine borné régulier. Nous considérons un problème aux limites de la forme

$$-\Delta u = c_\lambda(x)u + \mu(x)|\nabla u|^2 + h(x)\,, \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega)$$

où c_λ dépend d'un paramètre $\lambda \in \mathbb{R}$, les coefficients c_λ et h sont des fonctions dans $L^q(\Omega)$ avec $q > N/2$ et $\mu \in L^\infty(\Omega)$. Sous certaines hypothèses, mais sans imposer une condition de signe sur aucun des coefficients, nous obtenons une borne à priori supérieure sur les solutions. Notre preuve repose sur une nouvelle inégalité de Harnack au bord. Cette inégalité, qui est d'intérêt propre, est établie dans le cadre plus général du p-Laplacien. L'obtention d'une borne à priori nous permet de démontrer l'existence et la multiplicité de solutions.

1. Introduction and main results

The paper deals with the existence and multiplicity of solutions for boundary value problems of the form

$$(Q_\lambda) \qquad -\Delta u = c_\lambda(x)u + \mu(x)|\nabla u|^2 + h(x)\,, \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega)\,,$$

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with c_λ depending on a real parameter λ. Here $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with boundary $\partial\Omega$ of class $\mathcal{C}^{1,1}$, c_λ and h belong to $L^q(\Omega)$ for some $q > N/2$ and μ belongs to $L^\infty(\Omega)$.

This type of problem, which started to be studied by L. Boccardo, F. Murat and J.P. Puel in the 80's, has attracted a new attention these last years. Under the condition $c_\lambda \leq -\alpha_0 < 0$ a.e. in Ω for some $\alpha_0 > 0$, the existence of a solution of (Q_λ) is a particular case of the results of [7,8] and its uniqueness follows from [5,6]. The case $c_\lambda \equiv 0$ was studied in [1,16] and the existence requires some smallness condition on $\|\mu h\|_{N/2}$. The situation where one only requires $c_\lambda \leq 0$ a.e. in Ω (i.e. allowing parts of the domain where $c_\lambda \equiv 0$ and parts of it where $c_\lambda < 0$) proved to be more complex to treat. In the recent papers [4,12], the authors explicit sufficient conditions for the existence of solutions of (Q_λ). Moreover, in [4], the uniqueness of solution is established (see also [3] in that direction). All these results were obtained without requiring any sign conditions on μ and h.

In case $c_\lambda = \lambda c \gneqq 0$, as we shall discuss later, problem (Q_λ) behaves very differently and becomes much richer. Following [26], which considers a particular case, [21] studied (Q_λ) with $\mu(x) \equiv \mu > 0$ and $\lambda c \gneqq 0$ but without a sign condition on h. The authors proved the existence of at least two solutions when $\lambda > 0$ and $\|(\mu h)^+\|_{N/2}$ are small enough. The restriction μ constant was removed in [4] and extended to $\mu(x) \geq \mu_1 > 0$ a.e. in Ω, at the expense of adding the hypothesis $h \gneqq 0$. Next, in [14], assuming stronger regularity on c and h, the authors removed the condition $h \gneqq 0$. In this paper, it is also lightened that the structure of the set of solutions when $\lambda > 0$, crucially depends on the sign of the (unique) solution of (Q_0). Note that, in [12], the above results are extended to the p-Laplacian case. Also, in the frame of viscosity solutions and fully nonlinear equations, under corresponding assumptions, similar conclusions have been obtained very recently in [24].

We refer to [21] for an heuristic explanation on how the behavior of (Q_λ) is affected by the change of sign in front of the linear term. Actually, in the case where $\mu(x) \equiv \mu$ is a constant, it is possible to transform problem (Q_λ) into a new one which admits a variational formulation. When $c_\lambda \leq -\alpha_0 < 0$, the associated functional, defined on $H_0^1(\Omega)$, is coercive. If $c_\lambda \lneqq 0$, the coerciveness may be lost and when $c_\lambda \gneqq 0$, in fact as soon as $c_\lambda^+ \gneqq 0$, the functional is unbounded from below. In [21] this variational formulation was directly used to obtain the solutions. In [4,14] where μ is non constant, topological arguments, relying on the derivation of a priori bounds for certain classes of solutions, were used.

The only known results where c_λ may change sign are [13,20] (see also [17] for related problems). They both concern the case where μ is a positive constant. In [20], assuming $h \gneqq 0$, μh and c_λ^+ small in an appropriate sense, the existence of at least two non-negative solutions was proved. In [13], the authors show that the loss of positivity of the coefficient of u does not affect the structure of the set of solutions of (Q_λ) observed in [14] when $c_\lambda = \lambda c \gneqq 0$. Since μ is constant in [13,20], it is possible to treat the problem variationally. The main issue, to derive the existence of solutions, is then to show the boundedness of the Palais-Smale sequences.

When $c_\lambda \gneqq 0$, all the above mentioned results require either μ to be constant or to be uniformly bounded from below by a positive constant (or similarly bounded from above by a negative constant). In [29], assuming that the three coefficients functions are non-negative, a first attempt to remove these restrictions on μ is presented. Following the approach of [4], the proofs of the existence results reduce to obtaining a priori bounds on the non negative solutions of (Q_λ). First it is observed in [29] that a necessary condition is the existence of a ball $B(x_0, \rho) \subset \Omega$ and $\nu > 0$ such that $\mu \geq \nu$ and $c \geq \nu$ on $B(x_0, \rho)$. When $N = 2$ this condition also proves to be sufficient. If $N = 3$ or 4 the condition $\mu \geq \mu_0 > 0$ on a set $\omega \subset \Omega$ such that $\text{supp}(c) \subset \overline{\omega}$ permits to obtain the a priori bounds. Other sets of conditions are presented when $N = 3$ and $N = 5$. However, if the approach developed in [29], which relies on interpolation and elliptic estimates in weighted Lebesgue spaces, works well in low dimension, the possibility to extend it to dimension $N \geq 6$ is not apparent.

In this paper we pursue the study of (Q_λ) and consider situations where the three coefficients functions c_λ, μ and h may change sign. We define for $v \in L^1(\Omega)$, $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$. As observed already in [13], the structure of the solution set depends on the size of the positive hump (i.e. c_λ^+) but it is not affect by the size of the negative hump (i.e. c_λ^-). Hoping to clarify this, we now write c_λ under the form

$c_\lambda = \lambda c_+ - c_-$ and consider the problem

$$(P_\lambda) \qquad -\Delta u = (\lambda c_+(x) - c_-(x))u + \mu(x)|\nabla u|^2 + h(x)\,, \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega)\,,$$

under the assumption

$$(A_1) \qquad \begin{cases} \Omega \subset \mathbb{R}^N,\ N \geq 2,\ \text{is a bounded domain with boundary } \partial\Omega \text{ of class } \mathcal{C}^{1,1}, \\ c_+, c_-, h^+ \in L^q(\Omega) \text{ for some } q > N/2\,,\ \mu, h^- \in L^\infty(\Omega)\,, \\ c_+(x) \geq 0,\ c_-(x) \geq 0 \text{ and } c_-(x)c_+(x) = 0 \text{ a.e. in } \Omega, \\ |\Omega_+| > 0,\ \text{where } \Omega_+ := \mathrm{Supp}(c_+) \\ \text{there exists a } \varepsilon > 0 \text{ such that } \mu(x) \geq \mu_1 > 0 \text{ and } c_- = 0 \text{ in } \{x \in \Omega : d(x, \Omega_+) < \varepsilon\}. \end{cases}$$

For a definition of $\mathrm{Supp}(f)$ with $f \in L^p(\Omega)$, for some $p \geq 1$, we refer to [9, Proposition 4.17]. Note also that the condition that $c_- = 0$ on $\{x \in \Omega : d(x, \Omega_+) < \varepsilon\}$ for some $\varepsilon > 0$, is reminiscent of the so-called "thick zero set" condition first introduced in [2].

We also observe that, under the regularity assumptions of condition (A_1), any solution of (P_λ) belongs to $\mathcal{C}^{0,\tau}(\overline{\Omega})$ for some $\tau > 0$. This can be deduce from [22, Theorem IX-2.2], see also [3, Proposition 2.1].

As in [4, 14, 29] we obtain our results using a topological approach, relying thus on the derivation of a priori bounds. In that direction our main result is the following.

Theorem 1.1. *Assume (A_1). Then, for any $\Lambda_2 > \Lambda_1 > 0$, there exists a constant $M > 0$ such that, for each $\lambda \in [\Lambda_1, \Lambda_2]$, any solution of (P_λ) satisfies $\sup_\Omega u \leq M$.*

Having at hand this a priori bound, following the strategy of [4], we show the existence of a continuum of solutions of (P_λ). More precisely, defining

$$\Sigma := \{(\lambda, u) \in \mathbb{R} \times \mathcal{C}(\overline{\Omega}) : u \text{ solves } (P_\lambda)\}, \tag{1.1}$$

we prove the following theorem.

Theorem 1.2. *Assume (A_1) and suppose that (P_0) has a solution u_0 with $c_+u_0 \gneqq 0$. Then, there exists a continuum $\mathscr{C} \subset \Sigma$ such that the projection of $\mathscr{C}$ on the λ-axis is an unbounded interval $(-\infty, \overline{\lambda}]$ for some $\overline{\lambda} \in (0, +\infty)$ and $\mathscr{C}$ bifurcates from infinity to the right of the axis $\lambda = 0$. Moreover:*

1) *for all $\lambda \leq 0$, the problem (P_λ) has an unique solution u_λ and this solution satisfies $u_0 - \|u_0\|_\infty \leq u_\lambda \leq u_0$.*
2) *there exists $\lambda_0 \in (0, \overline{\lambda}]$ such that, for all $\lambda \in (0, \lambda_0)$, the problem (P_λ) has at least two solutions with $u_i \geq u_0$ for $i = 1, 2$.*

Remark 1.1.

(a) Theorem 1.2, 1) generalizes [4, Theorem 1.2].

(b) Note that problem (P_0) is given by

$$-\Delta u = -c_-(x)u + \mu(x)|\nabla u|^2 + h(x)\,, \qquad u \in H_0^1(\Omega) \cap L^\infty(\Omega)\,.$$

In [4, 12] the authors give sufficient conditions to ensure the existence of a solution of (P_0). Moreover, if $h \geq 0$ in Ω, [3, Lemma 2.2] implies that the solution of (P_0) is non-negative.

Let us give some ideas of the proofs. As we do not have global sign conditions, the approaches used in [4, 14, 29] to obtain the a priori bounds do not apply anymore and another strategy is required. To this aim, we further develop some techniques first sketched in the unpublished work [27]. These techniques, in the framework of viscosity solutions of fully nonlinear equations, now lies at the heart of the paper [24]. We also make use of some ideas from [17]. First we show, in Lemma 4.1, that it is sufficient to control the behavior of the solutions on $\overline{\Omega}_+$. By compactness, we are then reduced to study what happens around an (unknown) point $\overline{x} \in \overline{\Omega}_+$. We shall consider separately the alternative cases $\overline{x} \in \overline{\Omega}_+ \cap \Omega$ and $\overline{x} \in \overline{\Omega}_+ \cap \partial\Omega$. A local analysis is made respectively in a ball or a semiball centered at $\overline{x}$. If similar analysis, based on the use of Harnack type inequalities, had previously been performed in other contexts when $\overline{x} \in \Omega$, we believe it is not the case when $\overline{x} \in \partial\Omega$. For $\overline{x} \in \partial\Omega$, the key to our approach is the use of boundary weak Harnack inequality. Actually a major part of the paper is devoted to establishing this inequality. This is done in a more general

context than needed for (P_λ). In particular it also cover the case of the p-Laplacian with a zero order term. We believe that this "boundary weak Harnack inequality", see Theorem 3.1, is of independent interest and will proved to be useful in other settings. Its proof uses ideas introduced by B. Sirakov [28]. In [28] such type of inequalities is established for an uniformly elliptic operator and viscosity solutions. However, since our context is quite different, the result of [28] does not apply to our situation and we need to work out an adapted proof.

We now describe the organization of the paper. In Section 2, we present some preliminary results which are needed in the development of our proofs. In Section 3, we prove the boundary weak Harnack inequality for the p-Laplacian. The a priori bound, namely Theorem 1.1, is proved in Section 4. Finally Section 5 is devoted to the proof of Theorem 1.2.

Notation.

1) In $\mathbb{R}^N$, we use the notations $|x| = \sqrt{x_1^2 + \ldots + x_N^2}$ and $B_R(y) = \{x \in \mathbb{R}^N : |x - y| < R\}$.
2) We denote $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}^- = (-\infty, 0)$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$.
3) For h_1, $h_2 \in L^1(\Omega)$ we write
   - $h_1 \le h_2$ if $h_1(x) \le h_2(x)$ for a.e. $x \in \Omega$,
   - $h_1 \lneqq h_2$ if $h_1 \le h_2$ and $\text{meas}(\{x \in \Omega : h_1(x) < h_2(x)\}) > 0$.

2. Preliminary results

In this section, we collect some results which will play an important role throughout the work. First of all, let us consider the boundary value problem

$$-\Delta u + H(x, u, \nabla u) = f, \qquad u \in H_0^1(\omega) \cap L^\infty(\omega). \tag{2.1}$$

Here $\omega \subset \mathbb{R}^N$ is a bounded domain, $f \in L^1(\omega)$ and $H : \omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function.

Definition 2.1. We say that $\alpha \in H^1(\omega) \cap L^\infty(\omega)$ is a *lower solution* of (2.1) if $\alpha^+ \in H_0^1(\omega)$ and, for all $\varphi \in H_0^1(\omega) \cap L^\infty(\omega)$ with $\varphi \ge 0$, we have

$$\int_\omega \nabla\alpha\nabla\varphi\, dx + \int_\omega H(x, \alpha, \nabla\alpha)\varphi\, dx \le \int_\omega f(x)\varphi\, dx\,.$$

Similarly, $\beta \in H^1(\omega) \cap L^\infty(\omega)$ is an *upper solution* of (2.1) if $\beta^- \in H_0^1(\omega)$ and, for all $\varphi \in H_0^1(\omega) \cap L^\infty(\omega)$ with $\varphi \ge 0$, we have

$$\int_\omega \nabla\beta\nabla\varphi\, dx + \int_\omega H(x, \beta, \nabla\beta)\varphi\, dx \ge \int_\omega f(x)\varphi\, dx\,.$$

Next, we consider the boundary value problem

$$-\Delta u + a(x)u = b(x)\,, \qquad u \in H_0^1(\omega)\,, \tag{2.2}$$

under the assumption

$$\begin{cases} \omega \subset \mathbb{R}^N,\ N \ge 2\,,\ \text{is a bounded domain,} \\ a,\ b \in L^r(\omega)\ \text{for some}\ r > N/2. \end{cases} \tag{2.3}$$

Remark 2.1. With the regularity imposed in the following lemmas and in the absence of a gradient term in the equation, we do not need the lower and upper solutions to be bounded. The full Definition 2.1 will however be needed in other parts of the paper.

Lemma 2.1. (Local Maximum Principle) *Under the assumption (2.3), assume that $u \in H^1(\omega)$ is a lower solution of (2.2). For any ball $B_{2R}(y) \subset \omega$ and any $s > 0$, there exists $C = C(s, r, \|a\|_{L^r(B_{2R}(y))}, R) > 0$ such that*

$$\sup_{B_R(y)} u^+ \le C\Big[\Big(\int_{B_{2R}(y)} (u^+)^s dx\Big)^{1/s} + \|b^+\|_{L^r(B_{2R}(y))}\Big]\,.$$

Proof. See for instance [18, Theorem 8.17] and [23, Corollary 3.10]. □

Lemma 2.2. (Boundary Local Maximum Principle) *Under the assumption (2.3), assume that $u \in H^1(\omega)$ is a lower solution of (2.2) and let $x_0 \in \partial\omega$. For any $R > 0$ and any $s > 0$, there exists $C = C(s, r, \|a\|_{L^r(B_{2R}(x_0)\cap\omega)}, R) > 0$ such that*

$$\sup_{B_R(x_0)\cap\omega} u^+ \le C\Big[\Big(\int_{B_{2R}(x_0)\cap\omega} (u^+)^s dx\Big)^{1/s} + \|b^+\|_{L^r(B_{2R}(x_0)\cap\omega)}\Big].$$

Proof. See for instance [18, Theorem 8.25] and [23, Corollary 3.10 and Theorem 3.11]. □

Remark 2.2. Lemmas 2.1 and 2.2 proof's are done in [18] for $a \in L^\infty(\omega)$ and $s > 1$. Nevertheless, as it is remarked on page 193 of that book, the proof is valid for $a \in L^r(\omega)$ with $r > N/2$ and, following closely the proof of [23, Corollary 3.10], the proofs can be extended for any $s > 0$.

Lemma 2.3. (Weak Harnack Inequality) *Under the assumption (2.3), assume that $u \in H^1(\omega)$ is a non-negative upper solution of (2.2). Then, for any ball $B_{4R}(y) \subset \omega$ and any $1 \le s < \frac{N}{N-2}$ there exists $C = C(s, r, \|a\|_{L^r(B_{4R}(y))}, R) > 0$ such that*

$$\inf_{B_R(y)} u \ge C\Big[\Big(\int_{B_{2R}(y)} u^s dx\Big)^{1/s} - \|b^-\|_{L^r(B_{4R}(y))}\Big].$$

Proof. See for instance [18, Theorem 8.18] and [23, Theorem 3.13]. □

Now, inspired by [10, Lemma 3.2] (see also [15, Appendix A]), we establish the following version of the Brezis-Cabré Lemma.

Lemma 2.4. *Let $\omega \subset \mathbb{R}^N$, $N \ge 2$, be a bounded domain with boundary $\partial\omega$ of class $\mathcal{C}^{1,1}$ and let $a \in L^\infty(\omega)$ and $f \in L^1(\omega)$ be non-negative functions. Assume that $u \in H^1(\omega)$ is an upper solution of*

$$-\Delta u + a(x)u = f(x)\,, \quad u \in H^1_0(\omega)\,.$$

Then, for every $B_{2R}(y) \subset \omega$, there exists $C = C(R, y, \omega, \|a\|_\infty) > 0$ such that

$$\inf_\omega \frac{u(x)}{d(x,\partial\omega)} \ge C \int_{B_R(y)} f(x)\,dx\,.$$

Proof. First of all, as a and f are non-negative, by the weak maximum principle, it follows that

$$\inf_\omega \frac{u(x)}{d(x,\partial\omega)} \ge 0\,.$$

Now let $B_{2R}(y) \subset \omega$. By the above inequality, we can assume without loss of generality that

$$\int_{B_R(y)} f(x)\,dx > 0\,.$$

We split the proof into three steps.

Step 1: *There exists $c_1 = c_1(R, y, \omega, \|a\|_\infty) > 0$ such that*

$$\frac{u(x)}{d(x,\partial\omega)} \ge c_1 \int_{B_R(y)} f(x)\,dx\,, \quad \forall\, x \in \overline{B_{R/2}(y)}\,. \tag{2.4}$$

Since f is non-negative, observe that u is a non-negative upper solution of

$$-\Delta u + a(x)u = 0\,, \quad u \in H^1_0(\omega)\,.$$

Hence, by Lemma 2.3, there exists a constant $c_2 = c_2(R, \|a\|_\infty) > 0$ such that

$$u(x) \ge c_2 \int_{B_R(y)} u\,dx\,, \quad \forall\, x \in \overline{B_{R/2}(y)}\,. \tag{2.5}$$

Now, let us denote by ξ the solution of

$$\begin{cases} -\Delta\xi + \|a\|_\infty \xi = \chi_{B_R(y)}\,, & \text{in } \omega\,, \\ \xi = 0\,, & \text{on } \partial\omega\,. \end{cases} \tag{2.6}$$

By [11, Theorem 3], there exists a constant $c_3 = c_3(R, y, \omega, \|a\|_\infty) > 0$ such that, for all $x \in \omega$, $\xi(x) \geq c_3 d(x, \partial\omega)$. Thus, since $B_{2R}(y) \subset \omega$, f is non-negative and $d(x, \partial\omega) \geq R$ for $x \in B_R(y)$, it follows that

$$\int_{B_R(y)} u\,dx = \int_\omega u\big(-\Delta\xi + \|a\|_\infty \xi\big)\,dx \geq \int_\omega f(x)\,\xi\,dx \geq c_3 \int_\omega f(x)\,d(x,\partial\omega)\,dx \geq c_3 R \int_{B_R(y)} f(x)\,dx\,.$$

Hence, substituting the above information in (2.5) we obtain for $c_4 = c_2 c_3 R$

$$u(x) \geq c_4 \int_{B_R(y)} f(x)\,dx\,, \quad \forall\, x \in \overline{B_{R/2}(y)}\,, \tag{2.7}$$

from which, since $\omega \subset \mathbb{R}^N$ is bounded, (2.4) follows.

Step 2: *There exists $c_5 = c_5(R, y, \omega, \|a\|_\infty) > 0$ such that*

$$\frac{u(x)}{d(x,\partial\omega)} \geq c_5 \int_{B_R(y)} f(x)\,dx\,, \quad \forall\, x \in \omega \setminus \overline{B_{R/2}(y)}. \tag{2.8}$$

Let w be the unique solution of

$$\begin{cases} -\Delta w + \|a\|_\infty w = 0\,, & \text{in } \omega \setminus \overline{B_{R/2}(y)}\,, \\ w = 0\,, & \text{on } \partial\omega\,, \\ w = 1\,, & \text{on } \partial B_{R/2}(y)\,. \end{cases} \tag{2.9}$$

Still by [11, Theorem 3], there exists $c_6 = c_6(R, y, \omega, \|a\|_\infty) > 0$ such that $w(x) \geq c_6 d(x, \partial\omega)$ for all $x \in \omega \setminus \overline{B_{R/2}(y)}$. On the other hand, let us introduce

$$v(x) = \frac{u(x)}{c_4 \int_{B_R(y)} f(x)\,dx}\,,$$

with c_4 given in (2.7). Observe that v is an upper solution of (2.9). Hence, by the standard comparison principle, it follows that $v(x) \geq w(x)$ for all $x \in \omega \setminus \overline{B_{R/2}(y)}$ and (2.8) follows.

Step 3: *Conclusion.*

The result follows from (2.4) and (2.8). □

3. Boundary weak Harnack inequality

In this section we present a *boundary weak Harnack inequality* that will be central in the proof of Theorem 1.1. As we believe this type of inequality has its own interest, we establish it in the more general framework of the p-Laplacian. Recalling that $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ for $1 < p < \infty$, we introduce the boundary value problem

$$-\Delta_p u + a(x)|u|^{p-2}u = 0\,, \quad u \in W_0^{1,p}(\omega)\,. \tag{3.1}$$

Let us also recall that $u \in W^{1,p}(\omega)$ is an *upper solution of* (3.1) if $u^- \in W_0^{1,p}(\omega)$ and, for all $\varphi \in W_0^{1,p}(\omega)$ with $\varphi \geq 0$, it follows that

$$\int_\omega |\nabla u|^{p-2}\nabla u\,\nabla\varphi\,dx + \int_\omega a(x)|u|^{p-2}u\,\varphi\,dx \geq 0\,.$$

We then prove the following result.

Theorem 3.1. (Boundary Weak Harnack Inequality) Let $\omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with boundary $\partial\omega$ of class $\mathcal{C}^{1,1}$ and let $a \in L^\infty(\omega)$ be a non-negative function. Assume that u is a non-negative upper solution of (3.1) and let $x_0 \in \partial\omega$. Then, there exist $\overline{R} > 0$, $\varepsilon = \varepsilon(p, \overline{R}, \|a\|_\infty, \omega) > 0$ and $C = C(p, \overline{R}, \varepsilon, \|a\|_\infty, \omega) > 0$ such that, for all $R \in (0, \overline{R}]$,

$$\inf_{B_R(x_0)\cap\omega} \frac{u(x)}{d(x,\partial\omega)} \geq C\Big(\int_{B_R(x_0)\cap\omega} \Big(\frac{u(x)}{d(x,\partial\omega)}\Big)^\varepsilon dx\Big)^{1/\varepsilon}.$$

As already indicated, in the proof of Theorem 3.1 we shall make use of some ideas from [28].

Before going further, let us introduce some notation that we will be used throughout the section. We define

$$r := r(N,p) = \begin{cases} \dfrac{N(p-1)}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N, \end{cases}$$

and denote by $Q_\rho(y)$ the cube of center y and side of length ρ, i.e.

$$Q_\rho(y) = \{x \in \mathbb{R}^N : |x_i - y_i| < \rho/2 \text{ for } i = 1, \dots, N\}.$$

In case the center of the cube is ρe with $e = (0, 0, \dots, 1/2)$, we use the notation $Q_\rho = Q_\rho(\rho e)$.

Let us now introduce several auxiliary results that we shall need to prove Theorem 3.1. We begin recalling the following comparison principle for the p-Laplacian.

Lemma 3.2. [30, Lemma 3.1] *Let $\omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain and let $a \in L^\infty(\omega)$ be a non-negative function. Assume that $u, v \in W^{1,p}(\omega)$ satisfy (in a weak sense)*

$$\begin{cases} -\Delta_p u + a(x)|u|^{p-2}u \leq -\Delta_p v + a(x)|v|^{p-2}v\,, & \text{in } \omega\,, \\ u \leq v\,, & \text{on } \partial\omega\,. \end{cases}$$

Then, it follows that $u \leq v$.

As a second ingredient, we need the weak Harnack inequality.

Theorem 3.3. [23, Theorem 3.13] *Let $\omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain and let $a \in L^\infty(\omega)$ be a non-negative function. Assume that $u \in W^{1,p}(\omega)$ is a non-negative upper solution of*

$$-\Delta_p u + a(x)|u|^{p-2}u = 0\,, \quad u \in W_0^{1,p}(\omega)\,,$$

and let $Q_\rho(x_0) \subset \omega$. Then, for any $\sigma, \tau \in (0,1)$ and $\gamma \in (0, r)$, there exists $C = C(p, \gamma, \sigma, \tau, \rho, \|a\|_\infty) > 0$ such that

$$\inf_{Q_{\tau\rho}(x_0)} u \geq C\Big(\int_{Q_{\sigma\rho}(x_0)} u^\gamma\,dx\Big)^{1/\gamma}.$$

In the next result, we deduce a more precise information on the dependence of C with respect to ρ. This is closely related to [31, Theorem 1.2] where however the constant still depends on ρ.

Corollary 3.4. *Let a be a non-negative constant and $\gamma \in (0, r)$. There exists $C = C(p, \gamma, a) > 0$ such that, for all $0 < \tilde\rho \leq 1$, any $u \in W^{1,p}(Q_{\frac{3\tilde\rho}{2}}(e))$ non-negative upper solution of*

$$-\Delta_p u + a|u|^{p-2}u = 0\,, \quad u \in W_0^{1,p}(Q_{\frac{3\tilde\rho}{2}}(e)), \tag{3.2}$$

satisfies

$$\inf_{Q_{\tilde\rho}(e)} u \geq C\,\tilde\rho^{\,-N/\gamma}\Big(\int_{Q_{\tilde\rho}(e)} u^\gamma\,dx\Big)^{1/\gamma}.$$

Proof. Let $C = C(p, a, \gamma) > 0$ be the constant given by Theorem 3.3 applied with $\rho = \frac{3}{2}$ and $\sigma = \tau = \frac{2}{3}$. This means that if $v \in W^{1,p}(Q_{\frac{3}{2}}(e))$ is a non-negative upper solution of

$$-\Delta_p v + a|v|^{p-2}v = 0\,, \qquad v \in W_0^{1,p}(Q_{\frac{3}{2}}(e)), \tag{3.3}$$

then

$$\inf_{Q_1(e)} v(y) \geq C\Big(\int_{Q_1(e)} v^\gamma\,dy\Big)^{1/\gamma}.$$

As $0 < \tilde\rho \leq 1$, observe that if u is a non-negative upper solution of (3.2), then v defined by $v(y) = u(\tilde\rho y', \tilde\rho(y_N - \frac{1}{2}) + \frac{1}{2})$, where $y = (y', y_N)$ with $y' \in \mathbb{R}^{N-1}$, is a non-negative upper solution of (3.3). Thus, we can conclude that

$$\inf_{Q_{\tilde\rho}(e)} u(x) = \inf_{Q_1(e)} v(y) \geq C\Big(\int_{Q_1(e)} v^\gamma\,dy\Big)^{1/\gamma} = C\tilde\rho^{\,-N/\gamma}\Big(\int_{Q_{\tilde\rho}(e)} u^\gamma\,dx\Big)^{1/\gamma}.$$

□

Finally, we introduce a technical result of measure theory.

Lemma 3.5. [19, Lemma 2.1] *Let $E \subset F \subset Q_1$ be two open sets. Assume there exists $\alpha \in (0,1)$ such that:*

- $|E| \leq (1-\alpha)|Q_1|$.
- *For any cube $Q \subset Q_1$, $|Q \cap E| \geq (1-\alpha)|Q|$ implies $Q \subset F$.*

Then, it follows that $|E| \leq (1-c\alpha)|F|$ for some constant $c = c(N) \in (0,1)$.

Now, we can perform the proof of the main result. We prove the boundary weak Harnack inequality for cubes and as consequence we obtain the desired result.

Lemma 3.6 (Growth lemma). *Let a be a non-negative constant. Given $\nu > 0$, there exists $k = k(p,\nu,a) > 0$ such that, if $u \in W^{1,p}(Q_{\frac{3}{2}})$ is a non-negative upper solution of*

$$-\Delta_p u + a\,|u|^{p-2}u = 0\,, \quad u \in W_0^{1,p}(Q_{\frac{3}{2}})\,,$$

and the following inequality holds

$$|\{x \in Q_1 : u(x) > x_N\}| \geq \nu. \tag{3.4}$$

Then $u(x) > kx_N$ in Q_1.

Remark 3.1. Before we prove the Lemma, observe that there is no loss of generality in considering a a non-negative constant instead of $a \in L^\infty(Q_{\frac{3}{2}})$ non-negative. If $u \geq 0$ satisfies

$$-\Delta_p u + a(x)|u|^{p-2}u \geq 0\,, \quad \text{in } Q_{\frac{3}{2}}\,,$$

then u satisfies also

$$-\Delta_p u + \|a\|_\infty\,|u|^{p-2}u \geq 0\,, \quad \text{in } Q_{\frac{3}{2}}\,.$$

Proof. Let us define $S_\delta = Q_{\frac{3}{2}} \setminus Q_{\frac{3}{2}-\delta}\big(\frac{3}{2}e\big)$ and fix $c_1 = c_1(\nu) \in (0, \frac{1}{2})$ small enough in order to ensure that $|S_\delta| \leq \frac{\nu}{2}$ for any $0 < \delta \leq c_1$.

Step 1: *For all $\delta \in (0, c_1]$, it follows that $|\{x \in Q_{\frac{3}{2}-\delta}\big(\frac{3}{2}e\big) : u(x) > x_N\}| \geq \frac{\nu}{2}$.*

Directly observe that

$$\{x \in Q_1 : u(x) > x_N\} \subset \{x \in Q_{\frac{3}{2}} : u(x) > x_N\} \subset \{x \in Q_{\frac{3}{2}-\delta}\big(\frac{3}{2}e\big) : u(x) > x_N\} \cup S_\delta.$$

Hence, Step 1 follows from (3.4) and the choice of c_1.

Step 2: *For any $\varepsilon > 0$ and all $\delta \in (0, c_1]$, the following inequality holds*

$$\Big(\int_{Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right)} u^\varepsilon\,dx\Big)^{1/\varepsilon} \geq \frac{\delta}{2}\big(\frac{\nu}{2}\big)^{1/\varepsilon}. \tag{3.5}$$

Since $u \geq 0$ and, for any $x \in Q_{\frac{3}{2}-\delta}\big(\frac{3}{2}e\big)$ we have $x_N \geq \frac{\delta}{2}$, it follows that

$$\begin{aligned}\int_{Q_{\frac{3}{2}-\delta}(\frac{3}{2}e)} u^\varepsilon\,dx &\geq \int_{\{x \in Q_{\frac{3}{2}-\delta}(\frac{3}{2}e):\ u(x) \geq x_N\}} u^\varepsilon\,dx \\ &\geq \int_{\{x \in Q_{\frac{3}{2}-\delta}(\frac{3}{2}e): u(x) \geq x_N\}} \Big(\frac{\delta}{2}\Big)^\varepsilon\,dx = \Big(\frac{\delta}{2}\Big)^\varepsilon \Big|\Big\{x \in Q_{\frac{3}{2}-\delta}\big(\frac{3}{2}e\big) : u(x) \geq x_N\Big\}\Big|.\end{aligned}$$

Step 2 follows then from Step 1.

Step 3: *For any $\varepsilon \in (0,r)$ and all $\delta \in (0,c_1]$, there exists $C_\delta = C_\delta(p,\varepsilon,\delta,a) > 0$ such that*

$$\inf_{Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right)} \frac{u(x)}{x_N} \geq \frac{\delta\,C_\delta}{3}\Big(\frac{\nu}{2}\Big)^{1/\varepsilon}.$$

By Theorem 3.3 applied with $\rho = \frac{3}{2}$, $x_0 = \frac{3}{2}e$ and $\tau = \sigma = 1 - \frac{2}{3}\delta$, there exists a constant $C_\delta = C_\delta(p,\varepsilon,\delta,a) > 0$ such that

$$\inf_{Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right)} u(x) \geq C_\delta\Big(\int_{Q_{\frac{3}{2}-\delta}\left(\frac{3}{2}e\right)} u^\varepsilon\,dx\Big)^{1/\varepsilon}.$$

Since for all $x \in Q_{\frac{3}{2}-\delta}\big(\frac{3}{2}e\big)$ we have $x_N \le \frac{3}{2}$, Step 3 follows from the above inequality and Step 2.

Step 4: *Conclusion.*

We fix $\varepsilon \in (0,r)$, define $k_\delta = \frac{\delta\, C_\delta}{3}\big(\frac{\nu}{2}\big)^{1/\varepsilon}$ and introduce $\eta : [-\frac{3-2c_1}{4}, \frac{3-2c_1}{4}]^{N-1} \to \mathbb{R}$ a $\mathcal{C}^\infty$ function satisfying

$$\eta(x_1,\dots,x_{N-1}) = \begin{cases} 0, & \text{if } (x_1,\dots,x_{N-1}) \in \big[-\frac{1}{2},\frac{1}{2}\big]^{N-1}, \\ \frac{c_1}{2}, & \text{if } (x_1,\dots,x_{N-1}) \in \partial_{\mathbb{R}^{N-1}}\big(\big[-\frac{3-2c_1}{4},\frac{3-2c_1}{4}\big]^{N-1}\big), \end{cases}$$

and

$$0 \le \eta(x_1,\dots,x_{N-1}) \le \frac{c_1}{2} \qquad \text{for } (x_1,\dots,x_{N-1}) \in \big[-\tfrac{3-2c_1}{4},\tfrac{3-2c_1}{4}\big]^{N-1}.$$

Moreover, we consider the auxiliary function

$$v_\delta(x_1,\dots,x_N) = \frac{1}{\delta}\big(x_N - \eta(x_1,\dots,x_{N-1})\big)^2 + \big(x_N - \eta(x_1,\dots,x_{N-1})\big)$$

defined in

$$\omega_\delta = \Big\{(x_1,\dots,x_N) \in \big[-\tfrac{3-2c_1}{4},\tfrac{3-2c_1}{4}\big]^{N-1} \times \big[0,\tfrac{c_1}{2}\big] : \eta(x_1,\dots,x_{N-1}) \le x_N \le \frac{\delta}{2}\Big\}.$$

Observe that, in ω_δ, we have $0 \le x_N - \eta(x_1,\dots,x_{N-1}) \le \frac{\delta}{2}$. Hence, there exists $c_2 = c_2(p,\nu,a) \in (0,c_1]$ such that, for all $0 < \delta \le c_2$,

$$-\Delta_p v_\delta + a|v_\delta|^{p-2}v_\delta \le -\frac{2}{\delta}(p-1) + 2^{p-1}\Big|\sum_{i=1}^{N-1}\frac{\partial}{\partial x_i}\Big[\Big(\sum_{i=1}^{N-1}(\frac{\partial\eta}{\partial x_i})^2+1\Big)^{\frac{p-2}{2}}\frac{\partial\eta}{\partial x_i}\Big]\Big| + \frac{3a}{4}\delta \le 0\,, \qquad \text{in } \omega_\delta.$$

On the other hand, we define $u_\delta = \frac{2u}{k_\delta}$ and immediately observe that

$$-\Delta_p u_\delta + a|u_\delta|^{p-2}u_\delta \ge 0\,, \qquad \text{in } \omega_\delta\,.$$

Now, since by Step 3, we have

$$u_\delta \ge \frac{2k_\delta}{k_\delta}\frac{\delta}{2} = \delta \ge v_\delta, \qquad \text{for } x_N = \frac{\delta}{2},$$

it follows that

$$u_\delta \ge v_\delta \qquad \text{on } \partial\omega_\delta.$$

Then, applying Lemma 3.2, it follows that, for any $\delta \in (0,c_2]$, $v_\delta \le u_\delta$ in ω_δ. For $\delta = c_2/2$, we obtain in particular

$$u(x) \ge \frac{1}{2}k_{\frac{c_2}{2}}v_{\frac{c_2}{2}}(x) = \frac{1}{2}k_{\frac{c_2}{2}}\big(\frac{2}{c_2}x_N^2 + x_N\big) \ge \frac{1}{2}k_{\frac{c_2}{2}}x_N\,, \qquad \text{in } \omega_{\frac{c_2}{2}} \cap Q_1\,.$$

The result then follows from the above inequality and Step 3. □

Lemma 3.7. *Let $a \in L^\infty(Q_4)$ be a non-negative function. Assume that $u \in W^{1,p}(Q_4)$ is a non-negative upper solution of*

$$-\Delta_p u + a(x)|u|^{p-2}u = 0\,, \quad u \in W_0^{1,p}(Q_4)\,, \tag{3.6}$$

satisfying

$$\inf_{Q_1}\frac{u(x)}{x_N} \le 1\,.$$

Then, there exist $M = M(p,\|a\|_\infty) > 1$ and $\mu \in (0,1)$ such that

$$\big|\{x \in Q_1 : u(x)/x_N > M^j\}\big| < (1-\mu)^j\,, \quad \forall\, j \in \mathbb{N}\,. \tag{3.7}$$

Proof. Let us fix some notation that we use throughout the proof. We fix $\gamma \in (0,r)$ and consider $C_1 = C_1(p,\|a\|_\infty) > 0$ the constant given by Corollary 3.4. We introduce $\alpha \in (0,1)$ and fix $C_2 \in (0,1)$ the constant given by Lemma 3.5. Moreover, we choose $\nu = (1-\alpha)\big(\frac{1}{4}\big)^N$ and denote by $k = k(\nu,p,\|a\|_\infty) \in (0,1)$ the constant given by Lemma 3.6 applied to an upper solution of

$$-\Delta_p u + 2^p\|a\|_\infty|u|^{p-2}u = 0\,, \quad u \in W_0^{1,p}(Q_{\frac{3}{2}})\,, \tag{3.8}$$

with the chosen ν. Let us point out that, if u is a non-negative upper solution of (3.6), then u is a non-negative upper solution of (3.8). Finally, we consider

$$M \geq \max\Big\{\frac{1}{k}, \frac{4}{C_1}(1-\alpha)^{-1/\gamma}\Big\},$$

and we are going to show that (3.7) holds with $\mu = \alpha C_2$.

First of all, observe that $\{x \in Q_1 : u(x)/x_N > M\} \subset \{x \in Q_1 : ku(x) > x_N\}$. Hence, since $\inf_{Q_1} ku(x)/x_N \leq k$, Lemma 3.6 implies that

$$|\{x \in Q_1 : u(x)/x_N > M\}| \leq |\{x \in Q_1 : ku(x) > x_N\}| < \nu < 1-\alpha < 1 - C_2\alpha \tag{3.9}$$

and, in particular, (3.7) holds for $j = 1$. Now, let us introduce, for $j \in \mathbb{N} \setminus \{1\}$,

$$E = \{x \in Q_1 : u(x)/x_N > M^j\} \quad \text{and} \quad F = \{x \in Q_1 : u(x)/x_N > M^{j-1}\}.$$

Since $M > 1$ and $j \in \mathbb{N} \setminus \{1\}$, observe that (3.9) implies that

$$|E| = |\{x \in Q_1 : u(x)/x_N > M^j\}| \leq |\{x \in Q_1 : u(x)/x_N > M\}| \leq 1-\alpha, \tag{3.10}$$

and the first assumption of Lemma 3.5 is satisfied.

Claim: *For every cube $Q_\rho(x_0) \subset Q_1$ such that*

$$|E \cap Q_\rho(x_0)| \geq (1-\alpha)|Q_\rho(x_0)| = (1-\alpha)\rho^N. \tag{3.11}$$

we have $Q_\rho(x_0) \subset F$.

Let us denote $x_0 = (x_0', x_{0_N})$ with $x_0' \in \mathbb{R}^{N-1}$. We define the new variable $y = \big(\frac{x'-x_0'}{\rho'}, \frac{x_N}{\rho'}\big)$, where $\rho' = 2x_{0_N}$, and the rescaled function $v(y) = \frac{1}{\rho'}u(\rho' y' + x_0', \rho' y_N)$. Then v is a non-negative upper solution of

$$-\Delta_p v + 2^p\|a\|_\infty |v|^{p-2}v = 0, \quad \text{in } Q_{4/\rho'}\big(-x_0'/\rho', 2/\rho'\big). \tag{3.12}$$

Moreover, observe that

$$x \in E \cap Q_\rho(x_0) \quad \text{if and only if} \quad y \in \{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > y_N\},$$

and so, that (3.11) is equivalent to

$$|\{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > y_N\}| \geq (1-\alpha)|Q_{\rho/\rho'}(e)| = (1-\alpha)\big(\frac{\rho}{\rho'}\big)^N. \tag{3.13}$$

Observe also that the embedding $Q_\rho(x_0) \subset Q_1$ implies that $\rho \leq \rho' \leq 2-\rho$ and $|x_{0,i}| \leq \frac{1-\rho}{2}$ for $i \in \{1, \dots, N-1\}$. In particular, we have $Q_{\frac{3}{2}} \subset Q_{4/\rho'}\big(-x_0'/\rho', 2/\rho'\big)$. Hence v is an upper solution of (3.8).

Now, we distinguish two cases:

Case 1: $\rho \geq \rho'/4$. Observe that v/M^j is a non-negative upper solution of (3.8). Moreover, as $\rho \leq \rho'$, (3.13) implies that

$$|\{y \in Q_1 : v(y)/M^j > y_N\}| \geq |\{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > y_N\}| \geq \nu.$$

Hence, by Lemma 3.6, $v(y)/M^j > ky_N$ in Q_1 and so, by the definition of k, $v(y)/y_N > M^{j-1}$ in $Q_{\rho/\rho'}(e)$. This implies that $u(x)/x_N > M^{j-1}$ in $Q_\rho(x_0)$.

Case 2: $\rho < \rho'/4$. Recall that v/M^j is a non-negative upper solution of (3.8). Hence, v/M^j is also a non-negative upper solution of

$$-\Delta_p u + 2^p\|a\|_\infty |u|^{p-2}u = 0, \quad \text{in } Q_{\frac{3\rho}{2\rho'}}(e) \subset Q_{\frac{3}{2}},$$

Thus, by Corollary 3.4, we deduce that

$$\inf_{Q_{\rho/\rho'}(e)} \frac{v(y)}{M^j} \geq C_1\Big(\big(\frac{\rho}{\rho'}\big)^{-N} \int_{Q_{\rho/\rho'}(e)} \big(\frac{v}{M^j}\big)^\gamma dy\Big)^{1/\gamma}. \tag{3.14}$$

Now, let us introduce

$$G = \{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > 1/4\},$$

and, as $y_N > 1/4$ for all $y \in Q_{\rho/\rho'}(e)$, observe that (3.13) implies the following inequality

$$|G| \geq |\{y \in Q_{\rho/\rho'}(e) : v(y)/M^j > y_N\}| \geq (1-\alpha)\big(\frac{\rho}{\rho'}\big)^N .$$

Hence, we deduce that

$$\int_{Q_{\rho/\rho'}(e)} \big(\frac{v}{M^j}\big)^\gamma \, dy \geq \int_G \big(\frac{v}{M^j}\big)^\gamma \, dy > \big(\frac{1}{4}\big)^\gamma |G| \geq \big(\frac{1}{4}\big)^\gamma (1-\alpha)\big(\frac{\rho}{\rho'}\big)^N,$$

and so, by (3.14), that

$$\inf_{Q_{\rho/\rho'}(e)} \frac{v}{M^j} > \frac{C_1}{4}(1-\alpha)^{1/\gamma} .$$

Finally, using that $M \geq \frac{4}{C_1}(1-\alpha)^{-1/\gamma}$ and that $y_N \leq 1$ in $Q_{\rho/\rho'}(e)$, we deduce that $v(y) > M^{j-1} y_N$ in $Q_{\rho/\rho'}(e)$. Thus, we can conclude that $u(x)/x_N > M^{j-1}$ in $Q_\rho(x_0)$.

In both cases we prove that $u(x)/x_N > M^{j-1}$ in $Q_\rho(x_0)$. This means that $Q_\rho(x_0) \subset F$ and so, the Claim is proved.

Since (3.10) and the Claim hold, we can apply Lemma 3.5 and we obtain that $|E| \leq (1 - C_2\alpha)|F|$, i.e.

$$|\{x \in Q_1 : u(x)/x_N > M^j\}| \leq (1 - C_2\alpha)\, |\{x \in Q_1 : u(x)/x_N > M^{j-1}\}| , \quad \forall\, j \in \mathbb{N} \setminus \{1\} .$$

Iterating in j and using (3.9), the result follows with $\mu = C_2\alpha \in (0,1)$ depending only on N. □

Theorem 3.8 (**Boundary weak Harnack inequality for cubes**). *Let $a \in L^\infty(Q_4)$ be a non-negative function. Assume that $u \in W^{1,p}(Q_4)$ is a non-negative upper solution of*

$$-\Delta_p u + a(x)|u|^{p-2}u = 0 , \quad u \in W_0^{1,p}(Q_4),$$

Then, there exist $\varepsilon = \varepsilon(p, \|a\|_\infty) > 0$ and $C = C(p, \varepsilon, \|a\|_\infty) > 0$ such that

$$\inf_{Q_1} \frac{u(x)}{x_N} \geq C\Big(\int_{Q_1} \big(\frac{u(x)}{x_N}\big)^\varepsilon \, dx\Big)^{1/\varepsilon} .$$

Proof. Let us split the proof into three steps.

Step 1: *Assume that $\inf_{Q_1} \frac{u(x)}{x_N} \leq 1$. Then, there exist $\varepsilon = \varepsilon(p, \|a\|_\infty) > 0$ and $C = C(p, \varepsilon, \|a\|_\infty) > 0$ such that, for all $t \geq 0$,*

$$|\{x \in Q_1 : u(x)/x_N > t\}| \leq C \min\{1, t^{-2\varepsilon}\} .$$

Let us define the real valued function

$$f(t) = |\{x \in Q_1 : u(x)/x_N > t\}| ,$$

and let M and μ be the constants obtained in Lemma 3.7. We define

$$C = \max\{(1-\mu)^{-1}, M^{2\varepsilon}\} > 1 \quad \text{and} \quad \varepsilon = -\frac{1}{2}\frac{\ln(1-\mu)}{\ln M} > 0 .$$

If $t \in [0, M]$, we easily get

$$|\{x \in Q_1 : u(x)/x_N > t\}| \leq 1 \leq CM^{-2\varepsilon} \leq C \min\{1, t^{-2\varepsilon}\}.$$

Hence, let us assume $t > M > 1$. Without loss of generality, we assume $t \in [M^j, M^{j+1}]$ for some $j \in \mathbb{N}$, and it follows that

$$\frac{\ln t}{\ln M} - 1 \leq j \leq \frac{\ln t}{\ln M} .$$

Since f is non-increasing and $1 - \mu \in (0,1)$, the above inequality and Lemma 3.7 imply

$$f(t) \leq f(M^j) \leq (1-\mu)^j \leq (1-\mu)^{\frac{\ln t}{\ln M} - 1} . \tag{3.15}$$

Finally, observe that

$$\ln\Big((1-\mu)^{\frac{\ln t}{\ln M} - 1}\Big) = \Big(\frac{\ln t}{\ln M} - 1\Big)\ln(1-\mu) = \ln t \frac{\ln(1-\mu)}{\ln M} - \ln(1-\mu) \leq -2\varepsilon \ln t + \ln C = \ln(Ct^{-2\varepsilon}) . \tag{3.16}$$

The Step 1 then follows from (3.15), (3.16) and the fact that $\min\{1, t^{-2\varepsilon}\} = t^{-2\varepsilon}$ for $t \geq 1$.

Step 2: *Assume that* $\inf_{Q_1} \frac{u(x)}{x_N} \leq 1$. *Then, there exists* $C = C(p, \varepsilon, \|a\|_\infty) > 0$ such that

$$\int_{Q_1} \Big(\frac{u(x)}{x_N}\Big)^\varepsilon dx \leq C < +\infty\,. \tag{3.17}$$

Directly, applying [18, Lemma 9.7], we obtain that

$$\int_{Q_1} \Big(\frac{u(x)}{x_N}\Big)^\varepsilon dx = \varepsilon \int_0^\infty t^{\varepsilon-1} |\{x \in Q_1 : u(x)/x_N > t\}|\, dt\,.$$

Hence, (3.17) follows from Step 1.

Step 3: *Conclusion.*

Let us introduce the function

$$v = \frac{u}{\inf_{y\in Q_1} \frac{u(y)}{y_N} + \beta}\,,$$

where $\beta > 0$ is an arbitrary positive constant. Obviously, v satisfies the hypothesis of Step 2. Hence, applying Step 2, we obtain that

$$\int_{Q_1} \big(\frac{u(x)}{x_N}\big)^\varepsilon \Big(\frac{1}{\inf_{y\in Q_1} \frac{u(y)}{y_N} + \beta}\Big)^\varepsilon dx \leq C\,,$$

or equivalently that

$$\frac{1}{C^{1/\varepsilon}} \Big(\int_{Q_1} \big(\frac{u(x)}{x_N}\big)^\varepsilon dx \Big)^{1/\varepsilon} \leq \inf_{Q_1} \frac{u(x)}{x_N} + \beta\,.$$

Letting $\beta \to 0$ we obtain the desired result. □

Proof of Theorem 3.1. Thanks to the regularity of the boundary, there exists $\overline{R} > 0$ and a diffeomorphism φ such that $\varphi(B_{\overline{R}}(x_0) \cap \omega) \subset Q_1$ and $\varphi(B_{\overline{R}}(x_0) \cap \partial\omega) \subset \{x \in \partial Q_1 : x_N = 0\}$. The result then follows from Theorem 3.8. □

We end this section by presenting a corollary of Theorem 3.1. Consider the equation

$$-\Delta u + a(x)u = b(x)\,, \qquad u \in H_0^1(\omega)\,, \tag{3.18}$$

under the assumption

$$\begin{cases} \omega \subset \mathbb{R}^N,\ N \geq 2,\ \text{is a bounded domain with boundary } \partial\omega \text{ of class } \mathcal{C}^{1,1}\,, \\ a \in L^\infty(\omega)\,,\ b^- \in L^p(\omega) \text{ for some } p > N \text{ and } b^+ \in L^1(\omega)\,, \\ a \geq 0 \text{ a.e. in } \omega\,. \end{cases} \tag{3.19}$$

Corollary 3.9. *Under the assumption* (3.19)*, assume that* $u \in H^1(\omega)$ *is a non-negative upper solution of* (3.18) *and let* $x_0 \in \partial\omega$*. Then, there exist* $\overline{R} > 0$*,* $\varepsilon = \varepsilon(\overline{R}, \|a\|_\infty, \omega) > 0$*,* $C_1 = C_1(\overline{R}, \varepsilon, \|a\|_\infty, \omega) > 0$ *and* $C_2 = C_2(\omega, \|a\|_\infty) > 0$ *such that, for all* $R \in (0, \overline{R}]$*,*

$$\inf_{B_R(x_0)\cap\omega} \frac{u(x)}{d(x,\partial\omega)} \geq C_1 \Big(\int_{B_R(x_0)\cap\omega} \Big(\frac{u(x)}{d(x,\partial\omega)}\Big)^\varepsilon dx \Big)^{1/\varepsilon} - C_2 \|b^-\|_{L^p(\omega)}\,.$$

In order to prove Corollary 3.9 we need the following lemma

Lemma 3.10. *Let* $\omega \subset \mathbb{R}^N$*,* $N \geq 2$*, be a bounded domain with boundary* $\partial\omega$ *of class* $\mathcal{C}^{1,1}$ *and let* $a \in L^\infty(\omega)$ *and* $g \in L^p(\omega)$*,* $p > N$*, be non-negative functions. Assume that* $u \in H^1(\omega)$ *is a lower solution of*

$$-\Delta u + a(x)u = g(x)\,, \quad u \in H_0^1(\omega)\,.$$

Then there exists $C = C(\omega, \|a\|_\infty) > 0$ *such that*

$$\sup_\omega \frac{u(x)}{d(x,\partial\omega)} \leq C \|g\|_{L^p(\omega)}\,.$$

Proof. First of all, observe that it is enough to prove the result for v solution of

$$\begin{cases} -\Delta v + a(x)v = g(x)\,, & \text{in } \omega\,, \\ v = 0\,, & \text{on } \partial\omega\,. \end{cases}$$

as, by the standard comparison principle it follows that $u \le v$. Applying [18, Theorem 9.15 and Lemma 9.17] we deduce that $v \in W_0^{2,p}(\omega)$ and there exists $C_1 = C_1(\omega, \|a\|_\infty) > 0$ such that

$$\|v\|_{W^{2,p}(\omega)} \le C_1 \|g\|_{L^p(\omega)}\,.$$

Moreover, as $p > N$, by Sobolev's inequality, we have $C_2 = C_2(\omega, \|a\|_\infty)$ with

$$\|v\|_{\mathcal{C}^1(\overline{\omega})} \le C_2 \|g\|_{L^p(\omega)}\,,$$

and so, we easily deduce that

$$v(x) \le C_3 \|g\|_{L^p(\omega)} d(x, \partial\omega)\,, \quad \forall\, x \in \omega\,.$$

Hence, since $u \le v$, the result follows from the above inequality. □

Proof of Corollary 3.9. Let $w \ge 0$ be the solution of

$$\begin{cases} -\Delta w + a(x)w = b^-(x)\,, & \text{in } \omega\,, \\ w = 0\,, & \text{on } \partial\omega\,. \end{cases} \tag{3.20}$$

Observe that $v = u + w$ satisfies

$$\begin{cases} -\Delta v + a(x)v \ge 0, & \text{in } \omega\,, \\ v \ge 0\,, & \text{on } \partial\omega\,. \end{cases} \tag{3.21}$$

Hence, by Theorem 3.1, there exist $\overline{R} > 0$, $\varepsilon = \varepsilon(p, \overline{R}, \|a\|_\infty, \omega) > 0$ and $C = C(p, \overline{R}, \varepsilon, \|a\|_\infty, \omega) > 0$ such that, for all $R \in (0, \overline{R}]$,

$$\inf_{B_R(x_0)\cap\omega} \frac{v(x)}{d(x,\partial\omega)} \ge C \Big(\int_{B_R(x_0)\cap\omega} \Big(\frac{v(x)}{d(x,\partial\omega)} \Big)^\varepsilon dx \Big)^{1/\varepsilon}\,. \tag{3.22}$$

On the other hand, by Lemma 3.10, there exists $C_2 = C_2(\omega, \|a\|_\infty) > 0$ such that

$$\sup_\omega \frac{w(x)}{d(x,\partial\omega)} \le C_2 \|b^-\|_{L^p(\omega)}\,. \tag{3.23}$$

From (3.22), (3.23) and using that $u = v - w$, the corollary follows observing that $w \ge 0$ and hence $v \ge u$. □

4. A priori bound

This section is devoted to the proof of Theorem 1.1. As a first step we observe that, to obtain our a priori upper bound on the solutions of (P_λ), we only need to control the solutions on Ω^+. This can be proved under a weaker assumption than (A_1). More precisely, we assume

$$(B) \qquad \begin{cases} \Omega \subset \mathbb{R}^N,\ N \ge 2,\ \text{is a bounded domain with boundary } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ c_+,\ c_- \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > N/2\,,\ \mu \text{ belong to } L^\infty(\Omega)\,, \\ c_+(x) \ge 0,\ c_-(x) \ge 0 \text{ and } c_-(x)c_+(x) = 0 \text{ a.e. in } \Omega, \\ |\Omega_+| > 0,\ \text{where } \Omega_+ := \mathrm{Supp}(c_+), \end{cases}$$

and we prove the next result.

Lemma 4.1. *Assume that (B) holds. Then, there exists $M > 0$ such that, for any $\lambda \in \mathbb{R}$, any solution u of (P_λ) satisfies*

$$-\sup_{\Omega_+} u^- - M \ \le\ u \ \le \sup_{\Omega_+} u^+ + M.$$

Remark 4.1. Let us point out that if $c_+ \equiv 0$, i.e. $|\Omega_+| = 0$, the problem (P_λ) reduces to

$$-\Delta u = -c_-(x)u + \mu(x)|\nabla u|^2 + h(x), \quad u \in H_0^1(\Omega) \cap L^\infty(\Omega), \tag{4.1}$$

which is independent of λ. If (4.1) has a solution, by [4, Proposition 4.1] it is unique and so, we have an a priori bound.

Proof. In case problem (P_λ) has no solution for any $\lambda \in \mathbb{R}$, there is nothing to prove. Hence, we assume the existence of $\tilde{\lambda} \in \mathbb{R}$ such that ($P_{\tilde{\lambda}}$) has a solution $\tilde{u}$. We shall prove the result with $M := 2\|\tilde{u}\|_\infty$. Let u be an arbitrary solution of (P_λ).

Step 1: $u \leq \sup_{\Omega_+} u^+ + M$.

Setting $D := \Omega \backslash \overline{\Omega}_+$ we define $v = u - \sup\limits_{\partial D} u^+$. We then obtain

$$-\Delta v = -c_-(x)v + \mu(x)|\nabla v|^2 + h(x) - c_-(x)\sup_{\partial D} u^+ \leq -c_-(x)v + \mu(x)|\nabla v|^2 + h(x)\,, \quad \text{in } D\,.$$

As $v \leq 0$ on ∂D, the function v is a lower solution of

$$-\Delta z = -c_-(x)z + \mu(x)|\nabla z|^2 + h(x)\,, \qquad u \in H_0^1(D) \cap L^\infty(D). \tag{4.2}$$

Setting $\tilde{v} = \tilde{u} + \|\tilde{u}\|_\infty$ we observe that

$$-\Delta \tilde{v} = -c_-(x)\tilde{v} + \mu(x)|\nabla \tilde{v}|^2 + h(x) + c_-(x)\|\tilde{u}\|_\infty \geq -c_-(x)\tilde{v} + \mu(x)|\nabla \tilde{v}|^2 + h(x)\,, \quad \text{in } D\,,$$

and thus, as $\tilde{v} \geq 0$ on ∂D, the function $\tilde{v}$ is an upper solution of (4.2). By [3, Lemma 2.1], we know that u, $\tilde{u} \in H^1(\Omega) \cap W^{1,N}_{loc}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ and hence, v, $\tilde{v} \in H^1(D) \cap W^{1,N}_{loc}(D) \cap \mathcal{C}(\overline{D})$. Applying [3, Lemma 2.2] we conclude that $v \leq \tilde{v}$ in D namely, that

$$u - \sup_{\partial D} u^+ \leq \tilde{u} + \|\tilde{u}\|_\infty\,, \quad \text{in } D.$$

This gives that

$$u \leq \tilde{u} + \|\tilde{u}\|_\infty + \sup_{\partial D} u^+\,, \quad \text{in } D,$$

and hence

$$u \leq M + \sup_{\Omega_+} u^+\,, \quad \text{in } \Omega.$$

Step 2: $u \geq -\sup_{\Omega_+} u^- - M$.

We now define $v = u + \sup\limits_{\partial D} u^-$ and obtain $v \geq 0$ on ∂D as well as

$$-\Delta v = -c_-(x)v + \mu(x)|\nabla v|^2 + h(x) + c_-(x)\sup_{\partial D} u^- \geq -c_-(x)v + \mu(x)|\nabla v|^2 + h(x)\,, \quad \text{in } D\,.$$

Thus v is an upper solution of (4.2). Now defining $\tilde{v} = \tilde{u} - \|\tilde{u}\|_\infty$, again, we have $\tilde{v} \leq 0$ on ∂D as well as

$$-\Delta \tilde{v} = -c_-(x)\tilde{v} + \mu(x)|\nabla \tilde{v}|^2 + h(x) - c_-(x)\|\tilde{u}\|_\infty \leq -c_-(x)\tilde{v} + \mu(x)|\nabla \tilde{v}|^2 + h(x)\,, \quad \text{in } D\,.$$

Thus $\tilde{v}$ is a lower solution of (4.2). As previously we have that v, $\tilde{v} \in H^1(D) \cap W^{1,N}_{loc}(D) \cap \mathcal{C}(\overline{D})$ and applying [3, Lemma 2.2] we obtain that $\tilde{v} \leq v$ in D. Namely

$$\tilde{u} - \|\tilde{u}\|_\infty \leq u + \sup_{\partial D} u^-\,, \quad \text{in } D.$$

Thus

$$u \geq \tilde{u} - \|\tilde{u}\|_\infty - \sup_{\partial D} u^-\,, \quad \text{in } D,$$

and without restriction we get that

$$u \geq -\sup_{\Omega_+} u^- - M\,, \quad \text{in } \Omega,$$

ending the proof. □

Now, let $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be a solution of (P_λ). Following [4, Proposition 6.1], we introduce

$$w_i(x) = \frac{1}{\mu_i}\big(e^{\mu_i u(x)} - 1\big) \quad \text{and} \quad g_i(s) = \frac{1}{\mu_i}\ln(1 + \mu_i s), \qquad i = 1, 2\,, \tag{4.3}$$

where μ_1 is given in (A_1) and $\mu_2 = \operatorname{esssup} \mu(x)$. Observe that

$$u = g_i(w_i) \quad \text{and} \quad 1 + \mu_i w_i = e^{\mu_i u}, \qquad i = 1, 2\,,$$

and that, by standard computations,

$$-\Delta w_i = (1 + \mu_i w_i)\big[(\lambda c_+(x) - c_-(x))g_i(w_i) + h(x)\big] + e^{\mu_i u}|\nabla u|^2(\mu(x) - \mu_i). \tag{4.4}$$

Using (4.4) we shall obtain a uniform a priori upper bound on u in a neighborhood of any fixed point $\overline{x} \in \overline{\Omega}_+$. We consider the two cases $\overline{x} \in \overline{\Omega}_+ \cap \Omega$ and $\overline{x} \in \overline{\Omega}_+ \cap \partial\Omega$ separately.

Lemma 4.2. *Assume that* (A_1) *holds and that* $\overline{x} \in \overline{\Omega}_+ \cap \Omega$. *For each* $\Lambda_2 > \Lambda_1 > 0$, *there exist* $M_I > 0$ *and* $R > 0$ *such that, for any* $\lambda \in [\Lambda_1, \Lambda_2]$, *any solution* u *of* (P_λ) *satisfies* $\sup_{B_R(\overline{x})} u \le M_I$.

Proof. Under the assumption (A_1) we can find a $R > 0$ such that $\mu(x) \ge \mu_1 > 0$, $c_- \equiv 0$ in $B_{4R}(\overline{x}) \subset \Omega$ and $c_+ \gneqq 0$ in $B_R(\overline{x})$. For simplicity, in this proof, we denote $B_{mR} = B_{mR}(\overline{x})$, for $m \in \mathbb{N}$.

Since $c_- \equiv 0$ and $\mu(x) \ge \mu_1$ in B_{4R}, observe that (4.4) reduces to

$$-\Delta w_1 + \mu_1 h^-(x) w_1 \ge \lambda(1+\mu_1 w_1) c_+(x) g_1(w_1) + h^+(x)(1+\mu_1 w_1) - h^-(x)\,, \quad \text{in } B_{4R}. \tag{4.5}$$

Let z_2 be the solution of

$$-\Delta z_2 + \mu_1 h^-(x) z_2 = -\Lambda_2 c_+(x) \frac{e^{-1}}{\mu_1}\,, \qquad z_2 \in H_0^1(B_{4R}). \tag{4.6}$$

By classical regularity arguments (see for instance [22, Theorem III-14.1]), $z_2 \in \mathcal{C}(\overline{B_{4R}})$. Hence, there exists $D = D(\overline{x}, \mu_1, \Lambda_2, \|h^-\|_{L^q(B_{4R})}, \|c_+\|_{L^q(B_{4R})}, q, R) > 0$ such that

$$z_2 \ge -D \text{ in } B_{4R}. \tag{4.7}$$

Moreover, by the weak maximum principle [18, Theorem 8.1], we have that $z_2 \le 0$. Now defining $v_1 = w_1 - z_2 + \frac{1}{\mu_1}$, and since $\min_{[-1/\mu_i, +\infty[}(1+\mu_i s) g_i(s) = -\frac{e^{-1}}{\mu_i}$, we observe that v_1 satisfies

$$-\Delta v_1 + \mu_1 h^-(x) v_1 \ge \Lambda_1 c_+(x)(1+\mu_1 w_1) g_1(w_1)^+\,, \quad \text{in } B_{4R}. \tag{4.8}$$

Also, since $w_1 > -1/\mu_1$, we have $v_1 > 0$ in $\overline{B_{4R}}$. Note also that $0 < 1 + \mu_1 w_1 = \mu_1 v_1 + \mu_1 z_2$ in $\overline{B_{4R}}$. Now, we split the rest of the proof into four steps.

Step 1: *There exist* $C_1 = C_1(\overline{x}, \Lambda_1, \Lambda_2, R, \mu_1, q, \|h^-\|_{L^\infty(B_{4R})}, \|c_+\|_{L^q(B_{4R})}) > 0$ *such that*

$$k := \inf_{B_R} v_1(x) \le C_1. \tag{4.9}$$

In case $\mu_1 \inf_{B_R} v_1(x) \le 1 + \mu_1 D$, where D is given by (4.7), the Step 1 is proved. Hence, we assume that

$$\mu_1 v_1(x) \ge 1 + \mu_1 D, \qquad \forall\, x \in B_R. \tag{4.10}$$

In particular, $\mu_1 v_1 + \mu_1 z_2 \ge 1$ on B_R. Now, by Lemma 2.4 applied on (4.8) with $\omega = B_{4R}$, there exists $C = C(R, \|h^-\|_{L^\infty(B_{4R})}, \mu_1, \Lambda_1, \overline{x}) > 0$ such that,

$$\begin{aligned} k &\ge C \int_{B_R} c_+(y) \Big(\mu_1 v_1(y) + \mu_1 z_2(y)\Big) \ln\Big(\mu_1 v_1(y) + \mu_1 z_2(y)\Big)\, dy \\ &\ge C \int_{B_R} c_+(y) \left(\mu_1 k - \mu_1 D\right) \ln\big(\mu_1 k - \mu_1 D\big)\, dy \\ &= C(\mu_1 k - \mu_1 D) \ln\big(\mu_1 k - \mu_1 D\big) \|c_+\|_{L^1(B_R)}. \end{aligned}$$

As $c_+ \gneqq 0$ in B_R, comparing the growth in k of the various terms, we deduce that k must remain bounded and thus the existence of $C_1 = (\overline{x}, \Lambda_1, \Lambda_2, R, \mu_1, q, \|h^-\|_{L^\infty(B_{4R})}, \|c_+\|_{L^q(B_{4R})}) > 0$ such that (4.9) holds.

Step 2: *For any* $1 \le s < \frac{N}{N-2}$, *there exists* $C_2 = C_2(\overline{x}, \mu_1, R, s, \Lambda_1, \Lambda_2, q, \|h^-\|_{L^\infty(B_{4R})}, \|c_+\|_{L^q(B_{4R})}) > 0$ *such that*

$$\int_{B_{2R}} (1+\mu_1 w_1)^s\, dx \le C_2.$$

Applying Lemma 2.3 to (4.8), we deduce the existence of $C = C(s, \mu_1, R, \|h^-\|_{L^q(B_{4R})}) > 0$ such that

$$\Big(\int_{B_{2R}} v_1^s\, dx\Big)^{1/s} \le C \inf_{B_R} v_1\,.$$

The Step 2 follows from Step 1 observing that $0 \le 1 + \mu_1 w_1 = \mu_1 v_1 + \mu_1 z_2 \le \mu_1 v_1$.

Step 3: *For any $1 \le s < \frac{N}{N-2}$, we have, for the constant $C_2 > 0$ introduced in Step 2, that*

$$\int_{B_{2R}} \big(1+\mu_2 w_2\big)^{\frac{\mu_1 s}{\mu_2}} dx \le C_2.$$

This directly follows from Step 2 since, by the definition of w_i, we have

$$(1+\mu_2 w_2)^{\frac{\mu_1}{\mu_2}} = (e^{\mu_2 u})^{\frac{\mu_1}{\mu_2}} = e^{\mu_1 u} = (1+\mu_1 w_1).$$

Step 4: *Conclusion.*

We will show the existence of $C_3 = C_3(\overline{x}, \mu_1, \mu_2, R, \Lambda_1, \Lambda_2, q, \|h^-\|_{L^\infty(B_{4R})}, \|c_+\|_{L^q(B_{4R})}) > 0$ such that

$$\sup_{B_R} w_2 \le C_3\,.$$

Thus, thanks to the definition of w_2, we can conclude the proof. Let us fix $s \in [1, \frac{N}{N-2})$, $r \in (\frac{N}{2}, q)$ and $\alpha = \frac{(q-r)\mu_1 s}{\mu_2 q r}$ and let $c_\alpha > 0$ such that

$$\ln(1+x) \le (1+x)^\alpha + c_\alpha, \quad \forall\, x \ge 0.$$

We introduce the auxiliary functions

$$a(x) = \Lambda_2 c_+(x)(1+\mu_2 w_2)^\alpha + c_\alpha \Lambda_2 c_+(x) + \mu_2 h^+(x),$$
$$b(x) = \frac{\Lambda_2}{\mu_2} c_+(x)(1+\mu_2 w_2)^\alpha + c_\alpha \frac{\Lambda_2}{\mu_2} c_+(x) + h^+(x) + c_-(x)\frac{e^{-1}}{\mu_2},$$

and, as $\mu(x) \le \mu_2$, we deduce from (4.4) that w_2 satisfies

$$\begin{cases} -\Delta w_2 \le a(x) w_2 + b(x) & \text{in } \Omega\,, \\ \quad w_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, as $q/r > 1$, by Step 3 and Hölder inequality, it follows that

$$\begin{aligned}\int_{B_{2R}} (c_+(x)(1+\mu_2 w_2)^\alpha)^r dx &\le \|c_+\|^r_{L^q(B_{2R})} \Big(\int_{B_{2R}} (1+\mu_2 w_2)^{\frac{\alpha q r}{q-r}} dx \Big)^{\frac{q-r}{q}} \\ &\le \|c_+\|^r_{L^q(B_{2R})} \Big(\int_{B_{2R}} (1+\mu_2 w_2)^{\frac{\mu_1 s}{\mu_2}} dx \Big)^{\frac{q-r}{q}} \le C_2^{\frac{q-r}{q}} \|c_+\|^r_{L^q(B_{2R})}.\end{aligned}$$

Hence, there exists $D(\overline{x}, \mu_1, \mu_2, s, R, \Lambda_1, \Lambda_2, q, \|h^-\|_{L^\infty(B_{4R})}, \|c_+\|_{L^q(B_{4R})}, r, \|h^+\|_{L^q(B_{2R})}) > 0$ such that

$$\max\{\, \|a\|_{L^r(B_{2R})}, \|b\|_{L^r(B_{2R})}\} \le D\,. \tag{4.11}$$

Applying then Lemma 2.1, there exists $C(\overline{x}, \mu_1, \mu_2, s, R, \Lambda_1, \Lambda_2, q, \|h^-\|_{L^q(B_{4R})}, \|c_+\|_{L^q(B_{4R})}) > 0$ such that

$$\sup_{B_R} w_2^+ \le C\Big[\Big(\int_{B_{2R}} (w_2^+)^{\frac{\mu_1}{\mu_2} s} dx \Big)^{\frac{\mu_2}{\mu_1 s}} + \|b\|_{L^r(B_{2R})}\Big] \le C\Big[\Big(\int_{B_{2R}} (w_2^+)^{\frac{\mu_1}{\mu_2} s} dx \Big)^{\frac{\mu_2}{\mu_1 s}} + D\Big]\,.$$

On the other hand, by Step 3, we get

$$\int_{B_{2R}} (w_2^+)^{\frac{\mu_1}{\mu_2} s} dx \le C(\mu_1, \mu_2, s) \int_{B_{2R}} (1+\mu_2 w_2)^{\frac{\mu_1}{\mu_2} s} dx \le C(\mu_1, \mu_2, s)\, C_2\,,$$

and the result follows. □

Lemma 4.3. *Assume that (A_1) holds and that $\overline{x} \in \overline{\Omega}_+ \cap \partial\Omega$. For each $\Lambda_2 > \Lambda_1 > 0$, there exist $R > 0$ and $M_B > 0$ such that, for any $\lambda \in [\Lambda_1, \Lambda_2]$, any solution of ($P_\lambda$) satisfies $\sup_{B_R(\overline{x}) \cap \Omega} u \le M_B$.*

Proof. Let $\overline{R} > 0$ given by Theorem 3.1. Under the assumption (A_1), we can find $R \in (0, \overline{R}/2]$ and $\Omega_1 \subset \Omega$ with $\partial\Omega_1$ of class $\mathcal{C}^{1,1}$ such that $B_{2R}(\overline{x}) \cap \Omega \subset \Omega_1$ and $\mu(x) \ge \mu_1 > 0$, $c_- \equiv 0$ and $c_+ \gneqq 0$ in Ω_1.

Since $c_- \equiv 0$ and $\mu(x) \geq \mu_1$ in Ω_1, observe that (4.4) reduces to

$$-\Delta w_1 + \mu_1 h^-(x) w_1 \geq \lambda(1+\mu_1 w_1) c_+(x) g_1(w_1) + h^+(x)(1+\mu_1 w_1) - h^-(x)\,, \quad \text{in } \Omega_1 \tag{4.12}$$

Let z_2 be the solution of

$$-\Delta z_2 + \mu_1 h^-(x) z_2 = -\Lambda_2 c_+(x) \frac{e^{-1}}{\mu_1}\,, \qquad z_2 \in H_0^1(\Omega_1). \tag{4.13}$$

As in Lemma 4.2, $z_2 \in \mathcal{C}(\overline{\Omega_1})$ and there exists a $D = D(\mu_1, \Lambda_2, \|h^-\|_{L^q(B_{4R})}, \|c_+\|_{L^q(B_{4R})}, q, \Omega_1) > 0$ such that $-D \leq z_2 \leq 0$ on Ω_1. Now defining $v_1 = w_1 - z_2 + \frac{1}{\mu_1}$ we observe that v_1 satisfies

$$-\Delta v_1 + \mu_1 h^-(x) v_1 \geq \Lambda_1 c_+(x)(1+\mu_1 w_1) g_1(w_1)^+\,, \quad \text{in } \Omega_1. \tag{4.14}$$

and $v_1 > 0$ on $\overline{\Omega_1}$. Note also that $0 < 1 + \mu_1 w_1 = \mu_1 v_1 + \mu_1 z_2$ on $\overline{\Omega_1}$. Next, we split the rest of the proof into three steps.

Step 1: *There exists* $C_1 = C_1(\Omega_1, \overline{x}, \Lambda_1, \Lambda_2, R, \mu_1, q, \|h^-\|_{L^\infty(\Omega_1)}, \|c_+\|_{L^q(\Omega_1)}) > 0$ *such that*

$$\inf_{B_{2R}(\overline{x}) \cap \Omega_1} \frac{v_1(x)}{d(x, \partial\Omega_1)} \leq C_1\,.$$

Choose $R_2 > 0$ and $y \in \Omega$ such that $B_{4R_2}(y) \subset B_{2R}(\overline{x}) \cap \Omega$ and $c_+ \gneqq 0$ in $B_{R_2}(y)$. As in Step 1 of Lemma 4.2, there exists $C = C(\Omega_1, y, \Lambda_1, \Lambda_2, R_2, \mu_1, q, \|h^-\|_{L^\infty(\Omega_1)}, \|c_+\|_{L^q(\Omega_1)}) > 0$ such that

$$\inf_{B_{R_2}(y)} v_1(x) \leq C\,.$$

We conclude by observing, since $B_{4R_2}(y) \subset B_{2R}(\overline{x}) \cap \Omega_1$, that

$$\inf_{B_{2R}(\overline{x}) \cap \Omega_1} \frac{v_1(x)}{d(x, \partial\Omega_1)} \leq \inf_{B_{R_2}(y)} \frac{v_1(x)}{d(x, \partial\Omega_1)} \leq \frac{1}{3R_2} \inf_{B_{R_2}(y)} v_1(x).$$

Step 2: *There exist* $\varepsilon = \varepsilon(\overline{R}, \mu_1, \|h^-\|_{L^\infty(\Omega_1)}, \Omega_1) > 0$ *and* $C_2 = C_2(\overline{x}, \mu_1, R, \overline{R}, s, \Lambda_1, \Lambda_2, q, \|h^-\|_{L^\infty(\Omega_1)}, \|c_+\|_{L^q(\Omega_1)}) > 0$ *such that*

$$\Big(\int_{B_{2R}(\overline{x}) \cap \Omega} (1+\mu_1 w_1)^\varepsilon \, dx \Big)^{1/\varepsilon} \leq C_2.$$

By Theorem 3.1 applied on (4.14) and Step 1, we obtain constants $\varepsilon = \varepsilon(\overline{R}, \mu_1, \|h^-\|_{L^\infty(\Omega_1)}, \Omega_1) > 0$ and $C = C(\Omega_1, \overline{x}, \mu_1, \varepsilon, \overline{R}, \Lambda_1, \Lambda_2, q, \|h^-\|_{L^\infty(\Omega_1)}, \|c_+\|_{L^q(\Omega_1)}) > 0$ such that

$$\Big(\int_{B_{2R}(\overline{x}) \cap \Omega_1} \Big(\frac{v_1(x)}{d(x, \partial\Omega_1)} \Big)^\varepsilon dx \Big)^{1/\varepsilon} \leq C\,.$$

This clearly implies, since $\Omega_1 \subset \Omega$, that

$$\Big(\int_{B_{2R}(\overline{x}) \cap \Omega_1} v_1(x)^\varepsilon dx \Big)^{1/\varepsilon} \leq C \operatorname{diam}(\Omega)\,.$$

The Step 2 then follows observing that $0 \leq 1 + \mu_1 w_1 = \mu_1 v_1 + \mu_1 z_2 \leq \mu_1 v_1$ and taking into account that $B_{2R}(\overline{x}) \cap \Omega = B_{2R}(\overline{x}) \cap \Omega_1$.

Step 3: *Conclusion.*

Arguing exactly as in Step 3 and 4 of Lemma 4.2, using Lemma 2.2 and Step 2, we show the existence of $C_3 = C_3(\overline{x}, \mu_1, \mu_2, R, \Lambda_1, \Lambda_2, \|h^-\|_{L^\infty(\Omega_1)}, \|c_+\|_{L^q(B_{2R}(\Omega_1)}) > 0$ such that

$$\sup_{B_R(\overline{x}) \cap \Omega} w_2 \leq C_3\,.$$

Hence, the proof of the lemma follows by the definition of w_2. □

Proof of Theorem 1.1. Arguing by contradiction we assume the existence of sequences $\{\lambda_n\} \subset [\Lambda_1, \Lambda_2]$, $\{u_n\}$ solutions of (P_λ) for $\lambda = \lambda_n$ and of points $\{x_n\} \subset \Omega$ such that

$$u_n(x_n) = \max\{u_n(x) : x \in \overline{\Omega}\} \to \infty\,, \quad \text{as } n \to \infty\,. \tag{4.15}$$

Observe that Lemma 4.1 and (4.15) together imply the existence of a sequence of points $y_n \in \overline{\Omega}_+$ such that

$$u_n(y_n) = \max\{u_n(y) : y \in \overline{\Omega}_+\} \to \infty\,, \quad \text{as } n \to \infty\,. \tag{4.16}$$

Passing to a subsequence if necessary, we may assume that $\lambda_n \to \overline{\lambda} \in [\Lambda_1, \Lambda_2]$ and $y_n \to \overline{y} \in \overline{\Omega}_+$. Now, let us distinguish two cases:

- If $\overline{y} \in \overline{\Omega}_+ \cap \Omega$, Lemma 4.2 shows that we can find $R_I > 0$ and $M_I > 0$ such that, if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution of (P_λ), then $\sup_{B_{R_I}(\overline{y})} u \le M_I$. This contradicts (4.16).
- If $\overline{y} \in \overline{\Omega}_+ \cap \partial\Omega$, Lemma 4.3 shows that we can find $R_B > 0$ and $M_B > 0$ such that, if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution of (P_λ), then $\sup_{B_{R_B}(\overline{y}) \cap \Omega} u \le M_B$. Again, this contradicts (4.16).

As (4.16) cannot happen, the result follows. □

5. Proof of Theorem 1.2

Let us begin with a preliminary result.

Lemma 5.1. *Under the assumption (A_1), assume that (P_0) has a solution u_0 for which there exist $\overline{x} \in \Omega$ and $R > 0$ such that $c_+ u_0 \gneqq 0$, $c_- \equiv 0$ and $\mu \ge 0$ in $B_R(\overline{x})$. Then there exists $\overline{\Lambda} \in (0, \infty)$ such that, for $\lambda \ge \overline{\Lambda}$, the problem (P_λ) has no solution u with $u \ge u_0$ in $B_R(\overline{x})$.*

Proof. Let us introduce $\overline{c}(x) := \min\{c_+(x), 1\}$. Observe that $0 \lneqq \overline{c} \le c_+$ and define $\gamma_1^1 > 0$ as the first eigenvalue of the problem

$$\begin{cases} -\Delta\varphi = \gamma \overline{c}(x)\varphi & \text{in } B_R(\overline{x}), \\ \varphi = 0 & \text{on } \partial B_R(\overline{x}). \end{cases} \tag{5.1}$$

By standard arguments, there exists $\varphi_1^1 \in \mathcal{C}_0^1(\overline{B_R(\overline{x})})$ an associated first eigenfunction such that $\varphi_1^1(x) > 0$ for all $x \in B_R(\overline{x})$ and, denoting by n the outward normal to $\partial B_R(\overline{x})$, we also have

$$\frac{\partial \varphi_1^1(x)}{\partial n} < 0\,, \quad \text{on } \partial B_R(\overline{x}). \tag{5.2}$$

Now, let us introduce the function $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, defined as

$$\phi(x) = \begin{cases} \varphi_1^1(x)\,, & x \in B_R(\overline{x}), \\ 0 & x \in \Omega \setminus B_R(\overline{x}), \end{cases}$$

and suppose that u is a solution of (P_λ) such that $u \ge u_0$ in $B_R(\overline{x})$. First observe that, in view of (5.2) and as $u \ge u_0$ on $\overline{B_R(\overline{x})}$, there exists a constant $C > 0$ independent of u such that

$$\int_{\partial B_R(\overline{x})} u \frac{\partial \varphi_1^1}{\partial n}\, dS \le C. \tag{5.3}$$

Thus on one hand, using (5.1) and (5.3), we obtain

$$\begin{aligned} \int_\Omega \big(\nabla\phi\nabla u + c_-(x)\,\phi\, u\big)\, dx &= \int_{B_R(\overline{x})} \nabla\varphi_1^1 \nabla u\, dx = -\int_{B_R(\overline{x})} u\Delta\varphi_1^1\, dx + \int_{\partial B_R(\overline{x})} u \frac{\partial\varphi_1^1}{\partial n}\, dS \\ &\le -\int_{B_R(\overline{x})} u\Delta\varphi_1^1\, dx + C = \gamma_1^1 \int_{B_R(\overline{x})} \overline{c}(x)\,\varphi_1^1\, u\, dx + C \le \gamma_1^1 \int_\Omega c_+(x)\,\phi\, u\, dx + C. \end{aligned} \tag{5.4}$$

On the other hand, considering ϕ as test function in (P_λ) we observe that

$$\int_\Omega \big(\nabla\phi\nabla u + c_-(x)\,\phi\, u\big)\, dx = \lambda \int_\Omega c_+(x)\, u\, \phi\, dx + \int_\Omega \big(\mu(x)|\nabla u|^2 + h(x)\big)\phi\, dx\,. \tag{5.5}$$

From (5.4) and (5.5), we then deduce that, for a $D > 0$ independent of u.

$$\begin{aligned}(\gamma_1^1 - \lambda)\int_\Omega c_+(x)\,\phi\,u\,dx &\geq \int_\Omega \big(\mu(x)|\nabla u|^2 + h(x)\big)\phi\,dx - C \\ &= \int_{B_R(\overline{x})} \big(\mu(x)|\nabla u|^2 + h(x)\big)\varphi_1^1\,dx - C \geq -D.\end{aligned} \tag{5.6}$$

As $c_+u_0 \gneqq 0$ in $B_R(\overline{x})$, we have that

$$\int_\Omega c_+(x)\,\phi\,u\,dx \geq \int_\Omega c_+(x)\,\phi\,u_0\,dx > 0.$$

Hence, for $\lambda > \gamma_1^1$ large enough, we obtain a contradiction with (5.6). □

Proof of Theorem 1.2. We treat separately the cases $\lambda \leq 0$ and $\lambda > 0$.

Part 1: $\lambda \leq 0$.

Observe that for $\lambda \leq 0$ we have $\lambda c_+ - c_- \leq -c_-$ and hence the result follows from [4, Lemma 5.1, Proposition 4.1, Proposition 5.1, Theorem 2.2] as in the proof of [4, Theorem 1.2]. Moreover, observe that u_0 is an upper solution of (P_λ). Hence we conclude that $u_\lambda \leq u_0$ by [3, Lemmas 2.1 and 2.2].

Part 2: $\lambda > 0$.

Consider, for $\lambda \geq 0$ the modified problem

$$(\overline{P}_\lambda) \qquad -\Delta u + u = (\lambda c_+(x) - c_-(x) + 1)\,((u-u_0)^+ + u_0) + \mu(x)|\nabla u|^2 + h(x)\,, \quad u \in H_0^1(\Omega)\cap L^\infty(\Omega).$$

As in the case of (P_λ), any solution of $(\overline{P}_\lambda)$ belongs to $\mathcal{C}^{0,\tau}(\overline{\Omega})$ for some $\tau > 0$. Moreover, observe that u is a solution of $(\overline{P}_\lambda)$ if and only if it is a fixed point of the operator $\overline{T}_\lambda$ defined by $\overline{T}_\lambda : \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\overline{\Omega}) : v \mapsto u$ with u the solution of

$$-\Delta u + u - \mu(x)|\nabla u|^2 = (\lambda c_+(x) - c_-(x) + 1)\,((v-u_0)^+ + u_0) + h(x)\,, \quad u \in H_0^1(\Omega)\cap L^\infty(\Omega).$$

Applying [4, Lemma 5.2], we see that $\overline{T}_\lambda$ is completely continuous. Now, we denote

$$\overline{\Sigma} := \{(\lambda, u) \in \mathbb{R}\times\mathcal{C}(\overline{\Omega}) : u \text{ solves } (\overline{P}_\lambda)\}$$

and we split the rest of the proof into three steps.

Step 1: *If u is a solution of $(\overline{P}_\lambda)$ then $u \geq u_0$ and hence it is a solution of (P_λ).*

Observe that $(u-u_0)^+ + u_0 - u \geq 0$. Also we have that $\lambda c_+(x)((u-u_0)^+ + u_0) \geq \lambda c_+(x)u_0 \geq 0$. Hence, we deduce that a solution u of $(\overline{P}_\lambda)$ is an upper solution of

$$-\Delta u = -c_-(x)\,((u-u_0)^+ + u_0) + \mu(x)|\nabla u|^2 + h(x)\,, \quad u \in H_0^1(\Omega)\cap L^\infty(\Omega). \tag{5.7}$$

Then the result follows from [3, Lemmas 2.1 and 2.2] noting that u_0 is a solution of (5.7).

Step 2: *u_0 is the unique solution of $(\overline{P}_0)$ and $i(I - \overline{T}_0, u_0) = 1$.*

Again the uniqueness of the solution of $(\overline{P}_0)$ can be deduced from [3, Lemmas 2.1 and 2.2]. Now, in order to prove that $i(I - \overline{T}_0, u_0) = 1$, we consider the operator S_t defined by $S_t : \mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\overline{\Omega}) : v \mapsto u$ with u the solution of

$$-\Delta u + u - \mu(x)|\nabla u|^2 = t[(-c_-(x)+1)\,(u_0 + (v-u_0)^+ - (v-u_0-1)^+) + h(x)]\,, \quad u \in H_0^1(\Omega)\cap L^\infty(\Omega).$$

First, observe that there exists $R > 0$ such that, for all $t \in [0,1]$ and all $v \in \mathcal{C}(\overline{\Omega})$,

$$\|S_t v\|_\infty < R.$$

This implies that

$$\deg(I - S_1, B(0,R)) = \deg(I, B(0,R)) = 1.$$

By [3, Lemmas 2.1 and 2.2], we see that u_0 is the only fixed point of S_1. Hence, by the excision property of the degree, for all $\varepsilon > 0$ small enough, it follows that

$$\deg(I - S_1, B(u_0,\varepsilon)) = \deg(I - S_1, B(0,R)) = 1.$$

Thus, as for $\varepsilon < 1$, $S_1 = \overline{T}_0$, we conclude that

$$i(I - \overline{T}_0, u_0) = \lim_{\varepsilon \to 0} \deg(I - \overline{T}_0, B(u_0, \varepsilon)) = \lim_{\varepsilon \to 0} \deg(I - S_1, B(u_0, \varepsilon)) = 1.$$

Step 3: *Existence and behavior of the continuum.*

By [25, Theorem 3.2] (see also [4, Theorem 2.2]), there exists a continuum $\mathscr{C} \subset \overline{\Sigma}$ such that $\mathscr{C} \cap ([0, \infty) \times \mathcal{C}(\overline{\Omega}))$ is unbounded. By Step 1, we know that if $u \in \mathscr{C}$ then $u \geq u_0$ and is a solution of (P_λ). Thus applying Lemma 5.1, we deduce that $\mathrm{Proj}_{\mathbb{R}} \mathscr{C} \cap [0, \infty) \subset [0, \overline{\Lambda}]$. By Theorem 1.1 and Step 1, we deduce that for every $\Lambda_1 \in (0, \overline{\Lambda})$, there is an a priori bound on the solutions of $(\overline{P}_\lambda)$ for $\lambda \in [\Lambda_1, \overline{\Lambda}]$. Hence, the projection of $\mathscr{C} \cap ([\Lambda_1, \overline{\Lambda}) \times \mathcal{C}(\overline{\Omega}))$ on $\mathcal{C}(\overline{\Omega})$ is bounded, and so, we deduce that $\mathscr{C}$ emanates from infinity to the right of $\lambda = 0$. Finally, since $\mathscr{C}$ contains $(0, u_0)$ with u_0 the unique solution of (P_0), we conclude that there exists $\lambda_0 \in (0, \overline{\Lambda})$ such that problem $(\overline{P}_\lambda)$, and thus problem (P_λ), has at least two solutions satisfying $u \geq u_0$ for $\lambda \in (0, \lambda_0)$. □

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References

[1] B. Abdellaoui, A. Dall'Aglio, and I. Peral. Some remarks on elliptic problems with critical growth in the gradient. *J. Differential Equations*, 222(1):21–62, 2006.

[2] S. Alama and G. Tarantello. On semilinear elliptic equations with indefinite nonlinearities. *Calc. Var. Partial Differential Equations*, 1(4):439–475, 1993.

[3] D. Arcoya, C. De Coster, L. Jeanjean, and K. Tanaka. Remarks on the uniqueness for quasilinear elliptic equations with quadratic growth conditions. *J. Math. Anal. Appl.*, 420(1):772–780, 2014.

[4] D. Arcoya, C. De Coster, L. Jeanjean, and K. Tanaka. Continuum of solutions for an elliptic problem with critical growth in the gradient. *J. Funct. Anal.*, 268(8):2298–2335, 2015.

[5] G. Barles, A.-P. Blanc, C. Georgelin, and M. Kobylanski. Remarks on the maximum principle for nonlinear elliptic PDEs with quadratic growth conditions. *Ann. Scuola Norm. Sup. Pisa*, 28(3):381–404, 1999.

[6] G. Barles and F. Murat. Uniqueness and the maximum principle for quasilinear elliptic equations with quadratic growth conditions. *Arch. Rational Mech. Anal.*, 133(1):77–101, 1995.

[7] L. Boccardo, F. Murat, and J.-P. Puel. Existence de solutions faibles pour des équations elliptiques quasi-linéaires à croissance quadratique. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. IV (Paris, 1981/1982)*, volume 84 of *Res. Notes in Math.*, pages 19–73. Pitman, Boston, Mass.-London, 1983.

[8] L. Boccardo, F. Murat, and J.-P. Puel. L^∞ estimate for some nonlinear elliptic partial differential equations and application to an existence result. *SIAM J. Math. Anal.*, 23(2):326–333, 1992.

[9] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[10] H. Brezis and X. Cabré. Some simple nonlinear PDE's without solutions. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)*, 1(2):223–262, 1998.

[11] H. Brezis and L. Nirenberg. H^1 versus C^1 local minimizers. *C. R. Acad. Sci. Paris Sér. I Math.*, 317(5):465–472, 1993.

[12] C. De Coster and A. J. Fernández. Existence and multiplicity for elliptic p-Laplacian problems with critical growth in the gradient. *Calc. Var. Partial Differential Equations*, 57(3):57:89, 2018.

[13] C. De Coster and A. J. Fernández. Two solutions for an indefinite elliptic problem with critical growth in the gradient. *Preprint*. 2018.

[14] C. De Coster and L. Jeanjean. Multiplicity results in the non-coercive case for an elliptic problem with critical growth in the gradient. *J. Differential Equations*, 262(10):5231–5270, 2017.

[15] L. Dupaigne. *Stable solutions of elliptic partial differential equations*, volume 143 of Monographs and Surveys in Pure and Applied Mathematics. Chapman & Hall, 2011.

[16] V. Ferone and F. Murat. Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small. *Nonlinear Anal. (Ser. A: Theory Methods)*, 42(7):1309–1326, 2000.

[17] J. García-Melián, L. Iturriaga, and H. Ramos-Quoirin. A priori bounds and existence of solutions for slightly superlinear elliptic problems. *Adv. Nonlinear Stud.*, 15(4):923–938, 2015.

[18] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[19] C. Imbert and L. Silvestre. Estimates on elliptic equations that hold only where the gradient is large. *J. Eur. Math. Soc. (JEMS)*, 18(6):1321–1338, 2016.

[20] L. Jeanjean and H. Ramos Quoirin. Multiple solutions for an indefinite elliptic problem with critical growth in the gradient. *Proc. Amer. Math. Soc.*, 144(2):575–586, 2016.

[21] L. Jeanjean and B. Sirakov. Existence and multiplicity for elliptic problems with quadratic growth in the gradient. *Comm. Partial Differential Equations*, 38(2):244–264, 2013.

[22] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and quasilinear elliptic equations*. Translated from the Russian by Scripta Technica. Academic Press, New York-London, 1968.

[23] J. Malý and W. P. Ziemer. *Fine regularity of solutions of elliptic partial differential equations*, volume 51 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

[24] G. S. Nornberg and B. Sirakov. A priori bounds and multiplicity for fully nonlinear equations with quadratic growth in the gradient. *To appear in J. Funct. Anal.,* doi:10.1016/j.jfa.2018.06.017, 2018.

[25] P. H. Rabinowitz. A global theorem for nonlinear eigenvalue problems and applications. In *Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971)*, pages 11–36. Academic Press, New York, 1971.

[26] B. Sirakov. Solvability of uniformly elliptic fully nonlinear PDE. *Arch. Ration. Mech. Anal.*, 195(2):579–607, 2010.

[27] B. Sirakov. Uniform bounds via regularity estimates for elliptic PDE with critical growth in the gradient. *Preprint* https://arxiv.org/abs/1509.04495. September 2015.

[28] B. Sirakov. Boundary Harnack estimates and quantitative strong maximum principles for uniformly elliptic PDE. *To appear in Int. Math. Res. Not.,* doi:10.1093/imrn/rnx2017, 2017.

[29] P. Souplet. A priori estimates and bifurcation of solutions for an elliptic equation with semidefinite critical growth in the gradient. *Nonlinear Anal.*, 121:412–423, 2015.

[30] P. Tolksdorf. On the Dirichlet problem for quasilinear equations in domains with conical boundary points. *Comm. Partial Differential Equations*, 8(7):773–817, 1983.

[31] N. S. Trudinger. On Harnack type inequalities and their application to quasilinear elliptic equations. *Comm. Pure Appl. Math.*, 20:721–747, 1967.