Normal idempotent states on a locally compact quantum group

Interactions between Operator Space Theory and Quantum Probability with Applications to Quantum Information
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1. IDEMPOTENT STATES
   - Quasi-subgroups
   - Lattice of quasi-subgroups

2. OPEN QUASI-SUBGROUPS
   - Normal idempotent states
   - Applications

- Let $\mathbb{G}$ be a locally compact quantum group. A state ω on $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})$ is **idempotent** if

$$\omega * \omega = \omega,$$

where $*$ is the convolution: $\mu * \nu = (\mu \otimes \nu) \circ \Delta^{\mathrm{u}}$.
- Let $\mathrm{Idem}(\mathbb{G})$ be the set of all idempotent states on $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})$.

THEOREM (KAWADA-ITÔ, COHEN)

Let G be a locally compact group and let $\omega \in \mathrm{Idem}(G)$. Then there exists a unique compact subgroup K of G such that

$$\omega(f) = \int_K f(k)\, d\boldsymbol{h}_K(k), \qquad f \in \mathrm{C}_0(G)$$

($\boldsymbol{h}_K$ = *the Haar measure on K, $\mathrm{C}_0(G) = \mathrm{C}_0^{\mathrm{u}}(G)$ canonically*).

- If $\mathbb{G}$ is a locally compact quantum group and $\mathbb{K}$ is a compact quantum subgroup of $\mathbb{G}$ then
  - we have an epimorphism $\pi : \mathrm{C}_0^{\mathrm{u}}(\mathbb{G}) \twoheadrightarrow \mathrm{C}^{\mathrm{u}}(\mathbb{K})$,
  - $\omega = \boldsymbol{h}_{\mathbb{K}} \circ \pi$ is an idempotent state on $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})$.
- However, not every $\omega \in \mathrm{Idem}(\mathbb{G})$ arises this way (A. Pal).

- Given $\omega \in \mathrm{Idem}(\mathbb{G})$ we will say that ω corresponds to a compact quantum **quasi-subgroup** of $\mathbb{G}$.

- Let $\omega,\mu\in\mathrm{Idem}(\mathbb{G})$. We say that μ **dominates** ω if
$$\omega*\mu=\mu.$$
Notation: $\omega\preccurlyeq\mu$.
- For idempotent states ω,μ arising from compact quantum subgroups $\mathbb{H}$ and $\mathbb{K}$ we have
$$\Bigl(\omega\preccurlyeq\mu\Bigr)\Longleftrightarrow\Bigl(\mathbb{H}\subset\mathbb{K}\Bigr)$$
- Put $\mathrm{Idem}_0(\mathbb{G})=\mathrm{Idem}(\mathbb{G})\cup\{0\}$.

Theorem

Given $\omega,\mu\in\mathrm{Idem}(\mathbb{G})$ *there exist*
$$\omega\wedge\mu=\sup\bigl\{\nu\in\mathrm{Idem}(\mathbb{G})\,\big|\,\nu\preccurlyeq\omega,\ \nu\preccurlyeq\mu\bigr\}$$
and
$$\omega\vee\mu=\inf\bigl\{\nu\in\mathrm{Idem}_0(\mathbb{G})\,\big|\,\omega\preccurlyeq\nu,\ \mu\preccurlyeq\nu\bigr\}.$$

- A (left) **coideal** in $L^\infty(\mathbb{G})$ is a von Neumann subalgebra $\mathsf{N} \subset L^\infty(\mathbb{G})$ such that $\Delta_{\mathbb{G}}(\mathsf{N}) \subset L^\infty(\mathbb{G}) \bar{\otimes} \mathsf{N}$

Fact

There is a bijective correspondence between

- idempotent states on $\mathbb{G}$, and
- τ-invariant integrable coideals $\mathsf{N} \subset L^\infty(\mathbb{G})$.

The coideal N_ω corresponding to $\omega \in \mathrm{Idem}_0(\mathbb{G})$ is the range of the normal conditional expectation

$$\boldsymbol{E}_\omega : L^\infty(\mathbb{G}) \ni x \longmapsto \omega * x \in L^\infty(\mathbb{G}).$$

- We have
$$\mathsf{N}_{\omega \wedge \mu} = \mathsf{N}_\omega \vee \mathsf{N}_\mu$$
and
$$\mathsf{N}_{\omega \vee \mu} = \begin{cases} \mathsf{N}_\omega \cap \mathsf{N}_\mu & \text{when } \mathsf{N}_\omega \cap \mathsf{N}_\mu \text{ is integrable,} \\ \{0\} & \text{otherwise.} \end{cases}$$

- Let $\mathbb{H}$ and $\mathbb{K}$ be two compact quantum subgroups of $\mathbb{G}$.
- Let ω and μ be the corresponding idempotent states.
- Then
  - $\omega \wedge \mu$ is the Haar measure of $\mathbb{H} \cap \mathbb{K}$,
  - we have

    $$\omega \vee \mu = \text{Haar measure of } \overline{\langle \mathbb{H}, \mathbb{K} \rangle}$$

    when $\overline{\langle \mathbb{H}, \mathbb{K} \rangle}$ is compact, and

    $$\omega \vee \mu = 0$$

    otherwise.

DEFINITION

The quasi-subgroup corresponding to $\omega \wedge \mu$ is the **intersection** of quasi-subgroups related to ω and μ. In case $\omega \vee \mu$ is non-zero, w say that the corresponding quasi-subgroup is the quasi-subgroup **generated** by those of ω and μ.

Proposition

The operations

- $\mathrm{Idem}(\mathbb{G}) \times \mathrm{Idem}(\mathbb{G}) \ni (\omega, \mu) \mapsto \omega \wedge \mu \in \mathrm{Idem}(\mathbb{G})$,
- $\mathrm{Idem}_0(\mathbb{G}) \times \mathrm{Idem}_0(\mathbb{G}) \ni (\omega, \mu) \mapsto \omega \vee \mu \in \mathrm{Idem}_0(\mathbb{G})$

are commutative and associative.

Theorem

Let $\omega, \mu, \rho \in \mathrm{Idem}(\mathbb{G})$ *be such that*

1. $\rho \preccurlyeq \omega$,
2. $\mu * \rho = \rho * \mu$,
3. $\mathsf{N}_{\omega \wedge \mu} = (\mathsf{N}_\omega \mathsf{N}_\mu)^{\sigma-\text{c.l.s.}}$.

Then $\omega \wedge (\mu \vee \rho) = (\omega \wedge \mu) \vee \rho$.

- $\rho \preccurlyeq \omega$ means that the quasi-subgroup corresponding to ρ is contained in the one for ω,
- $\mu * \rho = \rho * \mu$ means that the quasi-subgroups corresponding to μ and ρ commute.

- We have $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G}) \twoheadrightarrow \mathrm{C}_0(\mathbb{G}) \subset L^\infty(\mathbb{G})$, so $L^\infty(\mathbb{G})_*$ maps into $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})^*$.
- This map is injective and its image is a closed ideal in the Banach algebra $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})^*$.
- Elements of $L^\infty(\mathbb{G})_*$ viewed in $\mathrm{C}_0^{\mathrm{u}}(\mathbb{G})^*$ are (sometimes) called **normal**.

Definition

We will say that a compact quasi-subgroup corresponding to $\omega \in \mathrm{Idem}(\mathbb{G})$ is **open** if ω is normal.

- Let $\mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ denote the set of normal idempotent states on $\mathbb{G}$.

Proposition

If $\mathbb{G}$ *is a discrete quantum group then* $\mathrm{Idem}_{\mathrm{nor}}(\mathbb{G}) = \mathrm{Idem}(\mathbb{G})$.

Proof.

The co-unit ε of $\mathbb{G}$ is normal and it is dominated by all idempotent states. So if $\omega \in \mathrm{Idem}(\mathbb{G})$ then

$$\omega * \varepsilon = \omega.$$

But normal states form an ideal, so ω is normal. □

Theorem

For any locally compact quantum group $\mathbb{G}$ *there is a bijection*

$$\mathrm{Idem}_{\mathrm{nor}}(\mathbb{G}) \ni \omega \longmapsto \widetilde{\omega} \in \mathrm{Idem}_{\mathrm{nor}}(\widehat{\mathbb{G}})$$

reversing natural orders and such that

$$\widetilde{\widetilde{\omega}} = \omega$$

for all $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$*.*

- On the level of coideals corresponding to idempotent states we have
$$\mathsf{N}_{\widetilde{\omega}} = \mathsf{N}_{\omega}{}' \cap L^{\infty}(\widehat{\mathbb{G}}).$$

- If $\omega,\mu\in\mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ and $\omega\vee\mu\neq 0$ then $\omega\vee\mu\in\mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$.
- However, $\omega\wedge\mu$ does not have to be normal.

Proposition

Let $\omega,\mu\in\mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$. *Then* $\omega\wedge\mu\in\mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ *if and only if* $\widetilde{\omega}\vee\widetilde{\mu}\neq 0$. *In this case we have*

$$\omega\wedge\mu=\widetilde{\widetilde{\omega}\vee\widetilde{\mu}}.$$

- By the work of Kalantar-Kasprzak-Skalski on open quantum subgroups of locally compact quantum groups we have a bijective correspondence between
  - normal open quantum subgroups of a l.c.q.g. $\mathbb{G}$,
  - normal compact quantum subgroups of $\widehat{\mathbb{G}}$

  given by
  $$\mathbb{G} \supset \mathbb{H} \longleftrightarrow \mathbb{K} \subset \widehat{\mathbb{G}},$$
  where $\widehat{\mathbb{H}} \cong \widehat{\mathbb{G}}/\mathbb{K}$.
- Our theorem gives a bijection between
  - compact open quasi-subgroups of $\mathbb{G}$,
  - compact open quasi-subgroups of $\widehat{\mathbb{G}}$.
- The latter is an extension of a special case of the former.

THEOREM

Let $\mathbb{G}$ *be a compact quantum group. Then*

1. *the following conditions are equivalent for* $\omega \in \mathrm{Idem}(\mathbb{G})$*:*
   1. $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$,
   2. $\dim \mathsf{N}_\omega < +\infty$,
   3. N_ω *has a finite-dimensional direct summand;*
2. *for any finite-dimensional coideal* $\mathsf{N} \subset L^\infty(\mathbb{G})$ *there exists* $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$ *such that* $\mathsf{N} = \mathsf{N}_\omega$*; in particular* N *is invariant under the scaling group.*

- The proof 3 $\Rightarrow$ 1 uses a strong result on ergodic actions of compact quantum groups: *if a compact quantum group acts ergodically on a von Neumann algebra* N *with a finite dimensional direct summand then* $\dim \mathsf{N} < +\infty$*.*

Theorem

Let $\mathbb{G}$ be a locally compact quantum group and let $\omega \in \mathrm{Idem}(\mathbb{G})$ be such that $\dim \mathsf{N}_\omega < +\infty$. Then $\mathbb{G}$ is compact and consequently $\omega \in \mathrm{Idem}_{\mathrm{nor}}(\mathbb{G})$.

Proof.

The coideal N_ω is integrable, so if $\dim \mathsf{N}_\omega < +\infty$, we have $\boldsymbol{h}(\mathbb{1}) < +\infty$, so that $\mathbb{G}$ is compact. The last statement follows from previous Theorem. □

- The above theorem corresponds to the elementary fact that if a quotient by a compact subgroup is finite then the original group must also be compact.
- In other words, the condition that $\dim \mathsf{N}_\omega < +\infty$, says that the corresponding quasi-subgroup is of “finite index”.

Thank you!