Jordan operator algebras, their operator space structure and noncommutative topology

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Abstract

In the first part we define Jordan operator algebras and describe some basic theory (joint with Wang). Then we turn to noncommutative topology in the C^*-algebraic sense of Akemann, L. G. Brown, Pedersen, etc, and as modified for not necessarily selfadjoint algebras by the authors with Read, Neal, Hay and other coauthors. We explain the latter ideas. We also discuss related topics such as noncommutative peak sets and peak interpolation, and hereditary subalgebras (HSAs) in these settings. In the last part (longer, and joint work with Matt Neal) we generalize all of this to Jordan operator algebras. Our breakthrough relies in part on establishing several variants of technical C^*-algebraic results of Brown relating to HSAs, proximinality, deeper facts about $L+L^*$ for a left ideal L in a C^*-algebra, noncommutative Urysohn lemmas, and other approximation results in C^*-algebras and various subspaces of C^*-algebras related to open and closed projections.

The operator space aspects will be emphasized.

Section I. Jordan operator algebras

Section II. Noncommutative topology for C^*-algebras and some results of Brown

Section III. Noncommutative topology for algebras of Hilbert space operators, and peak sets (B-Read, B-Neal, Hay)

Section IV. Noncommutative topology, peaking, and HSA theory for Jordan operator algebras

Section I. Jordan operator algebras (Joint with Z. Wang)

• Jordan algebras (Jordan-von Neumann-Wigner) arose in the context of formalizing quantum mechanics.

It was argued that since ordinary products xy of selfadjoints need not be selfadjoint, they do not make sense as observables, whereas squares x^2 (or equivalently Jordan products) do

Thus, JC*-algebras are subspaces of C^*-algebras closed under squares (or equivalently Jordan products $\frac{1}{2}(xy+yx)$), adjoints (and the norm topology)

Jordan operator algebras: subspaces of C^*-algebras closed under squares (or equivalently Jordan products $\frac{1}{2}(xy + yx)$)

• An interesting class of operator spaces, that has almost had no study until now (besides Arazy-Solel paper). Very many examples (give some momentarily).

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• We have a few abstract operator space characterizations of Jordan operator algebras (B-Wang, B-Neal):

Before we begin the first theorem we recall that B-Neal gave (two different) metric characterizations of unital operator spaces

Theorem (B-Wang) Let A be a unital operator space, a subspace of a unital operator space B. Let $m : A \times A \to B$ be a completely contractive bilinear map. Define $a \circ b = \frac{1}{2}(m(a,b) + m(b,a))$, and suppose that A is closed under this operation. Also, assume that $m(1,a) = m(a,1) = a$ for $a \in A$. Then A is a Jordan operator algebra with Jordan product $a \circ b$. Conversely, ...

The following very different characterization of Jordan operator algebras (which does not assume the existence of any kind of identity or approximate identity) is the Jordan algebra variant of the Kaneda-Paulsen theorem. Here $I(X)$ is the injective envelope, which is known to be a C^*-algebra if X is a unital operator space. The C^*-algebra generated by X inside $I(X)$ is called the C^*-envelope $C^*_{\rm e}(X)$.

Theorem (B-Neal) Let X be an operator space. The possibly nonassociative algebra products on X for which there exists a completely isometric Jordan homomorphism from X onto a Jordan operator algebra, are in a correspondence with the elements $z \in {\rm Ball}(I(X))$ such that $xz^*x \subset X$ for all $x \in X$. For such z the associated Jordan operator algebra product on X is $\frac{1}{2}(xz^*y + yz^*x)$ for $x, y \in X$ (viewing $X \subset I(X)$).

Corollary Let (X, u) be a unital operator space. The possibly nonassociative algebra products on X for which there exists a completely isometric Jordan homomorphism from X onto a Jordan operator algebra, are in a bijective correspondence with the elements $w \in \mathrm{Ball}(X)$ such that $xwx \subset X$ (multiplication taken in the C^*-algebra which is the C^*-envelope of (X, u) (or which is the injective envelope $I(X)$ with its unique C^*-algebra product for which u is the identity). For such w the associated Jordan operator algebra product on X is $\frac{1}{2}(xwy + ywx)$ for $x, y \in X$.

Corollary Let (X, u) be a unital operator space. The possibly nonassociative algebra products on X for which there exists a completely isometric Jordan homomorphism from X onto a Jordan operator algebra, are in a bijective correspondence with the elements $w \in \mathrm{Ball}(X)$ such that $xwx \subset X$ (multiplication taken in the C^*-algebra which is the C^*-envelope of (X, u) (or which is the injective envelope $I(X)$ with its unique C^*-algebra product for which u is the identity). For such w the associated Jordan operator algebra product on X is $\frac{1}{2}(xwy + ywx)$ for $x, y \in X$.

Corollary The range of a c. contractive projection P on a Jordan operator algebra, with product $P(xy)$, is (c. isometric Jordan isomorphic to) a Jordan operator algebra.

Some examples of Jordan operator algebras

- (Associative) operator algebras. JC*-algebras (closed selfadjoint subspaces of a C^*-algebra which are closed under squares)
- Consider $\{a \in A : \pi(a) = \theta(a)\}$ for a homomorphism $\pi : A \to A$ and an antihomomorphism $\theta : A \to A$, for an associative operator algebra A. Or consider $\{(\theta(a), \pi(a)) \in C \oplus^\infty D\}$ for a homomorphism $\pi : A \to C$ and antihomomorphism $\theta : A \to D$ for associative operator algebras C, D.
- The range of various natural classes of contractive (resp. c. contr) projections on operator algebras (or Jordan operator algebras) are Jordan operator algebras.
- Smallest Jordan operator subalgebra containing the generators, of well known C^*-algebras with n generators.

- Jordan operator algebras in a JC*-triple Z: Fix $y \in \mathrm{Ball}(Z)$, and define a bilinear map $m_y : Z \times Z \to Z$ by $m_y(x,z) = \frac{1}{2}(xy^*z + zy^*x)$. This is a Jordan operator algebra. Many such Jordan operator algebras are isometrically isomorphic as Jordan algebras , but are very different as operator spaces.

Later Jordan operator algebras with contractive approximate identity play a large role

- Jordan operator algebras with contractive approximate identity: If A is any Jordan operator algebra (say, a C^*-algebra), and E is any nonempty convex set of real positive (accretive) elements in A, then $\overline{\{xAx : x \in E\}}$ is an approximately unital Jordan operator algebra.

Definition. If A is a Jordan operator subalgebra of a C^*-algebra B then we say that a net (e_t) in $Ball(A)$ is a *B-relative partial cai* for A if $e_t a \to a$ and $a e_t \to a$ for all $a \in A$. Here we use usual product on B, which is not an element in A.

We now see that a 'relative cai' can be switched for a 'nonrelative' one:

Definition. A net (e_t) in $Ball(A)$ is a *partial cai* for A if for every C^*-algebra B containing A as a Jordan subalgebra, $e_t a \to a$ and $a e_t \to a$ for all $a \in A$, using the product on B. A is called *approximately unital* if it has a partial cai.

Moreover, a net (e_t) in $Ball(A)$ is a *Jordan cai* for A if $e_t a + a e_t \to 2a$ for all $a \in A$.

Theorem (B-Wang) If A is a Jordan operator subalgebra of a C^*-algebra B, then the following are equivalent:

(i) A has a partial cai.

(ii) A has a B-relative partial cai.

(iii) A has a Jordan cai.

(iv) A^{**} has an identity p of norm 1 with respect to usual product in B^{**}.

The operator space structure of Jordan operator algebras is occasionally frustrating. For example if A is a (nonassociative) Jordan operator algebra then $M_n(A)$ is not a Jordan operator algebra for $n \geq 2$.

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Unitization: From Meyer's theorem on the unitization of (associative) operator algebras we get:

Corollary The unitization A^1 of a Jordan operator algebra is unique up to isometric Jordan isomorphism.

(Indeed, If A and B are Jordan subalgebras of $B(H)$ and $B(K)$ respectively, with $I_H \notin A$, and if $T : A \to B$ is a contractive (resp. isometric) Jordan homomorphism, then there is a unital contractive (resp. isometric) Jordan homomorphism extending T from $A + \mathbb{C}I_H$ to $B + \mathbb{C}I_K$.)

Open: Is the unitization of a Jordan operator algebra unique up to completely isometric Jordan isomorphism?

(Think we proved this if A is approx unital.)

Proposition If J is an approximately unital closed Jordan ideal in a Jordan operator algebra A, then A/J is completely isometrically isomorphic to a Jordan operator algebra.

Open: If J is a closed Jordan ideal in a Jordan operator algebra A, then is A/J even isometrically isomorphic to a Jordan operator algebra?

Section II. Noncommutative topology for C^*-algebras and some results of Brown

Akemann's noncommutative topology:

A projection p in the second dual of a C^*-algebra B is open if it is a increasing (weak*) limit of positive elements in B

- Define projection q to be closed if $1 - q$ is open.

Exercise: A projection $q \in C_0(K)^{**}$ is open (resp. closed) iff it is the canonical image of the characteristic function of an open (resp. closed) set in locally compact Hausdorff K.

Thus topology has become the theory of a certain class of projections in the second dual B^{**}. This is Akemann's noncommutative topology.

We will not survey Akemann's noncommutative topology much today, but with the definitions above you should now try to prove noncommutative versions of the basic results in topology. Unions of sets are replaced by suprema $\vee_i p_i$ of projections, Intersections of sets are replaced by infima $\wedge_i p_i$ of projections.

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A projection $q \in B^{**}$ is compact if it is closed in $(B^1)^{**}$. Many equivalent conditions... .

Akemann's noncommutative Urysohn lemma (B unital case): Given p, q closed projections in B^{**}, with $pq = 0$ there exists $f \in B$ with $0 \leq f \leq 1$ and $fp = 0$ and $fq = q$.

- Similarly in the nonunital case but now one of the projections is compact

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• Similarly in the nonunital case but now one of the projections is compact

• A nice tool for C^*-algebras, and it obviously generalizes the classical Urysohn lemma

• In fact there is no other 'noncommutative topology' out there that has a Urysohn lemma, so for some 'noncommutative function theory' we have little choice; we have to go this way!

(Akemann's) open projections are sup's of support projections $s(x)$ for $x \in B_+$, and are just the support projections in a separable case. (Recall $s(x) = \mathrm{weak}^*\lim_n x^{\frac{1}{n}}$)

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• Another important picture of open projections: they are in one-to-one correspondence with the closed right ideals in B. Or the closed left ideals. Or hereditary subalgebras. So you can interpret the results above in terms of the one-sided ideal structure in B

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Hereditary subalg (or HSA) of B: approx unital subalg D s.t. $DBD \subset D$

Relations between some of these objects:

open projection p $\leftrightsquigarrow$ HSA $\{a \in B : pap = a\}$

$$\cup$$

support projection $s(x)$ $\leftrightsquigarrow$ HSA $\overline{xBx}$

(Go from right to left by taking w* limit of approximate identity, which is $(x^{\frac{1}{n}})$ in lower line)

• Support projections, closed and open projections, hereditary subalgebras (HSA's) appear all the time in modern C^*-algebra theory

- Noncommutative topology emerged out of ideas of Combes, Effros, Akemann. Some of the best C^*-algebraists took it up in the 1980s and early 1990s, getting many great results (Brown, Pedersen, Anderson, etc).

A good example is the 124 page Canadian J paper of Brown. In one section Brown considers various noncommutative Tietze theorems, and has several other technical results. We will mention this aspect today since it is very relevant to some of our new work described later

E.g. Lifting elements via the canonical quotient map $B \to B/J$, where J might possibly be $L, R, L+R, D = L \cap R$, etc. Here L is a left ideal, and R is the matching right ideal L^*.

In particular, lifting elements in $qBq \cong B/(L+L^*)$ for a closed projection q in B^{**} to elements in B (or $M(B)$) with 'values' in the same convex set, under some conditions.

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One immediately runs into important questions about proximinality, formulae for norm in B/J, lifting projections, etc.

Recall that a closed subspace E of a Banach space X is proximinal if for any $x \in X$ there exists $y \in E$ with the distance $d(x, E) = \|y - x\|$. That is, there is a closest point in E.

Theorem (Brown) A closed left ideal L, and $L + L^*$, are proximinal in B. Thus we can 'exactly lift' norm 1 elements via the quotient map $B \to B/L \cong Bq$ or $B \to B/(L + L^*) \cong qBq$. Here q is the closed projection corresponding to L.

- Such proximinality may be viewed as a simple case of the noncommutative Tietze theorem

In generalizing some of Brown's results in this paper to 'noncommutative function theory' settings we will want operator space versions of such results

E.g. Matrix completion problems in an operator space E: given a closed projection q, if [various] 'corners' with respect to q of an element of E are contractions, is there a contraction in E with those same corners?

Brown: Unfortunately, hereditary subalgebras (i.e. $L \cap L^*$ for a closed left ideal L) of a C^*-algebra B need not be proximinal in B

We will later give a positive result in this direction

Section III. Noncommutative topology, algebras of Hilbert space operators, and peak sets (B-Read, B-Neal, Hay)

Following the lead of Hay's thesis, we (with Read, Hay, Neal, and other coauthors) fused Akemann's noncommutative topology with the classical theory of peak sets, generalized peak sets, peak interpolation, etc, for function algebras. The latter topics are crucial tools for studying classical algebras of functions.

If A is a subalgebra of $C(K)$, then a peak set for A is a set of form $E = f^{-1}(\{1\})$ for a function $f \in A, \|f\| = 1$. By replacing f by $(1+f)/2$ we may assume also that $|f| < 1$ on E^c, in which case $f^n \to \chi_E$.

Say f peaks on E and write the characteristic function of E as $u(f)$

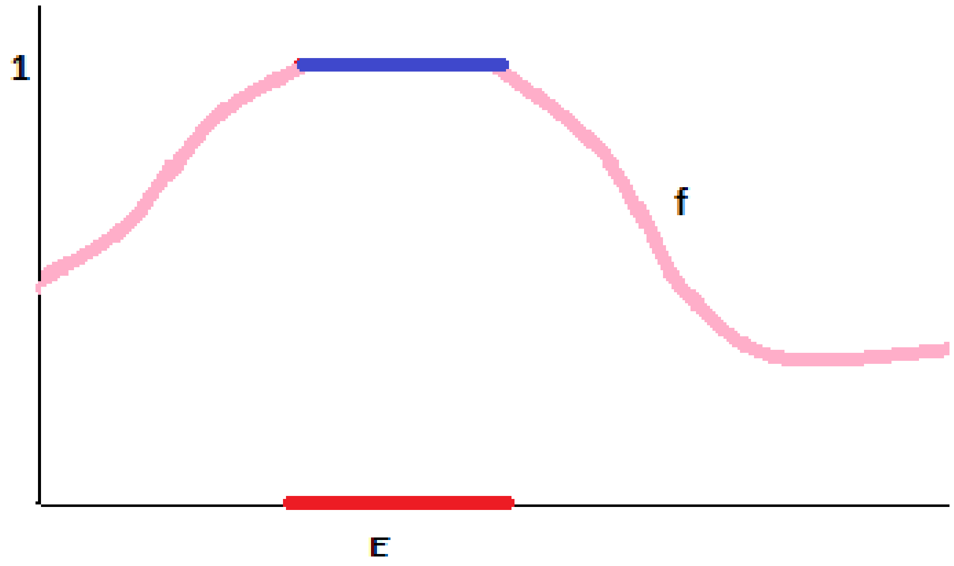


Figure 1: A peak set E

Noncommutative case: peak projections

If A is a subalgebra of a C^*-algebra B and $x \in \mathrm{Ball}(A)$ write $u(x)$ for $\mathrm{weak}^*\lim_n x^n$ if it exists in B^{**}... this is a peak projection. Say x peaks at $u(x)$.

• Peak projections in C^*-algebras are probably quite old but not called by this name (particularly peak projections of positive x, where $u(x) = \chi_{\{1\}}(x)$ in the Borel functional calculus)

Peak projections in operator algebras (subalgebras of C^*-algebras) are due to Damon Hay

- Need other equivalent definitions of peak projections to be optimally useful (Hay, B-Neal, etc).

Classical peak sets are crucially important for studying function algebras, where they arose in some of the deepest applications ...

The p-sets (or generalized peak sets) for a function algebra are just the intersections of peak sets. In the separable case they are just the peak sets.

- When applied in the case that the function algebra A is all of $C(K)$, one may view peak sets as important building blocks of the topology:

E.g. If K compact then closed sets in K are exactly the p-sets, and are just the peak sets if $C(K)$ separable. Similarly if K locally compact, but replace 'closed' by 'compact'.

Loosely speaking, complements of peak sets are support projections

So (and even in nc case) $s(x) = \text{weak}^*\lim_n x^{\frac{1}{n}}$, for real positive x

(Real positive for Hilbert space operators means $x + x^* \geq 0$; equivalently Re $\varphi(x) \geq 0$ for all states φ)

Part of the algebraic calculus: $u(x) = 1 - s(1 - x)$ for $x \in \text{Ball}(A)$ (one can see this in the picture:)

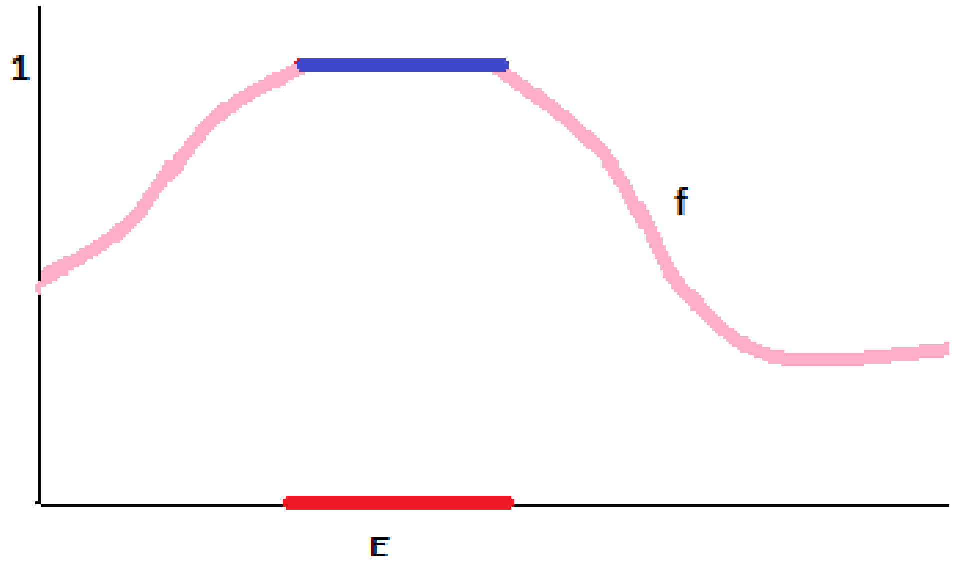


Figure 2: Peak set $E = u(f)$

Part of the algebraic calculus:

$$u(x) \wedge u(y) = u(\frac{x+y}{2})$$

$$s(x) \vee s(y) = s(\frac{x+y}{2})$$

A primer on classical peak interpolation:

A primer on classical peak interpolation:

- (Everything in this primer ‘goes noncommutative’)

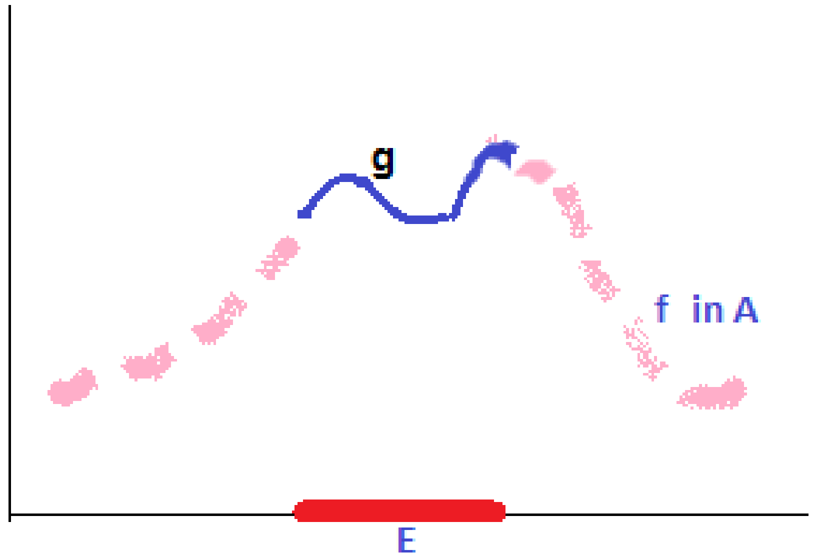
Peak interpolation is a way of building elements with prescribed behaviour

Importance of being able to build elements ... in C^*-algebras one has a good functional calculus and one can build elements using this and other tricks we are familiar with in C^*-algebras. But in more general spaces one does not have a very good functional calculus, and need other ways to build elements.

Setting for classical peak interpolation:

Given: fixed $A \subset C(K)$ as above, ...

... and one tries to build functions in A which have prescribed values or behaviour on a fixed closed subset E of K (or on several disjoint subsets), without increasing norms.



• The sets E that 'work' for this are the generalized peak sets, i.e. the intersections of peak sets

- In the separable case, they are just the peak sets (one doesnt need intersections)

Glicksberg has a beautiful characterization of generalized peak sets:

Glicksberg's peak set theorem (reformulated) For a unital subalgebra $A \subset C(K)$, the generalized peak sets are exactly the closed sets which, when viewed as elements of $C(K)^{**}$, are in the weak* closure of A in $C(K)^{**}$

A primary example of a peak interpolation result, which originated in results of Errett Bishop, says:

Theorem For a unital subalgebra $A \subset C(K)$, if h is a continuous strictly positive scalar valued function on K, then the continuous functions on a generalized peak set E which are restrictions of functions in A, and which are dominated in modulus by the 'control function' h on E, have extensions f in A with $|f(x)| \leq h(x)$ for all $x \in K$.

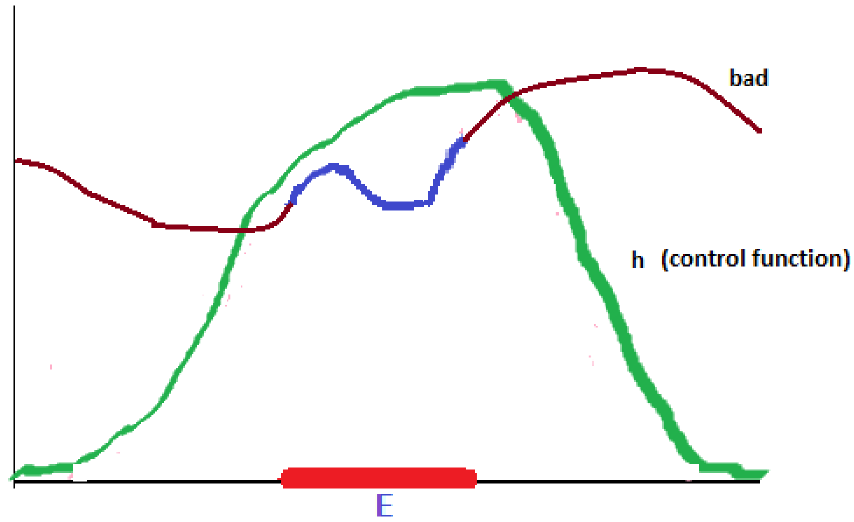


Figure 3: Extension dominated by control function

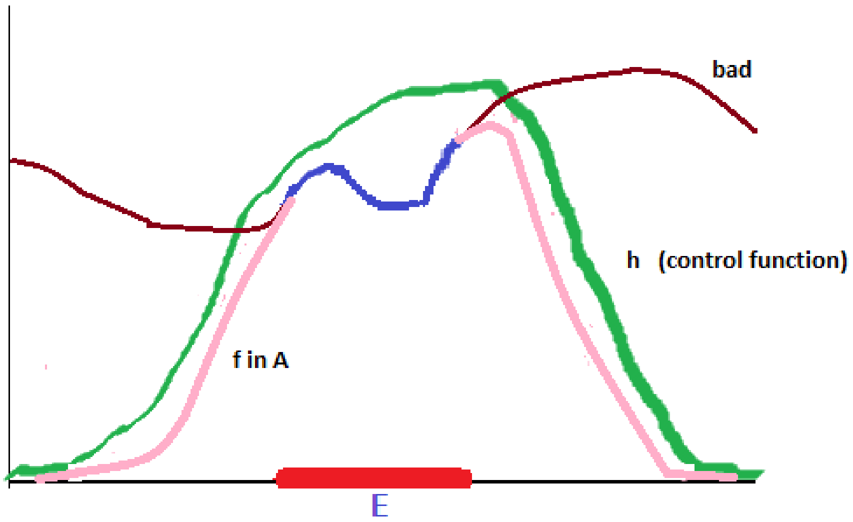


Figure 4: Extension dominated by control function

- Really it is $|f|$ that is dominated by h (the green curve)

Now replace function algebra $A \subset C(K)$, by an operator algebra A inside a C^*-algebra B.

Summary: Everything surveyed above works, the noncommutative topology, peak theory, etc. above, 'goes noncommutative' in a perfectly literal way (long series of papers by B-Read, B-Hay-Neal, B-Neal, ...). For lack of time won't state many results here. Its an extensive package, 7 papers or so stuffed full of theorems and results fusing noncommutative topology and classical peak set theory.

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$\exists$ Useful Kaplansky density theorems: density of (ball of) real positive elements in the (ball of) real positive elements in the bidual.

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The basic theory of hereditary subalgebras and open projections for C^*-algebras generalizes to operator algebras.

Theorem Let A be an operator algebra. There is a bijective correspondence between closed right ideals of A with left cai's, hereditary subalgebras in A, and left ideals of A ... . The support projection (limit of approximate identity above) of these three are the same.

Define an open projection to be such support projection, and a closed projection to be the complement $1 - p$ of an open projection p

Hay's theorem: Let A be a subalgebra of a C^*-algebra B. A projection $p \in A^{\perp\perp}$ is open w.r.t. A iff open w.r.t. B

This is not easy, but is a gateway to many noncommutative topology results.

Proposition Let A be a subalgebra of C^*-algebra B. The support projection of real positive $x \in A$ is open; and the matching HSA is of course $\overline{xAx}$. Conversely if A is separable (or under another 'countability' hypothesis), then every open projection is such support projection $s(x)$.

Proposition Let A be as above. Sup's of open projections relative to A are open.

Corollary Let A be as above. A projection in A^{**} is open if and only if it is an increasing weak* limit of support projections of real positive elements in A.

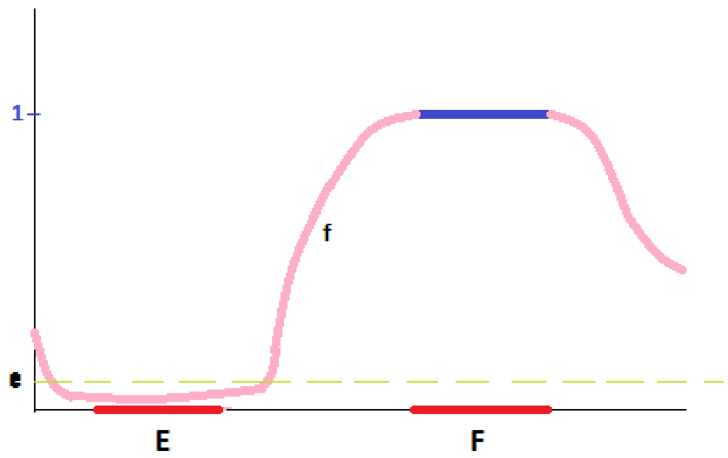
Corollary Let A be as above. The HSA's (resp. closed right ideals with left cai) are precisely the subsets of form $\overline{EAE}$ (resp. $\overline{EA}$) for a subset E of real positive elements in A).

Theorem. If A is a closed subalgebra of a C^*-algebra B, and q is a projection in A^{**} then q is compact in B^{**} in the sense of Akemann, iff q is a closed projection in $(A^1)^{**})$.

A peak projection $u(x)$ of $x \in A$, if it exists, is closed and compact.

Theorem. Compact projections in A^{**} are just the decreasing weak* limit of peak projections $u(x)$ for $x \in A$, and if A is separable they are just the peak projections

∃ Urysohn lemmas, which in the function algebra case would be where we find ‘nearly positive’ real positive functions in our function algebra A which are 1 on a closed set F and zero on a closed set E disjoint from F (or close to zero, depending on the type of closed set).



For example:

Intrinsic Urysohn theorem Let A be an operator algebra. Whenever a compact projection q in A^{**} is dominated by an open projection p in A^{**}, then there exists 'nearly positive' real positive $b \in A$ with $q = qbq, b = pbp$.

Plus strict Urysohn lemma ...

Some of our results seem new even when specialized to function algebras (subalgebras of a commutative C^*-algebra $C(K)$).

• The peak interpolation theorems which I stated for function algebras $A \subset C(K)$ generalize to subalgebras of C^*-algebras.

... so we can do the nc analogue of building functions in A which have prescribed values or behaviour on a fixed closed subset $E \subset K$...

...we can build operators in A which have prescribed behaviours on Akemann's noncommutative generalizations of closed sets, i.e. closed projections in the bidual of the C^*-algebra

Section IV. Noncommutative topology, peaking, and HSA theory for Jordan operator algebras (Joint with M. Neal)

Jordan operator algebras turn out to have an excellent noncommutative topology.

Here 'excellent' by definition (!) means what we (Hay, B-Hay-Neal, B-Neal, B-Read, and others) proved for subalgebras of $B(H)$, for Hilbert spaces H,

This includes for example Urysohn lemmas, peak interpolation, Kaplansky density, order theory (in the sense of B-Read "Order theory and interpolation in operator algebras"), etc.

- [B-Wang, Section I of this talk] uncovered the fact that there exists a large theory of such Jordan operator algebras; they are far more similar to associative operator algebras than was suspected, and we initiated the theory of such algebras.

- Unfortunately progress in [B-Wang] could only proceed to a certain point, because we were blocked by a couple of difficult issues. In particular, the noncommutative topology and peak set theory was blocked at an early stage.

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- Unfortunately progress in [B-Wang] could only proceed to a certain point, because we were blocked by a couple of difficult issues. In particular, the noncommutative topology and peak set theory was blocked at an early stage.

- [B-Neal 2017] solves these difficult points in the noncommutative topology, uusing more complicated variants of C^*-algebra arguments in Browns CJM paper mentioned earlier, and of some proofs of Hay, thus removing the blockage.

- To enable noncommutative topology for Jordan operator algebras, we first we need to prove some new results even for C^*-algebra, which we use in our breakthrough for the Jordan case:

• To enable noncommutative topology for Jordan operator algebras, we first we need to prove some new results even for C^*-algebra, which we use in our breakthrough for the Jordan case:

Theorem A hereditary subalgebra D in a C^*-algebra B is proximinal in $L + L^*$, where $L = \overline{AD}$ is the left ideal associated with D.

(recall Brown showed it need not be proximinal in B)

Theorem Let B be a C^*-algebra. A projection q in B^{**} is closed if and only if qB^*q is weak* closed in B^*.

We also proved an off-diagonal version of one of Brown's Tietze theorems, with a more complicated proof, which we need.

Theorem Let q be a closed projection in B^{**} for a C^*-algebra B. Let $\epsilon > 0$ be given and $x \in B$ with $\|x\| \leq 1 + \epsilon$, and $qxq = 0$, and the 1-2 and 2-1 corners of x with respect to q being contractions. Then for all $\epsilon' > 3\sqrt{2\epsilon + \epsilon^2}$ there exists $y \in B$ such that $y = q^\perp y q^\perp, \|y\| \leq \epsilon'$ and $\|x - y\| \leq 1$.

Proposition If B is a C^*-algebra and q is a closed projection in B^{**} supporting a closed left ideal L in B, then

$$(L + L^*)/D \cong \{qbq^\perp + q^\perp bq : b \in L + L^*\}$$

completely isometrically, where $D = L \cap L^*$ is the HSA supported by $q^\perp$.

• Also need some generalizations of lemmas of Hay. Some of these now become quite difficult, using the C^*-algebra results above in a complicated way. The first one is not very difficult, similar to the B-Hay proof of the analogous lemma in [Hay]:

Lemma (1st incarnation) Let X be a closed subspace of a C^*-algebra B. Let $q \in B^{**}$ be a closed projection such that $X^\perp \subset (qXq)_\perp$. Let $I = \{x \in X : qxq = 0\}$. Then qXq is completely isometric to X/I via the map $x + I \mapsto qxq$.

The condition $X^\perp \subset (qXq)_\perp$, if X is a Jordan subalgebra is simply saying that $q \in X^{\perp\perp}$.

Corollary Let X be a closed subspace of a unital C^*-algebra B and let q be a closed projection in B^{**} such that $X^\perp \subset (qXq)_\perp$ as in last Lemma. Suppose that $h \in B$ is strictly positive and commutes with q, and suppose that $a \in X$ with $a^*qa \leq h$. Then given $\epsilon > 0$ there exists $b \in X$ such that $qbq = qaq$ and $b^*b \leq h + \epsilon 1$.

(A 'preliminary ϵ-peak interpolation' result)

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Theorem (Lemma 2nd incarnation–harder) Let X be a closed subspace of a unital C^*-algebra B. Let $q \in B^{**}$ be a closed projection such that $qXq = (0)$, and $X^\perp \subset I^\perp$, where $I = \{[q, x, q^\perp] \in B^{**} : x \in X\}$. Let $J_X = \{x \in X : x = q^\perp x q^\perp\}$. Then $X/J_X \cong I_X$ completely isometrically.

This uses the new 'Brown-type' C^*-algebraic results above, such as our proximinality of HSA's in $L + L^*$.

Corollary Let X be a closed subspace of a unital C^*-algebra B. Let $q \in B^{**}$ be a closed projection such that $qXq = (0)$ and such that if $\psi \in B^*$ annihilates X, then ψ annihilates $\{[q, x, q^\perp] \in B^{**} : x \in X\}$. Then any element of X whose off-diagonal corners in its 2×2 matrix form with respect to the projection q, have norm ≤ 1, has the same off-diagonal corners as another element of X whose norm is close to 1.

Define a projection p in A^{**} to be open 'relative to' A iff there exists a real positive net $a_t \in \mathrm{Ball}(A)$ such that $pa_tp = a_t$ and $a_t \to p$ in the weak* topology.

Question: Is there a relation to open-ness with respect to a containing C^*-algebra? ($\exists$ Hay's theorem?)

Using the above results, and some rather technical checking that the conditions in those results hold, one may then deduce:

Theorem Let A be a closed Jordan subalgebra of a C^*-algebra B. A projection p in $A^{\perp\perp}$ is open in B^{**} iff p is open 'relative to' A. Thus $D = \{a \in A : pap = a\}$ is a hereditary subalgebra of A with support projection p.

This, the Jordan variant of Hay's theorem, is the breakthrough needed to enable noncommutative topology for Jordan operator algebras. And it enables a hereditary subalgebra theory there. E.g.:

Corollary Let A be a closed Jordan subalgebra of a C^*-algebra B. There is a bijective correspondence between HSA's in A and HSA's in B with support projection in $A^{\perp\perp}$. This correspondence takes such a HSA in B to its intersection with A.

One idea in the proof of the last theorem: ...

... at some point in the proof we get a space Z such that $A/Z \cong qAq$ completely isometrically, and we need to know $Z^{\perp\perp} = \{\eta \in A^{**} : q\eta q = 0\}$. Inclusion $\subset$ is clear, so suppose BWOC that the inclusion is proper. One can show that $A^{**}/Z^{\perp\perp}$ is a unital operator space. However we find an element for which the B-Neal metric characterization of unital operator spaces (below) fails; a contradiction. $\square$

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Theorem (B-Neal) If X is an operator space, $u \in X$, then (X, e) is a unital operator space (and e is unitary) if and only if

$$\max\{\|e + i^k x\| : k = 0, 1, 2, 3\} \geq \sqrt{1 + \|x\|}, \qquad x \in M_n(X),\ n \in \mathbb{N}.$$

In fact this only needs x of small norm (i.e. in a neighborhood of 0), so is a 'local' characterization

Applications of this breakthrough: ...

Remarkably, everything in our earlier noncommutative topology now works in the Jordan case... but for the deep reasons we mentioned (i.e. ultimately relying on our breakthrough above) .

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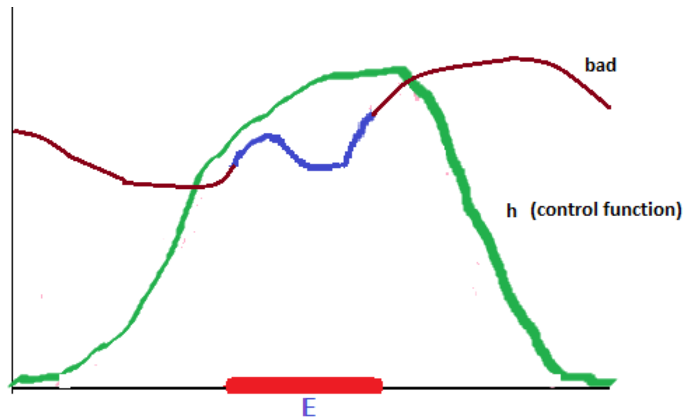
no time ... to give an idea of our results can simply flash up all the earlier slides we went through in the operator algebra case. I will just mention a couple:

E.g. the Urysohn lemmas (I mentioned three variants), the peak interpolation theorems, the algebraic calculus underlying topology. For example, Theorem (Noncommutative Glicksberg peak set theorem) Let A be a Jordan operator algebra in a C^*-algebra B. The compact projections in $\overline{A}^{w*}$ are precisely the decreasing limits (or infima) of peak projections. If A is separable, they are just the peak projections.

Theorem (Noncommutative Bishop type) Suppose that A is a Jordan operator algebra, a closed Jordan subalgebra of a C^*-algebra B. Suppose that q is a compact projection in A^{**}. If $b \in A$ with $bq = qb$, and $qb^*bq \leq qh$ for an invertible positive $h \in B$ which commutes with q, then there exists a real positive element $g \in A$ with $gq = qg = bq$, and $g^*g \leq h$.



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