Bi-Monotone Quantum Lévy Processes

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May 10th 2018
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Motivation

Scheme (works for tensor, Boolean, free, monotone)

Independence $\overset{CLT}{\rightsquigarrow}$ vacuum-distr. of Fock space operators

bi-freeness (Voiculescu 2014)

- free Fock space $\rightsquigarrow$ left & right free creation/annihilation
- bi-freeness: independence for *pairs* of operators
- Scheme works!

Aim of bi-monotone independence

- mon. Fock space $\rightsquigarrow$ mon. & anti-mon. creation/annihilation
- Find independence for pairs s.t. the scheme works!

Overview

Overview

Non-commutative probability: $*$-algebraic setting

Definition (non-commutative probability space)

pair $(\mathcal{A}, \Phi)$ with

- unital $*$-algebra $\mathcal{A}$
- state Φ on $\mathcal{A}$

Definition (random variable)

$*$-homomorphism $j\colon B \to \mathcal{A}$ (B is $*$-algebra)

- $\widetilde{B}$ unitization of B, $\widetilde{\jmath}$ unital extension of j
- $\Phi \circ \widetilde{\jmath}$ is called *distribution* of j
- selfadjoint $a \in \mathcal{A} \rightsquigarrow j_a\colon \mathbb{C}[x]_0 \to \mathcal{A}, x \mapsto a$
- distribution of $j_a \longleftrightarrow$ collection of *moments* $\big(\Phi(a^k)\big)_{k\in\mathbb{N}}$
- $*$-subalgebra $B \rightsquigarrow$ embedding $\iota\colon B \hookrightarrow \mathcal{A}$

Augmented algebras and unitization

Already for Boolean, monotone and anti-monotone product:

Take care with units!

Definition/Notation (augmented algebras)

- unital algebra with character (non-zero homomorphism to $\mathbb{C}$)
- every augmented algebra is the unitization of its augmentation ideal
- denote the augmentation ideal simply by B, the augmented algebra as $\widetilde{B} = \mathbb{C}1 \oplus B$.

Non-commutative independence

Fix product operation for states on unital (augmented) $*$-algebras

$$\bigtimes_i \tilde{B}_i' \ni (\varphi_i)_i \mapsto \bigodot_i \varphi_i \in \Big(\widetilde{\bigsqcup_i B_i}\Big)'$$

Definition ($\odot$-independence of random variables $j_i \colon B_i \to \mathcal{A}$)

$$\Phi \circ \widetilde{\bigsqcup_i j_i} = \bigodot_i (\Phi \circ \tilde{\jmath}_i)$$

joint distribution = product of marginals

Examples

Tensor, free, monotone, anti-monotone, Boolean

Non-commutative probability: $*$-algebraic setting

Definition (non-commutative probability space)

pair $(\mathcal{A}, \Phi)$ with

- unital $*$-algebra $\mathcal{A}$
- state Φ on $\mathcal{A}$

Definition (n-faced random variable)

$*$-homomorphism $j\colon B \to \mathcal{A}$
($B = B^{(1)} \sqcup \cdots \sqcup B^{(n)}$ is n-faced $*$-algebra)

- n-tuple of selfadjoint elements $a = (a^{(1)}, \ldots, a^{(n)}) \in \mathcal{A}^n \rightsquigarrow$ $j_a\colon \mathbb{C}\langle x_1, \ldots, x_n \rangle_0 \to \mathcal{A}, x_k \mapsto a^{(k)}$
- $a^\delta := a^{(\delta_1)} \cdots a^{(\delta_m)}$, $\delta \in [n]^*$
- distribution of $j_a \leftrightsquigarrow$ collection of *moments* $(\Phi(a^\delta))_{\delta \in [n]^*}$
- n-tuple of $*$-subalgebras $(B_1, \ldots, B_n) \rightsquigarrow \iota_1 \sqcup \cdots \sqcup \iota_n$

n-Independence

Fix product operation for states on unital (augmented) n-faced $*$-algebras

$$\bigtimes_i \widetilde{B}_i' \ni (\varphi_i)_i \mapsto \bigodot_i \varphi_i \in \Big(\widetilde{\bigsqcup_i B_i}\Big)'$$

Definition ($\odot$-independence of n-faced rv's $j_i\colon B_i \to \mathcal{A}$)

$$\Phi \circ \widetilde{\bigsqcup_i j_i} = \bigodot_i (\Phi \circ \tilde{\jmath}_i)$$

joint distribution = product of marginals

Examples

Bi-freeness, ??

Overview

Bi-monotone product I

Definition (bi-monotone product of pointed rep's)

For $(\pi_i\colon B_i \to L_{adj}(H_i), \Omega_i)$ pointed rep's on pre-HS define

$$(\pi_1 \bowtie \pi_2\colon B_1 \sqcup B_2 \to L_{adj}(H_1 \otimes H_2), \Omega_1 \otimes \Omega_2)$$

$$\pi_1 \bowtie \pi_2(b) := \begin{cases} \pi_1 \lhd \pi_2(b) & b \in B^\ell \\ \pi_1 \rhd \pi_2(b) & b \in B^r \end{cases} = \begin{cases} \pi_1(b) \otimes \mathrm{id} & b \in B_1^\ell \\ P_\Omega \otimes \pi_2(b) & b \in B_2^\ell \\ \pi_1(b) \otimes P_\Omega & b \in B_1^r \\ \mathrm{id} \otimes \pi_2(b) & b \in B_2^r \end{cases}$$

Definition (bi-monotone product of states on $\widetilde{B}_i$)

$$\varphi_1 \bowtie \varphi_2(b) := \langle \Omega, \pi_{\varphi_1} \bowtie \pi_{\varphi_2}(b)\Omega \rangle \quad \text{for all } b \in B_1 \sqcup B_2$$

whenever $\varphi_i = \langle \Omega_i, \pi_i(\cdot)\Omega_i \rangle$.

Bi-monotone product II

Theorem

The bi-monotone product of states (linear functionals) is

- unital in the sense that: $1 \bowtie \varphi = \varphi = \varphi \bowtie 1$
- associative
- universal in the sense that (for $*$-hom's $j_i \colon B_i \to A_i$)

$$(\varphi_1 \bowtie \varphi_2) \circ (\widetilde{j_1 \sqcup j_2}) = (\varphi_1 \circ \tilde{\jmath}_1) \bowtie (\varphi_2 \circ \tilde{\jmath}_2)$$

In short: $\bowtie$ is a positive 2-1-uau-product in the sense of Manzel & Schürmann (2017)

Overview

Bi-monotone partitions

Definition (bi-partition, ordered bi-partition)

Bi-partition of a set X:
set partition π of X together with a map $\delta\colon X \to \{\ell, r\}$

Ordered bi-partition of X:
bi-partition with total order between blocks

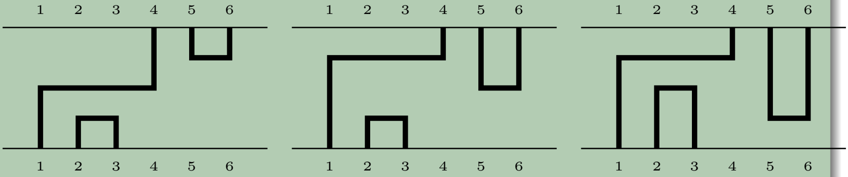
Definition (bi-monotone partition)

Bi-monotone partition of a totally ordered set X:
Ordered bi-partition π s.t. for elements $a < b < c$ of X and blocks $V, W \in \pi$:

$$a, c \in V \ (\textit{outer block}) \ \& \ b \in W \ (\textit{inner block})$$
$$\Downarrow$$
$$V \leqslant W \text{ if } \delta(b) = r \text{ and } V \geqslant W \text{ if } \delta(b) = \ell$$

Visualization

Three bi-monotone partitions



- horizontal lines indicate blocks of π
- height indicates block order
- vertical lines indicate δ ($\ell \hat{=}$ *downwards*, $r \hat{=}$ *upwards*)

Overview

Calculating Mixed moments

Lemma

- $A_1, \dots, A_n$ bi-monotonely independent
- $a_i \in A_{\varepsilon_i}^{\delta_i}$
- (π, δ) ordered bi-partition corresponding to $\big((\varepsilon_1, \dots, \varepsilon_m), (\delta_1, \dots, \delta_m)\big)$
- ρ partition obtained from π by dividing blocks at each crossing

$$\Longrightarrow \qquad \Phi(a_1 \cdots a_m) = \prod_{V \in \rho} \Phi(a_V)$$

Idea of proof

- V-line at j crossing H-line at $h \rightsquigarrow \pi(a_j)$ has P_Ω at leg h
- V-line at j not crossing H-line at $h \rightsquigarrow \pi(a_j)$ has id at leg h
- V-line at j touches H-line at $h \rightsquigarrow \pi(a_j)$ has $\pi_{\varepsilon_j}(a_j)$ at leg h

Bi-monotone CLT

Theorem

$(b_i)_{i\in\mathbb{N}}$, $b_i = (b_i^{(\ell)}, b_i^{(r)})$ sequence of pairs in ncps $\mathcal{A}$ s.t.

- $\Phi(b_i^{(j)}) = 0$ for all i, j (centered)
- $(b_i)_{i\in\mathbb{N}}$ bi-monotonely independent
- b_i identically distributed
- $\Phi(b_i^{(p)} b_i^{(q)}) = 1$ for $p, q \in \{\ell, r\}$

Then for $s_N := \frac{\sum_{n=1}^N b_n}{N^{1/2}}$

$$\lim_{N\to\infty} \Phi(s_N^\delta) = \frac{\#\mathrm{PP}_{\bowtie}(\delta)}{k!} \quad \text{for all } \delta \in \{\ell, r\}^{2k}.$$

Tools for proof.

Moment Lemma + CLT of Accardi, Hashimoto, Obata ('98) □

Overview

Monotone Fock Space

Definition (monotone Fock space, creation/annihilation)

- $\Gamma_{\mathrm{m}}([a,b)) := L^2(\Delta_{[a,b)})$ with

$$\Delta_{[a,b)} = \{(t_1,\ldots,t_n) \in \mathbb{R}^* \mid a \leqslant t_1 < \cdots < t_n < b\}$$

- $\ell^*(f)(g)(t_1,\ldots,t_n) = f(t_1)g(t_2,\ldots,t_n)$
- $\ell(f)(g)(t_1,\ldots,t_n) = \int_a^{t_1} \overline{f(\tau)}g(\tau,t_1,\ldots,t_n)\mathrm{d}\tau$
- $r^*(f)(g)(t_1,\ldots,t_n) = g(t_1,\ldots,t_{n-1})f(t_n)$
- $r(f)(g)(t_1,\ldots,t_n) = \int_{t_n}^{b} g(t_1,\ldots,t_n,\tau)\overline{f(\tau)}\mathrm{d}\tau$

Observation

- $\Gamma_{\mathrm{m}}(\mathrm{I}_1 \cup \mathrm{I}_2) \cong \Gamma_{\mathrm{m}}(\mathrm{I}_1) \otimes \Gamma_{\mathrm{m}}(\mathrm{I}_2)$ for $\mathrm{I}_1 < \mathrm{I}_2$
- $\ell^*(f) \cong \ell^*(f) \otimes \mathrm{id}$ for $\operatorname{supp}(f) \subset I_1$
- $\ell^*(f) \cong P_\Omega \otimes \ell^*(f)$ for $\operatorname{supp}(f) \subset I_2$

Bi-monotone Brownian motion

Operator processes

- $b_t^{\bowtie} = \big(\ell^*(1_{[0,t)}) + \ell(1_{[0,t)}), r^*(1_{[0,t)}) + r(1_{[0,t)})\big)$
- $b_t^{\lhd} + b_t^{\rhd} = \ell^*(1_{[0,t)}) + \ell(1_{[0,t)}) + r^*(1_{[0,t)}) + r(1_{[0,t)})$

First process: 2-dimensional, bi-monotonely independent increments! For $\delta \in \{\ell, r\}^n$

$$\Phi\left(b_t^{\bowtie\,\delta}\right) = \begin{cases} 0 & n \text{ odd} \\ \frac{\#\mathrm{PP}_{\bowtie}(\delta)}{k!} t^k & n = 2k \text{ even} \end{cases}$$

Second process: 1-dimensional, moments computed out of first process! For $n \in \mathbb{N}$

$$\Phi\big((b_t^{\lhd} + b_t^{\rhd})^n\big) = \begin{cases} 0 & n \text{ odd} \\ \frac{\#\mathrm{PP}_{\bowtie}(n)}{k!} t^k & n = 2k \text{ even} \end{cases}$$

Thank you!