Noncommutative maximal ergodic inequalities for some group actions

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Recall: Birkhoff ergodic theorem

Let (Ω, P) be a probability space and let $T : \Omega \to \Omega$ be a measure-preserving transformation. Define

$$A_N f = \frac{1}{N} \sum_{k=0}^{N-1} f \circ T^k, \quad f \in L_p(\Omega). (1 \leq p \leq \infty)$$

Theorem (Birkhoff) (1) $(A_N)_{N \geq 1}$ satisfies the weak type $(1,1)$ and strong type (p,p) inequalities, i.e.

$$\sup_{\lambda > 0} \lambda P(\{\sup_N |A_N f| \geq \lambda\}) \lesssim \|f\|_1, \quad f \in L_1(\Omega),$$

$$\|\sup_N |A_N f|\|_p \lesssim \|f\|_p, \quad f \in L_p(\Omega), \ p > 1.$$

(2) Let I be the σ-subalgebra of T-invariant sets. For all $f \in L_p(\Omega)$,

$$\lim_N A_N f = \mathbb{E}(f|I) \quad a.e.$$

Generalizations to the noncommutative setting

Aim: analogues for actions on von Neumann algebras and noncommutative L_p-spaces

- Lance (76') introduced an analogue for the noncommutative setting of a.e. convergence:
  Let $\mathcal{M}$ be a vNA equipped with a normal faithful semifinite trace φ. $(x_n)_{n\geq 1} \subset L_p(\mathcal{M})$ is said to converge almost uniformly (a.u. in short) to x if for every $\varepsilon > 0$ there is a projection $e \in \mathcal{M}$ such that

$$\varphi(e^{\perp}) < \varepsilon \quad \text{and} \quad \lim_{n\to\infty} \|(x_n - x)e\|_{\infty} = 0.$$

- Egorov: in the setting of classical probability spaces,

$$\text{a.u. convergence} \Leftrightarrow \text{ a.e. convergence.}$$

- Pisier, Junge: noncommutative maximal norms via $L_p(\mathcal{M}; \ell_\infty)$.

Junge-Xu's noncommutative ergodic theorem

Let $T \in Aut(\mathcal{M})$ s.t. $\varphi = \varphi \circ T$ (or a general Dunford-Schwartz operator). Let

$$A_N = \frac{1}{N} \sum_{k=0}^{N-1} T^k, \quad N \geq 1.$$

Theorem (Lance 76; Yeadon 77; Junge-Xu 07)

(1) weak type $(1,1)$ inequality: $\forall x \in L_1^+(\mathcal{M}), \lambda > 0, \exists\, e \in Proj(\mathcal{M})$ s.t.

$$\tau(1-e) \lesssim \frac{\|x\|_1}{\lambda}, \quad eA_N(x)e \leq \lambda e, N \geq 1.$$

strong type (p,p) inequalities $(p > 1)$: $\forall x \in L_p^+(\mathcal{M}), \exists\, a \in L_p^+(\mathcal{M})$

$$A_N(x) \leq a, \quad \|a\|_p \lesssim \|x\|_p, \quad N \geq 1.$$

(2) For $x \in L_p(\mathcal{M})$ $(1 \leq p < \infty)$

$$\lim_N A_N x = Px \ \text{ a.u.},$$

where P is the projection onto $Fix(T) \subset L_p(\mathcal{M})$.

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- Problem: more general actions beyond Junge-Xu's setting?

Recall: Calderon ergodic theorem for group actions

G: locally compact group, m: Haar measure, d: an invariant metric

Assume $B_r := \{g \in G : d(g, e) \leq r\}$ satisfy

- doubling condition: $m(B_{2r}) \leq Cm(B_r), \quad r > 0.$
- asymptotically invariance (or Følner condition): for every $g \in G$,

$$\lim_{r \to \infty} \frac{m((B_r g) \triangle B_r)}{m(B_r)} = 0.$$

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Theorem Assume that G acts on a probability space (Ω, P) by measure-preserving transformations $(T_g)_{g \in G}$. Let $\mathcal{I}$ be the σ-subalgebra of G-invariant sets. Then

$$A_r f := \frac{1}{m(B_r)} \int_{B_r} (f \circ T_g) dm(g), \quad f \in L_p(\Omega)$$

satisfies the weak type $(1, 1)$ and strong type (p, p) inequalities, and for every $f \in L_p(\Omega)$, we have

$$\lim_{r \to \infty} A_r f = \mathbb{E}(f | \mathcal{I}) \quad \text{a.e.}$$

Main result

G: locally compact group, m: Haar measure, d: an invariant metric
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- doubling condition: $m(B_{2r}) \leq Cm(B_r), \quad r > 0$.
- asymptotically invariance (or Følner condition): for every $g \in G$,

$$\lim_{r\to\infty} \frac{m((B_r g) \triangle B_r)}{m(B_r)} = 0.$$

Theorem (Hong-Liao-W.) Assume that $\alpha : G \to Aut(\mathcal{M})$ is a continuous homomorphism such that $\varphi = \varphi \circ \alpha_g$ for all $g \in G$. Then

$$A_r := \frac{1}{m(B_r)} \int_{B_r} \alpha_g dm(g), \quad r > 0,$$

satisfies the weak type $(1, 1)$ and strong type (p, p) inequalities, and for $x \in L_p(\mathcal{M})$ $(1 \leq p < \infty)$,

$$A_r x \to Px \ \text{ a.u.}, \quad \text{as } r \to \infty,$$

where P is the projection onto $\cap_{g\in G} Fix(\alpha_g) \subset L_p(\mathcal{M})$.

Ingredients: group-theoretic tools

We may use the group theory to prove the theorem for partial cases: G discrete group of polynomial growth generated by a finite subset $V = V^{-1}$, d word metric, $p > 1$:

- Gromov, Bass, Wolf,...: G contains a finitely generated nilpotent subgroup of finite index. We may view

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  but with mild noncommutative relations.
- Following this spirit, a careful study on the structure of G yields that

  $$V^n \hookrightarrow F \times [0, cn^{j_1}] \times \cdots \times [0, cn^{j_{d(G)}}] \subset F \times \mathbb{Z}^{d(G)}.$$
- Then it is easy to deduce the maximal inequalities for $A_n = \frac{1}{|V^n|} \sum_{g \in V^n} \alpha_g$ from Junge-Xu's maximal inequalities.

Ingredients: noncomm. Calderon transference principle

For the general cases, we may first reduce the problem to translation actions. Assume G is amenable.

- Calderon: maximal inequalities for translation action $G \times G \to G$
  $\Rightarrow$ maximal inequalities for general action $G \times (\Omega, P) \to (\Omega, P)$.

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Theorem (Hong-Liao-W.) Let $(\mu_n)_{n\geq 1}$ be a sequence of Radon probability measures on G. If

$$A'_n f = \int_G f(\cdot h) d\mu_n(h), \quad f \in L_p(G; L_p(\mathcal{M})), n \geq 1,$$

satisfies weak/strong (p, p) inequalities, then for a trace preserving action α on $\mathcal{M}$,

$$A_n x = \int_G \alpha_g x d\mu_n(g), \quad x \in L_p(\mathcal{M}), n \geq 1$$

satisfies the maximal inequalities of the same type.

Remark: For the strong type (p, p) inequalities, it suffices to assume that α is an action on $L_p(\mathcal{M})$ (for a fixed p).

Ingredients: probabilistic tools

- Mei 07: Classical $BMO(\mathbb{R}^n)$ is the intersection of $n+1$ copies of dyadic-BMO. Doob's martingale inequality implies the Hardy-Littlewood maximal inequality on $\mathbb{R}^n$.
- Naor-Tao 10, Hytonen-Kaimera 12: filtrations of random partitions of doubling metric spaces

Proposition Let (X, d, μ) be a doubling metric measure spaces. For some finite N and for each $1 \leq i \leq N$, one may construct a filtration $\{\mathcal{I}_n^{(i)} : n \in \mathbb{Z}\}$ of σ-subalgebras on X. For each r and $x \in X$, $\exists n(r), i$ s.t. for $f \in L_p(X; L_p(\mathcal{M}))_+$,

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} f \leq c\big(\mathbb{E}(\cdot|\mathcal{I}_{n(r)}^{(i)}) \otimes Id_{L_p(\mathcal{M})}\big)(f)(x).$$

Ingredients: Operator-valued Hardy-Littlewood maximal inequalities on doubling spaces

So by the noncommutative Doob martingale inequality (Cuculescu 71,Junge 02, Junge-Xu 07), we obtain

Corollary Let (X, d, μ) be a doubling metric measure spaces. Then

$$M_r f(x) := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f, \quad x \in X, r > 0, f \in L_p(X; L_p(\mathcal{M}))$$

satisfies the weak $(1,1)$ and strong (p,p) inequalities.

In particular, taking $(X, d, \mu) = (G, d, m)$ the group with doubling conditions, we obtain the desired maximal inequalities on G.

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Remark For $p > 1$, our previous transference methods also work for general positive actions on L_p-spaces (not necessarily arising from $Aut(\mathcal{M})$). But for $p = 1$ these do not hold. However in the case of word metrics, we may provide an alternative approach. ▷▷

Ingredients: Domination via Markov operators (case of word metrics)

G: group of polynomial growth wrt a symmetric cpt generating set V.
d: word metric.
Gaussian estimate in random walk theory: Let f be a "nice" density function with $supp(f) \supset V$. $\exists c > 0$, $\forall k$,

$$f^{\star k}(g) \geq \frac{ce^{-d(e,g)^2/k}}{m(B_{\sqrt{k}})}, \quad g \in B_k.$$

Proposition Let $T = \frac{1}{m(V)} \int_V \alpha_g dm(g)$. Then there exists a constant c such that

$$\frac{1}{m(V^n)} \int_{V^n} \alpha_g x dm(g) \leq \frac{c}{n^2} \sum_{k=1}^{2n^2} T^k x, \quad x \in L_p^+(\mathcal{M}).$$

Comments on individual ergodic theorems

- Junge,Xu,...: One may deduce the a.u. convergence from maximal ergodic inequalities for Dunford-Schwartz operators (which relies on the boundedness on $L_1 + L_\infty$).
- Hong-Liao-W: The same holds for general power bounded operators on a L_p-space for a fixed p (our new argument does not require boundedness on $L_1 + L_\infty$).
- Therefore the main theorem can be generalized for any fixed $1 < p < \infty$ and any action $\alpha : G \to B(L_p(\mathcal{M}))$ satisfying:
  - Continuity
  - Uniform boundedness: $\sup_{g\in G} \|\alpha_g : L_p(\mathcal{M}) \to L_p(\mathcal{M})\| < \infty$.
  - Positivity : for all $g \in G$, $\alpha_g x \geq 0$ if $x \geq 0$ in $L_p(\mathcal{M})$.

Comments on individual ergodic theorems

In particular, we have

Corollary Fix $1 < p < \infty$. Let $T : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ be a positive invertible operator with positive inverse such that $\sup_{n\in\mathbb{Z}} \|T^n\| < \infty$. Then

$$A_n = \frac{1}{2n+1} \sum_{k=-n}^{n} T^k, \quad n \in \mathbb{N}$$

satisfies strong (p,p) inequalities. For all $x \in L_p(\mathcal{M})$, $(A_n x)_{n\geq 1}$ converges b.a.u if $1 < p < 2$, and converges a.u if $p \geq 2$.

Thank you very much!