

On some applications of quantum Markov  
semigroups  
and related problems  
(based on joint work with Ami Viselter and Martijn Caspers)

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# Outline

My aim is to show how one can encode certain properties of an object (a von Neumann algebra, a quantum group) using **continuous cp evolutions, aka quantum Markov semigroups**, and discuss how these can be constructed either directly, or via their generators.

## Haagerup property

A discrete group  $G$  has the **Haagerup property** if it admits a mixing  $(C_0-)$  unitary representation which weakly contains the trivial representation.

In other words, we have  $\pi : G \rightarrow B(H)$  such that

$$\forall_{\xi, \eta \in H} \langle \xi, \pi(\cdot)\eta \rangle \in C_0(G),$$

but for some net of unit vectors  $(\xi_i)_{i \in I}$  such that

$$\forall_{g \in G} \pi(g)(\xi_i) - \xi_i \xrightarrow{i \in I} 0$$

In other words  $G$  admits a net of positive definite functions  $\phi_i$  which belong to  $C_0(G)$ , and yet converge pointwise to 1.

If  $G$  is countable, we can actually arrange  $I = \mathbb{R}_+$  and  $\phi_t = \exp(-t\psi)$ .

# Haagerup property via positive definite functions

## Theorem (Schönberg correspondence)

A symmetric function  $\psi : G \rightarrow \mathbb{R}$  ( $\psi(g) = \psi(g^{-1})$ ,  $g \in G$ ) is conditionally positive definite if and only if for each  $t \geq 0$  the function  $\varphi_t := e^{t\psi}$  is positive definite.

## Theorem

A countable group  $G$  has Haagerup property if and only if it admits a proper symmetric conditionally positive definite function.

Considering the associated Herz-Schur multipliers acting on  $VN(G)$ , i.e. maps of the form

$$M_t(\lambda_g) = \phi_t(g)\lambda_g, \quad g \in G,$$

we get the next reformulation.

# Haagerup property via von Neumann algebraic semigroup approximation

## Theorem

A countable group  $G$  has Haagerup property if and only if  $VN(G)$  admits a symmetric quantum Markov semigroup consisting of  $L^2$ -compact Herz-Schur multipliers.

We build the semigroup, first constructing  $\psi$  – which can be viewed as a ‘generating functional for the semigroup’.

# Quantum Markov semigroups

$M$  - von Neumann algebra, with a fixed normal semifinite faithful weight  $\phi$

## Definition

An operator  $T : M \rightarrow M$  will be called Markov if it is a positive contraction and in addition we also have the condition

$$\phi \circ T \leq \phi.$$

## Quantum Markov semigroups – $L^p$ -versions

**Tracial** case –  $L^p$ -spaces are certain completions of  $M$ , with the  $\|x\|_p := (\tau(|x|)^p)^{\frac{1}{p}}$ .

**Non-tracial state** case –  $L^p$ -spaces are either interpolation spaces (Araki, Kosaki, Izumi) or certain concrete spaces of operators (Hilsum, Haagerup).

**Non-tracial weight** case:

$L^p(M, \phi)$  – Haagerup  $L^p$ -space. We consider symmetric embeddings  $\iota_p : M^p \rightarrow L^p(M)$  : these are informally defined as

$$\iota_p(x) = D^{\frac{1}{2p}} x D^{\frac{1}{2p}}, \quad x \in M^{(p)}.$$

Here  $M^{(p)} \subset M$  is ‘the set of  $p$ -integrable elements’ and  $D$  can be thought of as ‘the density matrix of the weight’ – and formally is the unbounded generator of the implementing unitary group of the modular automorphism group of  $\phi$  in the core of  $M$ .

## Quantum Markov semigroups – $L^p$ -versions continued

Given a map  $T : M \rightarrow M$  we will say it is **KMS-symmetric** if its *KMS implementation*, the map  $T^{(2)} : \iota_2(M^{(2)}) \rightarrow L^2(M, \phi)$

$$T^{(2)}(\iota_2(x)) = \iota_2(Tx)$$

extends to a *bounded self-adjoint* operator.

Note that we require in particular that  $T^{(2)}(\iota_2(M^{(2)})) \subset \iota_2(M^{(2)})$ .

### Definition

A (quantum) KMS-symmetric Markov semigroup is a weak\*-continuous semigroup  $(T_t)_{t \geq 0^+}$  of normal KMS-symmetric Markov maps on  $M$ .

KMS-symmetry plus being a positive contraction itself yields the weight inequality. A family as above is **completely Markov**, if the same properties hold for  $P_t \otimes \text{id}_{M_n}$  for all  $n \in \mathbb{N}$ .



## Quantum Dirichlet forms

Definition (F.Cipriani, S.Goldstein+M.Lindsay)

A quantum Dirichlet form for  $(M, \phi)$  is a densely defined closed quadratic form on  $L^2(M)$  which is real (in a suitable sense) and satisfies the condition

$$Q(P_{\wedge}x) \leq Q(x), \quad x \in \text{Dom}(Q_{\mathbb{R}})$$

where  $P_{\wedge}$  is the orthogonal projection from  $L^2(M)_{\mathbb{R}}$  onto the 'interval'  $[0, D^{\frac{1}{2}}]$ .

Theorem (F.Cipriani, S.Goldstein+M.Lindsay, A.Viselter+AS)

There is a 1-1 correspondence between **completely Markov KMS-symmetric semigroups** and **quantum completely Dirichlet forms**.

# Haagerup property for von Neumann algebras

The following is one of several equivalent versions.

**Definition (M.Caspers + AS, R.Okayasu + R.Tomatsu)**

Let  $(M, \varphi)$  be a von Neumann algebra with a faithful normal semifinite weight. We say that  $(M, \varphi)$  has the Haagerup property if there exists a net of normal completely positive,  $\varphi$ -reducing maps  $(\Phi_i)_{i \in \mathcal{I}}$  on  $M$  such that the KMS-induced maps  $T_i$  on  $L^2(M, \varphi)$  are compact and the net  $(T_i)_{i \in \mathcal{I}}$  converges to  $I_{L^2(M, \varphi)}$  strongly.

**Theorem (C+S, O+T)**

The property above does not depend on the choice of the weight.

## Haagerup property via Markov semigroups

Let  $(M, \varphi)$  be a von Neumann algebra equipped with a faithful normal state.

### Definition

A quantum Markov semigroup  $\{\Phi_t : t \geq 0\}$  on  $(M, \varphi)$  is immediately  $L^2$ -compact if each of the maps  $\Phi_t^{(2)}$  with  $t > 0$  is compact.

The next result was inspired by the theorem for finite von Neumann algebras due to Jolissaint and Martin.

### Theorem (M.Caspers + AS)

The following are equivalent:

- ❶  $(M, \varphi)$  has the Haagerup property;
- ❷ there exists an immediately  $L^2$ -compact KMS-symmetric Markov semigroup  $\{\Phi_t : t \geq 0\}$  on  $M$ .

## Haagerup property via Dirichlet forms

### Theorem (M.Caspers + AS)

The following are equivalent:

- i  $(M, \varphi)$  has the Haagerup property;
- ii  $L^2(M, \varphi)$  admits an orthonormal basis  $(e_n)_{n \in \mathbb{N}}$  and a non-decreasing sequence of non-negative numbers  $(\lambda_n)_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow \infty} \lambda_n = +\infty$  and the prescription

$$Q(\xi) = \sum_{n=1}^{\infty} \lambda_n |\langle e_n, \xi \rangle|^2, \quad \xi \in \text{Dom } Q,$$

where  $\text{Dom } Q = \{\xi \in H_\varphi : \sum_{n=1}^{\infty} \lambda_n |\langle e_n, \xi \rangle|^2 < \infty\}$ , defines a completely Dirichlet form.

## LCQGs

$\mathbb{G}$  – locally compact quantum group à la Kustermans-Vaes

$L^\infty(\mathbb{G})$  – the von Neumann algebra, together with the *coproduct* (carrying all the information about  $\mathbb{G}$ )

$$\Delta : L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$$

and a canonical *right Haar weight*  $\phi$

$C_0(\mathbb{G})$  – the corresponding (reduced)  $C^*$ -object

$C_0^u(\mathbb{G})$  – the universal version of  $C_0(\mathbb{G})$ ,

$L^2(\mathbb{G})$  – the GNS Hilbert space of the right invariant Haar weight  $\phi$  on  $L^\infty(\mathbb{G})$

$$\begin{aligned} C_0(\mathbb{G}) &\subset L^\infty(\mathbb{G}) \\ L^2(\mathbb{G}) &\approx L^2(L^\infty(\mathbb{G}), \phi) \end{aligned}$$

## Dual groups

Each LCQG  $\mathbb{G}$  admits the dual LCQG  $\widehat{\mathbb{G}}$ .

$L^\infty(\widehat{\mathbb{G}})$ ,  $C_0(\widehat{\mathbb{G}})$  – subalgebras of  $B(L^2(\mathbb{G}))$

In particular for  $G$  – locally compact group

$$L^\infty(\widehat{G}) = VN(G), \quad C_0(\widehat{G}) = C_r^*(G), \quad C_0^u(\widehat{G}) = C^*(G)$$

## Simplifications in the compact case

### Definition

$\mathbb{G}$  is said to be **compact** if  $C_0(\mathbb{G})$  is unital (so written as  $C(\mathbb{G})$ ), equivalently, the weight  $\phi$  is a state.

Any compact quantum group can be described purely algebraically via the Hopf  $*$ -algebra  $\text{Pol}(\mathbb{G}) \subset C(\mathbb{G})$ , with the counit  $\epsilon$ .

# Convolution semigroups of states on compact quantum groups

A family  $(\mu_t)_{t \geq 0_+}$  of states on  $\text{Pol}(\mathbb{G})$  is called a **convolution semigroup of states** if

- ❶  $\mu_{t+s} = \mu_t \star \mu_s := (\mu_t \otimes \mu_s) \circ \Delta, \quad t, s \geq 0;$
- ❷  $\mu_t(a) \xrightarrow{t \rightarrow 0^+} \mu_0(a) := \epsilon(a), \quad a \in \text{Pol}(\mathbb{G}).$

Such convolution semigroups admit generating functionals:

$$\gamma(a) = \lim_{t \rightarrow 0^+} \frac{\mu_t(a) - \epsilon(a)}{t}, \quad a \in \text{Pol}(\mathbb{G}).$$

We associate to it a convolution semigroup of operators  $(R_{\mu_t})_{t \geq 0_+}$  on  $\text{Pol}(\mathbb{G})$ :

$$R_{\mu_t} := (\mu_t \otimes \text{id}) \circ \Delta$$

These extend to operators on  $L^\infty(\mathbb{G})$  which form a Markov semigroup. The corresponding Dirichlet forms contain  $\text{Pol}(\mathbb{G})$  in the domain and can be characterised/studied in the purely algebraic manner (see Cipriani, Franz, Kula).



## Convolution semigroups of states revisited

$\mathbb{G}$  – locally compact quantum group

A family  $(\mu_t)_{t \geq 0_+}$  of states on  $C_0^u(\mathbb{G})$  is called a **convolution semigroup of states** if

$$\text{i} \quad \mu_{t+s} = \mu_t \star \mu_s := (\mu_t \otimes \mu_s) \circ \Delta, \quad t, s \geq 0;$$

$$\text{ii} \quad \mu_t(a) \xrightarrow{t \rightarrow 0^+} \mu_0(a) := \epsilon(a), \quad a \in C_0^u(\mathbb{G}).$$

We no longer have the 'algebraic domain' such as  $\text{Pol}(\mathbb{G})$ . Generating functionals are densely defined, but that is all we know a priori (however: very recently we showed that the domain of the generating functional always contains a dense  $*$ -subalgebra).

## Convolution operators – revisited once again

### Theorem (A.Viselter + AS)

Let  $\mu \in S(C_0^u(\mathbb{G}))$ . The associated operator  $R_\mu : L^\infty(\mathbb{G}) \rightarrow L^\infty(\mathbb{G})$  (which can be informally thought of as the map  $(\mu \otimes \text{id}) \circ \Delta$ ) is unital, completely positive,  $\phi$ -preserving. Its KMS implementation (acting on  $L^2(\mathbb{G})$ ) is always bounded and belongs to  $L^\infty(\hat{\mathbb{G}})$ . The map  $R_\mu$  is KMS-symmetric iff  $\mu = \mu \circ R^u$ .

Here  $R^u$  is the **universal unitary antipode** of  $C_0^u(\mathbb{G})$ .

# Convolution semigroups on locally compact quantum groups – generators

## Theorem (A. Viselter + AS)

Let  $\mathbb{G}$  be a locally compact quantum group. There exist 1 – 1 correspondences between:

- i  $w^*$ -continuous convolution semigroups  $(\mu_t)_{t \geq 0}$  of  $R^u$ -invariant states of  $C_0^u(\mathbb{G})$ ;
- ii  $C_0^*$ -semigroups  $(T_t)_{t \geq 0}$  of normal, unital, completely positive maps on  $L^\infty(\mathbb{G})$  that are KMS-symmetric with respect to  $\phi$  and satisfy  $\Delta \circ T_t = (T_t \otimes \text{id}) \circ \Delta$  for every  $t \geq 0$ ;
- iii completely Dirichlet forms  $Q$  on  $L^2(\mathbb{G})$  with respect to  $\phi$  that are invariant under  $\mathcal{U}(L^\infty(\hat{\mathbb{G}})')$  (modulo multiplication of forms by a positive number).

## Applications to geometric properties of quantum groups

### Theorem (D.Kyed, Viselter + AS)

Let  $\mathbb{G}$  be a second countable locally compact quantum group. Then  $\hat{\mathbb{G}}$  has Property (T) of Kazhdan if and only if every convolution semigroup of  $R^u$ -invariant states on  $C_0^u(\mathbb{G})$  has a bounded generator.

### Theorem (M.Daws+P.Fima+S.White+AS, A.Viselter+AS)

Let  $\mathbb{G}$  be a second countable locally compact quantum group. Then  $\hat{\mathbb{G}}$  has the Haagerup property if and only if there exists a convolution semigroup of  $R^u$ -invariant states on  $C_0^u(\mathbb{G})$  such that the  $L^2$ -implementations of the associated convolution operators, acting on  $L^2(\mathbb{G})$ , in fact belong to  $C_0(\hat{\mathbb{G}})$ .

## Ongoing work

In the compact case every ‘generating functional’, i.e. a hermitian, conditionally positive functional on the algebra  $\text{Pol}(\mathbb{G})$ , vanishing at 1, indeed generates a convolution semigroup of states.

Suppose we have a dense unital  $*$ -subalgebra  $A$  in the (unitization) of  $C_0^u(\mathbb{G})$ , and a functional  $\gamma : A \rightarrow \mathbb{C}$  with the properties as above – what to assume about  $A$  to guarantee that  $L$  ‘generates’ a convolution semigroup of states.

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