Invariant subspaces for $H^2$ spaces of $\sigma$-finite algebras

L E Labuschagne

School of Mathematical and Statistical Sciences
North-West University (Potchefstroom Campus)

Oberwolfach, May 2018
Outline

1. The classical roots
2. Mildly noncommutative precepts
3. Wildly noncommutative precepts
4. A very general Beurling Theorem
Invariant subspaces for $H^2$

The classical roots

$H^p$ spaces of the disc

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and $\mathbb{T}$ the unit circle.

**Definition**

$f : \mathbb{D} \rightarrow \mathbb{C}$ belongs to $H^\infty(\mathbb{D})$ iff $f$ is analytic and bounded on $\mathbb{D}$.

The space $H^\infty(\mathbb{D})$ may be realised as a subspace of $L^\infty(\mathbb{T})$ by the following process:

- By taking radial limits every $f \in H^\infty$ defines a corresponding function $\tilde{f}$ on $\mathbb{T}$;
- By an extension of the Cauchy Integration formulae, $f$ may similarly be recovered from $\tilde{f}$;
- $\sup_{z \in \mathbb{D}} |f(z)| = \text{ess sup}_{z \in \mathbb{T}} |\tilde{f}(z)|$.

$H^p(\mathbb{T})$ is then simply the closure in $L^p(\mathbb{T})$ of $H^\infty(\mathbb{T})$. 
Invariant subspaces for $H^2$

The classical roots

$H^p$ spaces of the disc

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and $\mathbb{T}$ the unit circle.

**Definition**

$f : \mathbb{D} \to \mathbb{C}$ belongs to $H^\infty(\mathbb{D})$ iff $f$ is analytic and bounded on $\mathbb{D}$.

The space $H^\infty(\mathbb{D})$ may be realised as a subspace of $L^\infty(\mathbb{T})$ by the following process:

- By taking radial limits every $f \in H^\infty$ defines a corresponding function $\tilde{f}$ on $\mathbb{T}$;
- By an extension of the Cauchy Integration formulae, $f$ may similarly be recovered from $\tilde{f}$;
- $\sup_{z \in \mathbb{D}} |f(z)| = \text{ess sup}_{z \in \mathbb{T}} |\tilde{f}(z)|$.

$H^p(\mathbb{T})$ is then simply the closure in $L^p(\mathbb{T})$ of $H^\infty(\mathbb{T})$. 
**Invariant subspaces for $H^2$**

**The classical roots**

**$H^p$ spaces of the disc**

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and $\mathbb{T}$ the unit circle.

**Definition**

$f : \mathbb{D} \to \mathbb{C}$ belongs to $H^\infty(\mathbb{D})$ iff $f$ is analytic and bounded on $\mathbb{D}$.

The space $H^\infty(\mathbb{D})$ may be realised as a subspace of $L^\infty(\mathbb{T})$ by the following process:

- By taking radial limits every $f \in H^\infty$ defines a corresponding function $\tilde{f}$ on $\mathbb{T}$;
- By an extension of the Cauchy Integration formulae, $f$ may similarly be recovered from $\tilde{f}$;
- $\sup_{z \in \mathbb{D}} |f(z)| = \text{ess sup}_{z \in \mathbb{T}} |\tilde{f}(z)|$.

$H^p(\mathbb{T})$ is then simply the closure in $L^p(\mathbb{T})$ of $H^\infty(\mathbb{T})$. 
Invariant subspaces for $H^2$

The classical roots

$H^p$ spaces of the disc

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and $\mathbb{T}$ the unit circle.

**Definition**

$f : \mathbb{D} \to \mathbb{C}$ belongs to $H^\infty(\mathbb{D})$ iff $f$ is analytic and bounded on $\mathbb{D}$.

The space $H^\infty(\mathbb{D})$ may be realised as a subspace of $L^\infty(\mathbb{T})$ by the following process:

- By taking radial limits every $f \in H^\infty$ defines a corresponding function $\tilde{f}$ on $\mathbb{T}$;
- By an extension of the Cauchy Integration formulae, $f$ may similarly be recovered from $\tilde{f}$;
- $\sup_{z \in \mathbb{D}} |f(z)| = \text{ess sup}_{z \in \mathbb{T}} |\tilde{f}(z)|$.

$H^p(\mathbb{T})$ is then simply the closure in $L^p(\mathbb{T})$ of $H^\infty(\mathbb{T})$. 
Invariant subspaces for $H^2$

The classical roots

$H^p$ spaces of the disc

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and $\mathbb{T}$ the unit circle.

**Definition**

$f : \mathbb{D} \to \mathbb{C}$ belongs to $H^\infty(\mathbb{D})$ iff $f$ is analytic and bounded on $\mathbb{D}$.

The space $H^\infty(\mathbb{D})$ may be realised as a subspace of $L^\infty(\mathbb{T})$ by the following process:

- By taking radial limits every $f \in H^\infty$ defines a corresponding function $\tilde{f}$ on $\mathbb{T}$;
- By an extension of the Cauchy Integration formulae, $f$ may similarly be recovered from $\tilde{f}$;
- $\sup_{z \in \mathbb{D}} |f(z)| = \text{ess sup}_{z \in \mathbb{T}} |\tilde{f}(z)|$.

$H^p(\mathbb{T})$ is then simply the closure in $L^p(\mathbb{T})$ of $H^\infty(\mathbb{T})$. 
Invariant subspaces for $H^2$

The classical roots

$H^p$ spaces of the disc

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and $\mathbb{T}$ the unit circle.

**Definition**

$f : \mathbb{D} \to \mathbb{C}$ belongs to $H^\infty(\mathbb{D})$ iff $f$ is analytic and bounded on $\mathbb{D}$.

The space $H^\infty(\mathbb{D})$ may be realised as a subspace of $L^\infty(\mathbb{T})$ by the following process:

- By taking radial limits every $f \in H^\infty$ defines a corresponding function $\tilde{f}$ on $\mathbb{T}$;
- By an extension of the Cauchy Integration formulae, $f$ may similarly be recovered from $\tilde{f}$;
- $\sup_{z \in \mathbb{D}} |f(z)| = \text{ess sup}_{z \in \mathbb{T}} |\tilde{f}(z)|$.

$H^p(\mathbb{T})$ is then simply the closure in $L^p(\mathbb{T})$ of $H^\infty(\mathbb{T})$. 
Invariant subspaces for $H^2$

The classical roots

$H^p$ spaces of the disc

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and $\mathbb{T}$ the unit circle.

**Definition**

$f : \mathbb{D} \rightarrow \mathbb{C}$ belongs to $H^\infty(\mathbb{D})$ iff $f$ is analytic and bounded on $\mathbb{D}$.

The space $H^\infty(\mathbb{D})$ may be realised as a subspace of $L^\infty(\mathbb{T})$ by the following process:

- By taking radial limits every $f \in H^\infty$ defines a corresponding function $\tilde{f}$ on $\mathbb{T}$;
- By an extension of the Cauchy Integration formulae, $f$ may similarly be recovered from $\tilde{f}$;
- $\sup_{z \in \mathbb{D}} |f(z)| = \operatorname{ess \ sup}_{z \in \mathbb{T}} |\tilde{f}(z)|$.

$H^p(\mathbb{T})$ is then simply the closure in $L^p(\mathbb{T})$ of $H^\infty(\mathbb{T})$. 
Invariant subspaces for $H^2$

The classical roots

$H^p$ spaces of the disc

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and $\mathbb{T}$ the unit circle.

**Definition**

$f : \mathbb{D} \to \mathbb{C}$ belongs to $H^\infty(\mathbb{D})$ iff $f$ is analytic and bounded on $\mathbb{D}$.

The space $H^\infty(\mathbb{D})$ may be realised as a subspace of $L^\infty(\mathbb{T})$ by the following process:

- By taking radial limits every $f \in H^\infty$ defines a corresponding function $\tilde{f}$ on $\mathbb{T}$;
- By an extension of the Cauchy Integration formulae, $f$ may similarly be recovered from $\tilde{f}$;
- $\sup_{z \in \mathbb{D}} |f(z)| = \text{ess sup}_{z \in \mathbb{T}} |\tilde{f}(z)|$.

$H^p(\mathbb{T})$ is then simply the closure in $L^p(\mathbb{T})$ of $H^\infty(\mathbb{T})$. 
Invariant subspaces for $H^2$

The classical roots

**Wk* Dirichlet algebras**

Let $X$ be a probability space. A weak* closed unital-subalgebra $A$ of $L^\infty(X)$, is called wk* sub-Dirichlet if:

$$\int fg = \int f \int g, \quad f, g \in A.$$  \hspace{1cm} (1)

$A$ exhibits $H^\infty$-like behaviour iff $A + \bar{A}$ is wk* dense in $L^\infty(X)$. Such algebras are called wk* Dirichlet algebras.

- When this condition holds we will write $H^\infty(A)$ for $A$, and $H^p(A)$ ($1 \leq p < \infty$) for the closure of $A$ in $L^p(X)$. More generally $[S]_p$ will be the norm-closure of $S \subset L^p$ in the $p$-norm.
- For $A_0 = \{ f \in A : \int f = 0 \}$, similarly write $H^p_0(A)$ for $[A_0]_p$. 
Let $X$ be a probability space. A weak* closed unital-subalgebra $A$ of $L^\infty(X)$, is called \textit{wk* sub-Dirichlet} if:

$$\int fg = \int f \int g, \quad f, g \in A. \quad (1)$$

$A$ exhibits $H^\infty$-like behaviour iff $A + \bar{A}$ is wk* dense in $L^\infty(X)$. Such algebras are called \textit{wk* Dirichlet} algebras.

- When this condition holds we will write $H^\infty(A)$ for $A$, and $H^p(A)$ ($1 \leq p < \infty$) for the closure of $A$ in $L^p(X)$. More generally $[S]_p$ will be the norm-closure of $S \subset L^p$ in the $p$-norm.
- For $A_0 = \{ f \in A : \int f = 0 \}$, similarly write $H^p_0(A)$ for $[A_0]_p$. 
Invariant subspaces for $H^2$

The classical roots

Wk* Dirichlet algebras

Let $X$ be a probability space. A weak* closed unital-subalgebra $A$ of $L^\infty(X)$, is called wk* sub-Dirichlet if:

$$\int fg = \int f \int g, \quad f, g \in A. \quad (1)$$

$A$ exhibits $H^\infty$-like behaviour iff $A + \overline{A}$ is wk* dense in $L^\infty(X)$. Such algebras are called wk* Dirichlet algebras.

- When this condition holds we will write $H^\infty(A)$ for $A$, and $H^p(A)$ $(1 \leq p < \infty)$ for the closure of $A$ in $L^p(X)$. More generally $[S]_p$ will be the norm-closure of $S \subset L^p$ in the $p$-norm.
- For $A_0 = \{ f \in A : \int f = 0 \}$, similarly write $H^p_0(A)$ for $[A_0]_p$. 
Invariant subspaces for $H^2$

The classical roots

**Wk* Dirichlet algebras**

Let $X$ be a probability space. A weak* closed unital-subalgebra $A$ of $L^\infty(X)$, is called *wk* sub-Dirichlet if:

$$\int fg = \int f \int g, \quad f, g \in A.$$  \hfill (1)

$A$ exhibits $H^\infty$-like behaviour iff $A + \bar{A}$ is wk* dense in $L^\infty(X)$. Such algebras are called *wk* Dirichlet algebras.

- When this condition holds we will write $H^\infty(A)$ for $A$, and $H^p(A)$ ($1 \leq p < \infty$) for the closure of $A$ in $L^p(X)$. More generally $[S]_p$ will be the norm-closure of $S \subset L^p$ in the $p$-norm.
- For $A_0 = \{f \in A : \int f = 0\}$, similarly write $H^p_0(A)$ for $[A_0]_p$. 
Wk* Dirichlet algebras

Let $X$ be a probability space. A weak* closed unital-subalgebra $A$ of $L^\infty(X)$, is called wk* sub-Dirichlet if:

$$\int fg = \int f \int g, \quad f, g \in A.$$  \hspace{1cm} (1)

$A$ exhibits $H^\infty$-like behaviour iff $A + \bar{A}$ is wk* dense in $L^\infty(X)$. Such algebras are called wk* Dirichlet algebras.

- When this condition holds we will write $H^\infty(A)$ for $A$, and $H^p(A)$ ($1 \leq p < \infty$) for the closure of $A$ in $L^p(X)$. More generally $[S]_p$ will be the norm-closure of $S \subset L^p$ in the $p$-norm.

- For $A_0 = \{f \in A : \int f = 0\}$, similarly write $H^p_0(A)$ for $[A_0]_p$. 
Invariant subspaces for $H^2$

The classical roots

**Wk* Dirichlet algebras**

Let $X$ be a probability space. A weak* closed unital-subalgebra $A$ of $L^\infty(X)$, is called wk* sub-Dirichlet if:

$$\int fg = \int f \int g, \quad f, g \in A.$$  

(1)

$A$ exhibits $H^\infty$-like behaviour iff $A + \bar{A}$ is wk* dense in $L^\infty(X)$. Such algebras are called wk* Dirichlet algebras.

- When this condition holds we will write $H^\infty(A)$ for $A$, and $H^p(A)$ ($1 \leq p < \infty$) for the closure of $A$ in $L^p(X)$. More generally $[S]_p$ will be the norm-closure of $S \subset L^p$ in the $p$-norm.
- For $A_0 = \{f \in A : \int f = 0\}$, similarly write $H^p_0(A)$ for $[A_0]_p$. 
Invariant subspaces for $H^2$

The classical roots

Fun facts about $A$

For wk* sub-Dirichlet algebras the following are equivalent:

- $A + \overline{A}$ is wk* dense in $L^\infty(X)$.
- Validity of Szegö's formula: $\forall g \in L^1_+(X)$,
  $\exp \int \log g = \inf \{ \int |1 - f|^2 g : f \in A, \int f = 0 \}$.
- Unique state extension: If $g \in L^1(X)$ is nonnegative with
  $\int fg = \int f$ for all $f \in A$, then $g = 1$ a.e.
- Gleason-Whitney property: there is a unique Hahn-Banach extension to $L^\infty(X)$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
- Beurling's theorem: every simply $A$-invariant subspace $K$ of $L^2(X)$, is of the form $u[A]_2$ for some unimodular $u$.
- Plus about 6 other conditions.
Fun facts about $A$

For wk* sub-Dirichlet algebras the following are equivalent:

- $A + \bar{A}$ is wk* dense in $L^\infty(X)$.
- Validity of Szegö’s formula: $\forall g \in L^1_+(X)$, $\exp \int \log g = \inf \{\int |1 - f|^2 g : f \in A, \int f = 0\}$.
- Unique state extension: If $g \in L^1(X)$ is nonnegative with $\int fg = \int f$ for all $f \in A$, then $g = 1$ a.e.
- Gleason-Whitney property: there is a unique Hahn-Banach extension to $L^\infty(X)$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
- Beurling’s theorem: every simply $A$-invariant subspace $K$ of $L^2(X)$, is of the form $u[A]_2$ for some unimodular $u$.
- Plus about 6 other conditions.
Invariant subspaces for $H^2$

The classical roots

Fun facts about $A$

For wk* sub-Dirichlet algebras the following are equivalent:

- $A + \overline{A}$ is wk* dense in $L^\infty(X)$.

- **Validity of Szegö’s formula:** $\forall g \in L^1_+(X),$ 
  \[ \exp \int \log g = \inf \left\{ \int |1 - f|^2 g : f \in A, \int f = 0 \right\}. \]

- **Unique state extension:** If $g \in L^1(X)$ is nonnegative with $\int fg = \int f$ for all $f \in A$, then $g = 1$ a.e.

- **Gleason-Whitney property:** there is a unique Hahn-Banach extension to $L^\infty(X)$ of any weak* continuous functional on $A$, and this extension is weak* continuous.

- **Beurling’s theorem:** every simply $A$-invariant subspace $K$ of $L^2(X)$, is of the form $u[A]_2$ for some unimodular $u$.

- Plus about 6 other conditions.
Invariant subspaces for $H^2$

The classical roots

Fun facts about $A$

For wk* sub-Dirichlet algebras the following are equivalent:

- $A + \bar{A}$ is wk* dense in $L^\infty(X)$.
- **Validity of Szegö’s formula:** $\forall g \in L^1_+(X)$, 
  $\exp \int \log g = \inf \{ \int |1 - f|^2 g : f \in A, \int f = 0 \}$.
- **Unique state extension:** If $g \in L^1(X)$ is nonnegative with 
  $\int fg = \int f$ for all $f \in A$, then $g = 1$ a.e.
- **Gleason-Whitney property:** there is a unique Hahn-Banach 
  extension to $L^\infty(X)$ of any weak* continuous functional on 
  $A$, and this extension is weak* continuous.
- **Beurling’s theorem:** every simply $A$-invariant subspace $K$ 
  of $L^2(X)$, is of the form $u[A]_2$ for some unimodular $u$.
- Plus about 6 other conditions.
Fun facts about $A$

For wk* sub-Dirichlet algebras the following are equivalent:

- $A + \bar{A}$ is wk* dense in $L^\infty(X)$.
- **Validity of Szegö’s formula:** $\forall g \in L^1_+(X), \quad \exp \int \log g = \inf \{ \int |1 - f|^2g : f \in A, \int f = 0 \}$.
- **Unique state extension:** If $g \in L^1(X)$ is nonnegative with $\int fg = \int f$ for all $f \in A$, then $g = 1$ a.e.
- **Gleason-Whitney property:** there is a unique Hahn-Banach extension to $L^\infty(X)$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
- **Beurling’s theorem:** every simply $A$-invariant subspace $K$ of $L^2(X)$, is of the form $u[A]_2$ for some unimodular $u$.
- Plus about 6 other conditions.
For wk* sub-Dirichlet algebras the following are equivalent:

- $A + \overline{A}$ is wk* dense in $L^\infty(X)$.

- **Validity of Szegö’s formula:** $\forall g \in L^1_+(X)$,
  $\exp \int \log g = \inf \{ \int |1 - f|^2 g : f \in A, \int f = 0 \}$.

- **Unique state extension:** If $g \in L^1(X)$ is nonnegative with
  $\int fg = \int f$ for all $f \in A$, then $g = 1$ a.e.

- **Gleason-Whitney property:** there is a unique Hahn-Banach extension to $L^\infty(X)$ of any weak* continuous functional on $A$, and this extension is weak* continuous.

- **Beurling’s theorem:** every simply $A$-invariant subspace $K$ of $L^2(X)$, is of the form $u[A]_2$ for some unimodular $u$.

- Plus about 6 other conditions.
Invariant subspaces for $H^2$

The classical roots

Fun facts about $A$

For wk* sub-Dirichlet algebras the following are equivalent:

- $A + \overline{A}$ is wk* dense in $L^\infty(X)$.
- **Validity of Szegö’s formula:** $\forall g \in L^1_+(X)$,
  \[ \exp \int \log g = \inf \{ \int |1 - f|^2 g : f \in A, \int f = 0 \}. \]
- **Unique state extension:** If $g \in L^1(X)$ is nonnegative with
  $\int fg = \int f$ for all $f \in A$, then $g = 1$ a.e.
- **Gleason-Whitney property:** there is a unique Hahn-Banach
  extension to $L^\infty(X)$ of any weak* continuous functional on
  $A$, and this extension is weak* continuous.
- **Beurling’s theorem:** every simply $A$-invariant subspace $K$
  of $L^2(X)$, is of the form $u[A]_2$ for some unimodular $u$.
- Plus about 6 other conditions.
Invariant subspaces for $H^2$

Mildly noncommutative precepts

Quantising $L^\infty(\mathbb{T})$

Context:
- $M$ a (semi)finite von Neumann algebra, equipped with a faithful normal (semi)finite trace $\tau_M = \tau$. (Example: $B(H)$ equipped with $\text{Tr}$.)
- $\tilde{M}$ the $\tau_M$-measurable operators affiliated to $M$, i.e. all operators $a$ affiliated to $M$, such that for every $\varepsilon > 0$ there exists a projection $e \in M$ with $\tau(1-e) \leq \varepsilon$, and $ae \in M$.

Dictionary: $M = L^\infty(M, \tau)$ and $L^p(M, \tau) = \{ a \in \tilde{M} : \tau(|a|^p) < \infty \}$ for $p > 0$. It turns out that $M_\ast \equiv L^1(M, \tau)$. 
Invariant subspaces for $H^2$
Mildly noncommutative precepts

Quantising $L^\infty(\mathbb{T})$

Context:

- $M$ a (semi)finite von Neumann algebra, equipped with a faithful normal (semi)finite trace $\tau_M = \tau$. (Example: $B(H)$ equipped with $\text{Tr}$.)

- $\tilde{M}$ the $\tau_M$-measurable operators affiliated to $M$, i.e. all operators $a$ affiliated to $M$, such that for every $\varepsilon > 0$ there exists a projection $e \in M$ with $\tau(1 - e) \leq \varepsilon$, and $ae \in M$.

Dictionary: $M = L^\infty(M, \tau)$ and $L^p(M, \tau) = \{ a \in \tilde{M} : \tau(|a|^p) < \infty \}$ for $p > 0$. It turns out that $M_* \equiv L^1(M, \tau)$. 
Quantising $L^\infty(\mathbb{T})$

**Context:**
- $M$ a (semi)finite von Neumann algebra, equipped with a faithful normal (semi)finite trace $\tau_M = \tau$. (Example: $B(H)$ equipped with $\text{Tr}$.)
- $\tilde{M}$ the $\tau_M$-measurable operators affiliated to $M$, i.e. all operators $a$ affiliated to $M$, such that for every $\varepsilon > 0$ there exists a projection $e \in M$ with $\tau(1 - e) \leq \varepsilon$, and $ae \in M$.

**Dictionary:** $M = L^\infty(M, \tau)$ and $L^p(M, \tau) = \{ a \in \tilde{M} : \tau(|a|^p) < \infty \}$ for $p > 0$. It turns out that $M_* \equiv L^1(M, \tau)$. 
Quantising $L^\infty(\mathbb{T})$

Context:

- $M$ a (semi)finite von Neumann algebra, equipped with a faithful normal (semi)finite trace $\tau_M = \tau$. (Example: $B(H)$ equipped with $\text{Tr}$.)
- $\tilde{M}$ the $\tau_M$-measurable operators affiliated to $M$, i.e. all operators $a$ affiliated to $M$, such that for every $\varepsilon > 0$ there exists a projection $e \in M$ with $\tau(1 - e) \leq \varepsilon$, and $ae \in M$.

Dictionary: $M = L^\infty(M, \tau)$ and $L^p(M, \tau) = \{ a \in \tilde{M} : \tau(|a|^p) < \infty \}$ for $p > 0$. It turns out that $M_* \equiv L^1(M, \tau)$. 
Quantising $L^\infty(\mathbb{T})$

Context:

- $M$ a (semi)finite von Neumann algebra, equipped with a faithful normal (semi)finite trace $\tau_M = \tau$. (Example: $B(H)$ equipped with $\text{Tr}$.)

- $\tilde{M}$ the $\tau_M$-measurable operators affiliated to $M$, i.e. all operators $a$ affiliated to $M$, such that for every $\varepsilon > 0$ there exists a projection $e \in M$ with $\tau(1 - e) \leq \varepsilon$, and $ae \in M$.

Dictionary: $M = L^\infty(M, \tau)$ and $L^p(M, \tau) = \{a \in \tilde{M} : \tau(|a|^p) < \infty\}$ for $p > 0$. It turns out that $M_* \equiv L^1(M, \tau)$. 
Quantising $L^\infty(\mathbb{T})$

Context:

- $M$ a (semi)finite von Neumann algebra, equipped with a faithful normal (semi)finite trace $\tau_M = \tau$. (Example: $B(H)$ equipped with $\text{Tr}$.)
- $\tilde{M}$ the $\tau_M$-measurable operators affiliated to $M$, i.e. all operators $a$ affiliated to $M$, such that for every $\varepsilon > 0$ there exists a projection $e \in M$ with $\tau(1 - e) \leq \varepsilon$, and $ae \in M$.

Dictionary: $M = L^\infty(M, \tau)$ and $L^p(M, \tau) = \{ a \in \tilde{M} : \tau(|a|^p) < \infty \}$ for $p > 0$. It turns out that $M_* \equiv L^1(M, \tau)$. 
Quantising $L^\infty(\mathbb{T})$

Context:

- $M$ a (semi)finite von Neumann algebra, equipped with a faithful normal (semi)finite trace $\tau_M = \tau$. (Example: $B(H)$ equipped with $\text{Tr}$.)
- $\tilde{M}$ the $\tau_M$-measurable operators affiliated to $M$, i.e. all operators $a$ affiliated to $M$, such that for every $\varepsilon > 0$ there exists a projection $e \in M$ with $\tau(1 - e) \leq \varepsilon$, and $ae \in M$.

Dictionary: $M = L^\infty(M, \tau)$ and $L^p(M, \tau) = \{ a \in \tilde{M} : \tau(|a|^p) < \infty \}$ for $p > 0$. It turns out that $M_* \equiv L^1(M, \tau)$. 
Quantising $L^\infty(\mathbb{T})$

Context:
- $M$ a (semi)finite von Neumann algebra, equipped with a faithful normal (semi)finite trace $\tau_M = \tau$. (Example: $B(H)$ equipped with $\text{Tr}$.)
- $\tilde{M}$ the $\tau_M$-measurable operators affiliated to $M$, i.e. all operators $a$ affiliated to $M$, such that for every $\varepsilon > 0$ there exists a projection $e \in M$ with $\tau(1 - e) \leq \varepsilon$, and $ae \in M$.

Dictionary: $M = L^\infty(M, \tau)$ and $L^p(M, \tau) = \{ a \in \tilde{M} : \tau(|a|^p) < \infty \}$ for $p > 0$. It turns out that $M_* \equiv L^1(M, \tau)$. 
Tracial subalgebras

Let $M$ be a finite von Neumann algebra equipped with a faithful normal tracial state $\tau_M$.

A tracial subalgebra of $M$ is a wk* closed unital subalgebra $A$ of $M$ for which the trace preserving faithful normal conditional expectation $\mathcal{E} : M \to A \cap A^* = D$ satisfies:

$$\mathcal{E}(a_1 a_2) = \mathcal{E}(a_1) \mathcal{E}(a_2), \quad a_1, a_2 \in A.$$ (2)

A tracial subalgebra for which $A + A^*$ is weak* dense in $M$, is maximal as a tracial subalgebra (Exel, 1988). The tracial subalgebras satisfying this weak* density criterion are said to be finite maximal subdiagonal subalgebras. These are our noncommutative $H^\infty$'s.
Invariant subspaces for $H^2$
Mildly noncommutative precepts

Tracial subalgebras

Let $M$ be a finite von Neumann algebra equipped with a faithful normal tracial state $\tau_M$. A **tracial subalgebra** of $M$ is a wk* closed unital subalgebra $A$ of $M$ for which the trace preserving faithful normal conditional expectation $\mathcal{E} : M \to A \cap A^* = D$ satisfies:

$$\mathcal{E}(a_1 a_2) = \mathcal{E}(a_1) \mathcal{E}(a_2), \quad a_1, a_2 \in A.$$  \hspace{1cm} (2)

A tracial subalgebra for which $A + A^*$ is weak* dense in $M$, is maximal as a tracial subalgebra (Exel, 1988). The tracial subalgebras satisfying this weak* density criterion are said to be **finite maximal subdiagonal subalgebras**. These are our noncommutative $H^\infty$’s.
Tracial subalgebras

Let $M$ be a finite von Neumann algebra equipped with a faithful normal tracial state $\tau_M$.
A tracial subalgebra of $M$ is a wk* closed unital subalgebra $A$ of $M$ for which the trace preserving faithful normal conditional expectation $\mathcal{E} : M \to A \cap A^* = D$ satisfies:

$$\mathcal{E}(a_1 a_2) = \mathcal{E}(a_1) \mathcal{E}(a_2), \quad a_1, a_2 \in A.$$  \hspace{1cm} (2)

A tracial subalgebra for which $A + A^*$ is weak* dense in $M$, is maximal as a tracial subalgebra (Exel, 1988). The tracial subalgebras satisfying this weak* density criterion are said to be finite maximal subdiagonal subalgebras. These are our noncommutative $H^\infty$’s.
Tracial subalgebras

Let $M$ be a finite von Neumann algebra equipped with a faithful normal tracial state $\tau_M$.

A tracial subalgebra of $M$ is a wk* closed unital subalgebra $A$ of $M$ for which the trace preserving faithful normal conditional expectation $\mathcal{E} : M \rightarrow A \cap A^* = D$ satisfies:

$$\mathcal{E}(a_1 a_2) = \mathcal{E}(a_1) \mathcal{E}(a_2), \quad a_1, a_2 \in A.$$

(2)

A tracial subalgebra for which $A + A^*$ is weak* dense in $M$, is maximal as a tracial subalgebra (Exel, 1988). The tracial subalgebras satisfying this weak* density criterion are said to be finite maximal subdiagonal subalgebras. These are our noncommutative $H^\infty$’s.
Fun facts about tracial subalgebras

For any tracial subalgebra $A$ of $M$, the following are equivalent (BL):

- $A + A^{*w*} = M$.
- $A$ satisfies a Szegö-like formula formulated in terms of the Fuglede-Kadison determinant.
- Unique state extension: $A + A_0^*$ is a dense subspace of $L^2(M)$, and any $g \in L^1(M)_+$ in the annihilator of $A_0$, is in $L^1(D)$.
- Gleason-Whitney property: there is a unique Hahn-Banach extension to $M$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
- A noncommutative version of Beurling’s theorem holds.
- Plus about 6 other conditions.
Fun facts about tracial subalgebras

For any tracial subalgebra $A$ of $M$, the following are equivalent (BL):

- $A + A^*_{w^*} = M$.
- $A$ satisfies a Szegö-like formula formulated in terms of the Fuglede-Kadison determinant.
- Unique state extension: $A + A_0^*$ is a dense subspace of $L^2(M)$, and any $g \in L^1(M)_+$ in the annihilator of $A_0$, is in $L^1(D)$.
- Gleason-Whitney property: there is a unique Hahn-Banach extension to $M$ of any weak$^*$ continuous functional on $A$, and this extension is weak$^*$ continuous.
- A noncommutative version of Beurling’s theorem holds.
- Plus about 6 other conditions.
Fun facts about tracial subalgebras

For any tracial subalgebra $A$ of $M$, the following are equivalent (BL):

- $A + A^* = M$.
- $A$ satisfies a Szegö-like formula formulated in terms of the Fuglede-Kadison determinant.
- Unique state extension: $A + A^*_0$ is a dense subspace of $L^2(M)$, and any $g \in L^1(M)_+$ in the annihilator of $A_0$, is in $L^1(D)$.
- Gleason-Whitney property: there is a unique Hahn-Banach extension to $M$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
- A noncommutative version of Beurling’s theorem holds.
- Plus about 6 other conditions.
Fun facts about tracial subalgebras

For any tracial subalgebra $A$ of $M$, the following are equivalent (BL):

- $A + A^* = M$.
- $A$ satisfies a Szegö-like formula formulated in terms of the Fuglede-Kadison determinant.
- **Unique state extension:** $A + A_0^*$ is a dense subspace of $L^2(M)$, and any $g \in L^1(M)_{+}$ in the annihilator of $A_0$, is in $L^1(D)$.
- Gleason-Whitney property: there is a unique Hahn-Banach extension to $M$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
- A noncommutative version of Beurling’s theorem holds.
- Plus about 6 other conditions.
Fun facts about tracial subalgebras

For any tracial subalgebra $A$ of $M$, the following are equivalent (BL):

- $A + A^* \subseteq M$.
- $A$ satisfies a Szegö-like formula formulated in terms of the Fuglede-Kadison determinant.
- **Unique state extension**: $A + A_0^*$ is a dense subspace of $L^2(M)$, and any $g \in L^1(M)_+$ in the annihilator of $A_0$, is in $L^1(D)$.
- **Gleason-Whitney property**: there is a unique Hahn-Banach extension to $M$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
- A noncommutative version of Beurling’s theorem holds.
- Plus about 6 other conditions.
Fun facts about tracial subalgebras

For any tracial subalgebra $A$ of $M$, the following are equivalent (BL):

- $A + A^* = M$.
- $A$ satisfies a Szegö-like formula formulated in terms of the Fuglede-Kadison determinant.
- Unique state extension: $A + A_0^*$ is a dense subspace of $L^2(M)$, and any $g \in L^1(M)_+$ in the annihilator of $A_0$, is in $L^1(D)$.
- Gleason-Whitney property: there is a unique Hahn-Banach extension to $M$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
- A noncommutative version of Beurling’s theorem holds.
- Plus about 6 other conditions.
Fun facts about tracial subalgebras

For any tracial subalgebra $A$ of $M$, the following are equivalent (BL):

- $A + A^* = M$.
- $A$ satisfies a Szegö-like formula formulated in terms of the Fuglede-Kadison determinant.
- **Unique state extension**: $A + A_0^*$ is a dense subspace of $L^2(M)$, and any $g \in L^1(M)_+$ in the annihilator of $A_0$, is in $L^1(D)$.
- **Gleason-Whitney property**: there is a unique Hahn-Banach extension to $M$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
- A noncommutative version of Beurling’s theorem holds.
- Plus about 6 other conditions.
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Haagerup $L^p$-spaces in 2 minutes

A von Neumann algebra $M$ equipped with an $fns$ weight $\nu$, $M$ embeds into a semifinite von Neumann algebra $N = M \rtimes_{\nu} \mathbb{R}$ admitting an operator valued weight $T : \hat{N} \to \hat{M}$ and a one-parameter group of $*$-automorphisms $\{\theta_s\}$ ($s \in \mathbb{R}$) for which

- there exists a canonical trace satisfying $\tau_N \circ \theta_s = e^{-s} \tau_N$,
- $L^\infty(M) = M = \{a \in \hat{N} : \theta_s(a) = a \text{ for all } s \in \mathbb{R}\}$ and $L^1(M) = M_* = \{a \in \hat{N} : \theta_s(a) = e^{-s} a \text{ for all } s \in \mathbb{R}\}$.

Definition: $L^p(M) = \{a \in \hat{N} : \theta_s(a) = e^{-s/p} a \text{ for all } s \in \mathbb{R}\}$

Convention: Given a normal weight $\omega$ on $M$ write $\tilde{\omega}$ for $\omega \circ T$
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Haagerup $L^p$-spaces in 2 minutes

A von Neumann algebra $M$ equipped with an $fns$ weight $\nu$, $M$ embeds into a semifinite von Neumann algebra $N = M \rtimes_\nu \mathbb{R}$ admitting an operator valued weight $T : \hat{N} \to \hat{M}$ and a one-parameter group of *-automorphisms $\{\theta_s\} (s \in \mathbb{R})$ for which

- there exists a canonical trace satisfying $\tau_N \circ \theta_s = e^{-s} \tau_N$,
- $L^\infty(M) = M = \{a \in \hat{N} : \theta_s(a) = a \text{ for all } s \in \mathbb{R}\}$ and $L^1(M) = M_* = \{a \in \hat{N} : \theta_s(a) = e^{-s} a \text{ for all } s \in \mathbb{R}\}$.

Definition: $L^p(M) = \{a \in \hat{N} : \theta_s(a) = e^{-s/p} a \text{ for all } s \in \mathbb{R}\}$

Convention: Given a normal weight $\omega$ on $M$ write $\tilde{\omega}$ for $\omega \circ T$
A von Neumann algebra $M$ equipped with an $fns$ weight $\nu$, $M$ embeds into a semifinite von Neumann algebra $N = M \rtimes_\nu \mathbb{R}$ admitting an operator valued weight $T : \hat{N} \to \hat{M}$ and a one-parameter group of *-automorphisms $\{\theta_s\}$ ($s \in \mathbb{R}$) for which

- there exists a canonical trace satisfying $\tau_N \circ \theta_s = e^{-s}\tau_N$,
- $L^\infty(M) = M = \{ a \in \hat{N} : \theta_s(a) = a \text{ for all } s \in \mathbb{R} \}$ and $L^1(M) = M_* = \{ a \in \hat{N} : \theta_s(a) = e^{-s}a \text{ for all } s \in \mathbb{R} \}$.

Definition: $L^p(M) = \{ a \in \hat{N} : \theta_s(a) = e^{-s/p}a \text{ for all } s \in \mathbb{R} \}$

Convention: Given a normal weight $\omega$ on $M$ write $\tilde{\omega}$ for $\omega \circ T$
A von Neumann algebra $M$ equipped with an \textit{fns} weight $\nu$, $M$ embeds into a semifinite von Neumann algebra $N = M \rtimes_{\nu} \mathbb{R}$ admitting an \textbf{operator valued weight} $T : \hat{N} \to \hat{M}$ and a \textbf{one-parameter group of *-automorphisms} $\{\theta_s\} (s \in \mathbb{R})$ for which

- there exists a canonical trace satisfying $\tau_N \circ \theta_s = e^{-s} \tau_N$,
- $L^\infty(M) = M = \{ a \in \hat{N} : \theta_s(a) = a \text{ for all } s \in \mathbb{R} \}$ and $L^1(M) = M_* = \{ a \in \hat{N} : \theta_s(a) = e^{-s} a \text{ for all } s \in \mathbb{R} \}$.

\textbf{Definition:} $L^p(M) = \{ a \in \hat{N} : \theta_s(a) = e^{-s/p} a \text{ for all } s \in \mathbb{R} \}$

\textbf{Convention:} Given a normal weight $\omega$ on $M$ write $\tilde{\omega}$ for $\omega \circ T$
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Haagerup $L^p$-spaces in 2 minutes

A von Neumann algebra $M$ equipped with an $fns$ weight $\nu$, $M$ embeds into a semifinite von Neumann algebra $N = M \rtimes_\nu \mathbb{R}$ admitting an operator valued weight $T : \hat{N} \to \hat{M}$ and a one-parameter group of $*$-automorphisms $\{\theta_s\} (s \in \mathbb{R})$ for which

- there exists a canonical trace satisfying $\tau_N \circ \theta_s = e^{-s} \tau_N$,
- $L^\infty(M) = M = \{a \in \hat{N} : \theta_s(a) = a \text{ for all } s \in \mathbb{R}\}$ and $L^1(M) = M_* = \{a \in \hat{N} : \theta_s(a) = e^{-s} a \text{ for all } s \in \mathbb{R}\}$.

Definition: $L^p(M) = \{a \in \hat{N} : \theta_s(a) = e^{-s/p} a \text{ for all } s \in \mathbb{R}\}$

Convention: Given a normal weight $\omega$ on $M$ write $\tilde{\omega}$ for $\omega \circ T$. 
Haagerup $L^p$-spaces in 2 minutes

A von Neumann algebra $M$ equipped with an \textit{fns} weight $\nu$, $M$ embeds into a semifinite von Neumann algebra $N = M \rtimes_\nu \mathbb{R}$ admitting an operator valued weight $T : \hat{N} \to \hat{M}$ and a one-parameter group of *-automorphisms $\{\theta_s\}$ ($s \in \mathbb{R}$) for which

- there exists a canonical trace satisfying $\tau_N \circ \theta_s = e^{-s} \tau_N$,
- $L^\infty(M) = M = \{a \in \hat{N} : \theta_s(a) = a$ for all $s \in \mathbb{R}\}$ and $L^1(M) = M_* = \{a \in \hat{N} : \theta_s(a) = e^{-s} a$ for all $s \in \mathbb{R}\}$.

Definition: $L^p(M) = \{a \in \hat{N} : \theta_s(a) = e^{-s/p} a$ for all $s \in \mathbb{R}\}$

Convention: Given a normal weight $\omega$ on $M$ write $\tilde{\omega}$ for $\omega \circ T$
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Haagerup $L^p$-spaces in 2 minutes

A von Neumann algebra $M$ equipped with an $fns$ weight $\nu$, $M$ embeds into a semifinite von Neumann algebra $N = M \rtimes \nu \mathbb{R}$ admitting an operator valued weight $T : \hat{N} \to \hat{M}$ and a one-parameter group of $*$-automorphisms $\{\theta_s\}$ ($s \in \mathbb{R}$) for which

- there exists a canonical trace satisfying $\tau_N \circ \theta_s = e^{-s} \tau_N$,
- $L^\infty(M) = M = \{a \in \tilde{N} : \theta_s(a) = a \text{ for all } s \in \mathbb{R}\}$ and $L^1(M) = M_* \equiv \{a \in \tilde{N} : \theta_s(a) = e^{-s}a \text{ for all } s \in \mathbb{R}\}$.

Definition: $L^p(M) = \{a \in \tilde{N} : \theta_s(a) = e^{-s/p}a \text{ for all } s \in \mathbb{R}\}$

Convention: Given a normal weight $\omega$ on $M$ write $\tilde{\omega}$ for $\omega \circ T$. 
A von Neumann algebra $M$ equipped with an $fns$ weight $\nu$, $M$ embeds into a semifinite von Neumann algebra $N = M \rtimes_\nu \mathbb{R}$ admitting an operator valued weight $T : \hat{N} \rightarrow \hat{M}$ and a one-parameter group of $*$-automorphisms $\{\theta_s\} (s \in \mathbb{R})$ for which

- there exists a canonical trace satisfying $\tau_N \circ \theta_s = e^{-s} \tau_N$,
- $L^\infty(M) = M = \{a \in \tilde{N} : \theta_s(a) = a \text{ for all } s \in \mathbb{R}\}$ and $L^1(M) = M_* = \{a \in \tilde{N} : \theta_s(a) = e^{-s} a \text{ for all } s \in \mathbb{R}\}$.

Definition: $L^p(M) = \{a \in \tilde{N} : \theta_s(a) = e^{-s/p} a \text{ for all } s \in \mathbb{R}\}$

Convention: Given a normal weight $\omega$ on $M$ write $\hat{\omega}$ for $\omega \circ T$
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Haagerup $L^p$-spaces in 2 minutes

A von Neumann algebra $M$ equipped with an fns weight $\nu$, $M$ embeds into a semifinite von Neumann algebra $N = M \rtimes_{\nu} \mathbb{R}$ admitting an operator valued weight $T : \hat{N} \to \hat{M}$ and a one-parameter group of *-automorphisms $\{\theta_s\} (s \in \mathbb{R})$ for which

- there exists a canonical trace satisfying $\tau_N \circ \theta_s = e^{-s} \tau_N$,
- $L^\infty(M) = M = \{a \in \hat{N} : \theta_s(a) = a \text{ for all } s \in \mathbb{R}\}$ and $L^1(M) = M_* \equiv \{a \in \hat{N} : \theta_s(a) = e^{-s} a \text{ for all } s \in \mathbb{R}\}$.

Definition: $L^p(M) = \{a \in \hat{N} : \theta_s(a) = e^{-s/p} a \text{ for all } s \in \mathbb{R}\}$

Convention: Given a normal weight $\omega$ on $M$ write $\tilde{\omega}$ for $\omega \circ T$
A von Neumann algebra $M$ equipped with an $fns$ weight $\nu$, $M$ embeds into a semifinite von Neumann algebra $N = M \rtimes_{\nu} \mathbb{R}$ admitting an operator valued weight $T : \hat{N} \to \hat{M}$ and a one-parameter group of $*$-automorphisms $\{\theta_s\} (s \in \mathbb{R})$ for which

- there exists a canonical trace satisfying $\tau_N \circ \theta_s = e^{-s} \tau_N$,
- $L^\infty(M) = M = \{a \in \hat{N} : \theta_s(a) = a \text{ for all } s \in \mathbb{R}\}$ and $L^1(M) = M_* \equiv \{a \in \hat{N} : \theta_s(a) = e^{-s} a \text{ for all } s \in \mathbb{R}\}$.

**Definition:** $L^p(M) = \{a \in \hat{N} : \theta_s(a) = e^{-s/p} a \text{ for all } s \in \mathbb{R}\}$

**Convention:** Given a normal weight $\omega$ on $M$ write $\tilde{\omega}$ for $\omega \circ T$. 

Invariant subspaces for $H^2$
Wildly noncommutative precepts 

Haagerup $L^p$-spaces in 2 minutes
A von Neumann algebra $M$ equipped with an \textit{fns} weight $\nu$, $M$ embeds into a semifinite von Neumann algebra $N = M \rtimes_{\nu} \mathbb{R}$ admitting an operator valued weight $T : \hat{N} \rightarrow \hat{M}$ and a one-parameter group of *-automorphisms $\{\theta_s\}$ ($s \in \mathbb{R}$) for which

- there exists a canonical trace satisfying $\tau_N \circ \theta_s = e^{-s} \tau_N$,
- $L^\infty(M) = M = \{a \in \hat{N} : \theta_s(a) = a \text{ for all } s \in \mathbb{R}\}$ and $L^1(M) = M_* \equiv \{a \in \hat{N} : \theta_s(a) = e^{-s}a \text{ for all } s \in \mathbb{R}\}$.

\textbf{Definition:} $L^p(M) = \{a \in \hat{N} : \theta_s(a) = e^{-s/p}a \text{ for all } s \in \mathbb{R}\}$

\textbf{Convention:} Given a normal weight $\omega$ on $M$ write $\tilde{\omega}$ for $\omega \circ T$.
**$H^\infty$-spaces : general case 1**

Let $D \subset M$ be a von Neumann subalgebra of a $\sigma$-finite von Neumann algebra $M$ equipped with a faithful normal state $\nu$, which admits a weak*-continuous contractive projection $\mathcal{E}$ onto $D$ satisfying $\nu \circ \mathcal{E} = \nu$. (The canonical conditional expectation from $M$ onto $D$.)

As before we say that a unital weak* closed subalgebra $A \subset M$ is subdiagonal with respect to $\mathcal{E}$ if

- $A \cap A^* = D$,
- $\mathcal{E}(a_1a_2) = \mathcal{E}(a_1)\mathcal{E}(a_2)$, $a_1, a_2 \in A$,
- and $A + A^*$ is weak* dense in $M$.

In the $\sigma$-finite context, such subdiagonal algebras are not automatically maximal!!
Let $D \subset M$ be a von Neumann subalgebra of a $\sigma$-finite von Neumann algebra $M$ equipped with a faithful normal state $\nu$, which admits a weak*-continuous contractive projection $E$ onto $D$ satisfying $\nu \circ E = \nu$. (The canonical conditional expectation from $M$ onto $D$.)

As before we say that a unital weak* closed subalgebra $A \subset M$ is subdiagonal with respect to $E$ if

- $A \cap A^* = D$,
- $E(a_1 a_2) = E(a_1) E(a_2), \quad a_1, a_2 \in A$,
- and $A + A^*$ is weak* dense in $M$.

In the $\sigma$-finite context, such subdiagonal algebras are not automatically maximal!!
$H^\infty$-spaces : general case 1

Let $D \subset M$ be a von Neumann subalgebra of a $\sigma$-finite von Neumann algebra $M$ equipped with a faithful normal state $\nu$, which admits a weak*-continuous contractive projection $E$ onto $D$ satisfying $\nu \circ E = \nu$. (The canonical conditional expectation from $M$ onto $D$.)

As before we say that a unital weak* closed subalgebra $A \subset M$ is subdiagonal with respect to $E$ if

- $A \cap A^* = D$,
- $E(a_1 a_2) = E(a_1) E(a_2)$, $a_1, a_2 \in A$,
- and $A + A^*$ is weak* dense in $M$.

In the $\sigma$-finite context, such subdiagonal algebras are not automatically maximal!!
Let $D \subset M$ be a von Neumann subalgebra of a $\sigma$-finite von Neumann algebra $M$ equipped with a faithful normal state $\nu$, which admits a weak*-continuous contractive projection $E$ onto $D$ satisfying $\nu \circ E = \nu$. (The canonical conditional expectation from $M$ onto $D$.)

As before we say that a unital weak* closed subalgebra $A \subset M$ is subdiagonal with respect to $E$ if

- $A \cap A^* = D$,
- $E(a_1 a_2) = E(a_1) E(a_2)$, $a_1, a_2 \in A$,
- and $A + A^*$ is weak* dense in $M$.

In the $\sigma$-finite context, such subdiagonal algebras are not automatically maximal!!
Invariant subspaces for $H^2$
Wildly noncommutative precepts

$H^\infty$-spaces : general case 1

Let $D \subset M$ be a von Neumann subalgebra of a $\sigma$-finite von Neumann algebra $M$ equipped with a faithful normal state $\nu$, which admits a weak*-continuous contractive projection $E$ onto $D$ satisfying $\nu \circ E = \nu$. (The canonical conditional expectation from $M$ onto $D$.)

As before we say that a unital weak* closed subalgebra $A \subset M$ is subdiagonal with respect to $E$ if

- $A \cap A^* = D$,
- $E(a_1 a_2) = E(a_1) E(a_2)$, $a_1, a_2 \in A$,
- and $A + A^*$ is weak* dense in $M$.

In the $\sigma$-finite context, such subdiagonal algebras are not automatically maximal!!
Let $D \subset M$ be a von Neumann subalgebra of a $\sigma$-finite von Neumann algebra $M$ equipped with a faithful normal state $\nu$, which admits a weak*-continuous contractive projection $E$ onto $D$ satisfying $\nu \circ E = \nu$. (The canonical conditional expectation from $M$ onto $D$.)

As before we say that a unital weak* closed subalgebra $A \subset M$ is subdiagonal with respect to $E$ if

- $A \cap A^* = D$,
- $E(a_1 a_2) = E(a_1) E(a_2)$, $a_1, a_2 \in A$,
- and $A + A^*$ is weak* dense in $M$.

In the $\sigma$-finite context, such subdiagonal algebras are not automatically maximal!!
Invariant subspaces for $H^2$
Wildly noncommutative precepts

$H^\infty$-spaces : general case 1

Let $D \subset M$ be a von Neumann subalgebra of a $\sigma$-finite von Neumann algebra $M$ equipped with a faithful normal state $\nu$, which admits a weak*-continuous contractive projection $E$ onto $D$ satisfying $\nu \circ E = \nu$. (The canonical conditional expectation from $M$ onto $D$.)

As before we say that a unital weak* closed subalgebra $A \subset M$ is subdiagonal with respect to $E$ if

- $A \cap A^* = D$,
- $E(a_1 a_2) = E(a_1) E(a_2)$, $a_1, a_2 \in A$,

and $A + A^*$ is weak* dense in $M$.

In the $\sigma$-finite context, such subdiagonal algebras are not automatically maximal!!
Let $D \subset M$ be a von Neumann subalgebra of a $\sigma$-finite von Neumann algebra $M$ equipped with a faithful normal state $\nu$, which admits a weak*‐continuous contractive projection $E$ onto $D$ satisfying $\nu \circ E = \nu$. (The canonical conditional expectation from $M$ onto $D$.)

As before we say that a unital weak* closed subalgebra $A \subset M$ is subdiagonal with respect to $E$ if

- $A \cap A^* = D$,
- $E(a_1 a_2) = E(a_1) E(a_2)$, $a_1, a_2 \in A$,
- and $A + A^*$ is weak* dense in $M$.

In the $\sigma$-finite context, such subdiagonal algebras are not automatically maximal!!
$H^\infty$-spaces : general case 1

Let $D \subset M$ be a von Neumann subalgebra of a $\sigma$-finite von Neumann algebra $M$ equipped with a faithful normal state $\nu$, which admits a weak*-continuous contractive projection $\mathcal{E}$ onto $D$ satisfying $\nu \circ \mathcal{E} = \nu$. (The canonical conditional expectation from $M$ onto $D$.)

As before we say that a unital weak* closed subalgebra $A \subset M$ is subdiagonal with respect to $\mathcal{E}$ if

- $A \cap A^* = D$,
- $\mathcal{E}(a_1 a_2) = \mathcal{E}(a_1) \mathcal{E}(a_2)$, $a_1, a_2 \in A$,
- and $A + A^*$ is weak* dense in $M$.

In the $\sigma$-finite context, such subdiagonal algebras are not automatically maximal!!
Invariant subspaces for $H^2$
Wildly noncommutative precepts

$H^\infty$-spaces : general case 1

Let $D \subset M$ be a von Neumann subalgebra of a $\sigma$-finite von Neumann algebra $M$ equipped with a faithful normal state $\nu$, which admits a weak*-continuous contractive projection $E$ onto $D$ satisfying $\nu \circ E = \nu$. (The canonical conditional expectation from $M$ onto $D$.)

As before we say that a unital weak* closed subalgebra $A \subset M$ is subdiagonal with respect to $E$ if

1. $A \cap A^* = D$,
2. $E(a_1 a_2) = E(a_1) E(a_2)$, $a_1, a_2 \in A$,
3. and $A + A^*$ is weak* dense in $M$.

In the $\sigma$-finite context, such subdiagonal algebras are not automatically maximal!!
Invariant subspaces for $H^2$
Wildly noncommutative precepts

$H^\infty$-spaces: general case 2


Let $M$ be $\sigma$-finite, and $A \subset M$ subdiagonal. Then $A$ is maximal subdiagonal if and only if $\sigma^\nu_t(A) = A$ for all $t \in \mathbb{R}$.

As before it is the maximal subdiagonal subalgebras that are our noncommutative $H^\infty$'s.
Invariant subspaces for $H^2$
Wildly noncommutative precepts

$H^\infty$-spaces: general case 2


Let $M$ be $\sigma$-finite, and $A \subset M$ subdiagonal. Then $A$ is maximal subdiagonal if and only if $\sigma^\nu_t(A) = A$ for all $t \in \mathbb{R}$.

As before it is the maximal subdiagonal subalgebras that are our noncommutative $H^\infty$'s.

Let $M$ be $\sigma$-finite, and $A \subset M$ subdiagonal. Then $A$ is maximal subdiagonal if and only if $\sigma_t^\nu(A) = A$ for all $t \in \mathbb{R}$.

As before it is the *maximal* subdiagonal subalgebras that are our noncommutative $H^\infty$'s.
Analytically conditioned subalgebras

Based on the maximality criteria for the $\sigma$-finite case, we say that a weak*-closed unital subalgebra $A \subset M$ is an analytically conditioned subalgebra if

1. $A = \sigma^\nu_t(A)$ for each $t \in \mathbb{R}$
2. and if the conditional expectation $\mathcal{E}$ onto $A \cap A^* = D$ leaving $\nu$ invariant, is multiplicative on $A$.

Is it possible for the earlier equivalences to survive the transition to the $\sigma$-finite case?

Problem: Type III $\sigma$-finite algebras necessarily do not admit a Fuglede-Kadison determinant. So there is no analogue of the Szegö formula in this setting!!!
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Analytically conditioned subalgebras

Based on the maximality criteria for the $\sigma$-finite case, we say that a weak*-closed unital subalgebra $A \subset M$ is an analytically conditioned subalgebra if

1. $A = \sigma^\nu_t(A)$ for each $t \in \mathbb{R}$

2. and if the conditional expectation $\mathcal{E}$ onto $A \cap A^* = D$ leaving $\nu$ invariant, is multiplicative on $A$.

Is it possible for the earlier equivalences to survive the transition to the $\sigma$-finite case?

Problem: Type III $\sigma$-finite algebras necessarily do not admit a Fuglede-Kadison determinant. So there is no analogue of the Szegö formula in this setting!!!
Analytically conditioned subalgebras

Based on the maximality criteria for the $\sigma$-finite case, we say that a weak*-closed unital subalgebra $A \subset M$ is an analytically conditioned subalgebra if

(1) $A = \sigma_t^\nu(A)$ for each $t \in \mathbb{R}$

(2) and if the conditional expectation $\mathcal{E}$ onto $A \cap A^* = D$ leaving $\nu$ invariant, is multiplicative on $A$.

Is it possible for the earlier equivalences to survive the transition to the $\sigma$-finite case?

Problem: Type III $\sigma$-finite algebras necessarily do not admit a Fuglede-Kadison determinant. So there is no analogue of the Szegö formula in this setting!!!
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Analytically conditioned subalgebras

Based on the maximality criteria for the $\sigma$-finite case, we say that a weak*-closed unital subalgebra $A \subset M$ is an analytically conditioned subalgebra if

1. $A = \sigma^\nu_t(A)$ for each $t \in \mathbb{R}$
2. and if the conditional expectation $\mathcal{E}$ onto $A \cap A^* = D$ leaving $\nu$ invariant, is multiplicative on $A$.

Is it possible for the earlier equivalences to survive the transition to the $\sigma$-finite case?

Problem: Type III $\sigma$-finite algebras necessarily do not admit a Fuglede-Kadison determinant. So there is no analogue of the Szegö formula in this setting!!!
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Analytically conditioned subalgebras

Based on the maximality criteria for the $\sigma$-finite case, we say that a weak*-closed unital subalgebra $A \subset M$ is an **analytically conditioned** subalgebra if

1. $A = \sigma^\nu_t(A)$ for each $t \in \mathbb{R}$
2. and if the conditional expectation $\mathcal{E}$ onto $A \cap A^* = D$ leaving $\nu$ invariant, is multiplicative on $A$.

Is it possible for the earlier equivalences to survive the transition to the $\sigma$-finite case?

**Problem:** Type III $\sigma$-finite algebras necessarily do not admit a Fuglede-Kadison determinant. So there is no analogue of the Szegö formula in this setting!!!
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Analytically conditioned subalgebras

Based on the maximality criteria for the $\sigma$-finite case, we say that a weak*-closed unital subalgebra $A \subset M$ is an analytically conditioned subalgebra if

1. $A = \sigma^\nu_t(A)$ for each $t \in \mathbb{R}$
2. and if the conditional expectation $\mathcal{E}$ onto $A \cap A^* = D$ leaving $\nu$ invariant, is multiplicative on $A$.

Is it possible for the earlier equivalences to survive the transition to the $\sigma$-finite case?

Problem: Type III $\sigma$-finite algebras necessarily do not admit a Fuglede-Kadison determinant. So there is no analogue of the Szegö formula in this setting!!!
The Haagerup reduction theorem

Construct the $\sigma$-finite von Neumann super-algebra $R = M \rtimes_\nu \mathbb{Q}_d$ of $M$. ($\mathbb{Q}_d$=diadic rationals)

Note that $\Phi : R \to M$ for some faithful normal conditional expectation.

$R = \bigcup_{n=1}^{\infty} R_n^{w*}$ for some sequence $R_1 \subset R_2 \subset R_3 \subset \ldots$ of von Neumann algebras each of which is finite and is the image of a faithful normal conditional expectation $\Phi_n : R \to R_n$ for which $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$. 
Invariant subspaces for $H^2$
Wildly noncommutative precepts

The Haagerup reduction theorem

Construct the $\sigma$-finite von Neumann super-algebra $R = M \rtimes_{\nu} \mathbb{Q}_d$ of $M$. ($\mathbb{Q}_d$=diadic rationals)

Note that $\Phi : R \to M$ for some faithful normal conditional expectation.

$R = \bigcup_{n=1}^{\infty} R_n^{w*}$ for some sequence $R_1 \subset R_2 \subset R_3 \subset \ldots$ of von Neumann algebras each of which is finite and is the image of a faithful normal conditional expectation $\Phi_n : R \to R_n$ for which $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$. 
Invariant subspaces for $H^2$
Wildly noncommutative precepts

The Haagerup reduction theorem

Construct the $\sigma$-finite von Neumann super-algebra $R = M \rtimes_\nu \mathbb{Q}_d$ of $M$. ($\mathbb{Q}_d$=diadic rationals)

Note that $\Phi : R \to M$ for some faithful normal conditional expectation.

$R = \overline{\bigcup_{n=1}^{\infty} R_n}^{w^*}$ for some sequence $R_1 \subset R_2 \subset R_3 \subset \ldots$ of von Neumann algebras each of which is finite and is the image of a faithful normal conditional expectation $\Phi_n : R \to R_n$ for which $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$. 
Invariant subspaces for $H^2$
Wildly noncommutative precepts

The Haagerup reduction theorem

Construct the $\sigma$-finite von Neumann super-algebra
$R = M \rtimes_\nu \mathbb{Q}_d$ of $M$. ($\mathbb{Q}_d$=diadic rationals)

Note that $\Phi : R \to M$ for some faithful normal conditional expectation.

$R = \bigcup_{n=1}^{\infty} R_n^{w*}$ for some sequence $R_1 \subset R_2 \subset R_3 \subset \ldots$ of von Neumann algebras each of which is finite and is the image of a faithful normal conditional expectation $\Phi_n : R \to R_n$ for which $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$. 
Invariant subspaces for $H^2$
Wildly noncommutative precepts

The Haagerup reduction theorem

Construct the $\sigma$-finite von Neumann super-algebra

$R = M \rtimes_\nu \mathbb{Q}_d$ of $M$. ($\mathbb{Q}_d=$diadic rationals)

Note that $\Phi : R \to M$ for some faithful normal conditional expectation.

$R = \bigcup_{n=1}^{\infty} R_n^{w^*}$ for some sequence $R_1 \subset R_2 \subset R_3 \subset \ldots$ of von Neumann algebras each of which is finite and is the image of a faithful normal conditional expectation $\Phi_n : R \to R_n$ for which $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$. 
The Haagerup reduction theorem

Construct the $\sigma$-finite von Neumann super-algebra
$R = M \rtimes_\nu \mathbb{Q}_d$ of $M$. ($\mathbb{Q}_d$=diadic rationals)

Note that $\Phi : R \to M$ for some faithful normal conditional expectation.

$R = \bigcup_{n=1}^{\infty} R_n^{w^*}$ for some sequence $R_1 \subset R_2 \subset R_3 \subset \ldots$ of von Neumann algebras each of which is finite and is the image of a faithful normal conditional expectation $\Phi_n : R \to R_n$ for which $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$.

$A \subset M$ maximal subdiagonal $\Rightarrow \hat{A} \subset R$ maximal subdiagonal, where formally $\hat{A} = A \rtimes_\nu \mathbb{Q}_d$. Moreover $\Phi$ maps $\hat{A}$ onto $A$, and $\hat{A} \cap \hat{A}^*$ onto $A \cap A^*$[Quanhua Xu, 2005].

In addition the subalgebras $\hat{A}_n = \hat{A} \cap R_n \subset R_n$, are each maximal subdiagonal in $R_n$, with $\bigcup_{n=1}^{\infty} \hat{A}_n$ weak*-*dense in $\hat{A}$.
The Haagerup reduction theorem

Construct the $\sigma$-finite von Neumann super-algebra $R = M \rtimes \nu \mathbb{Q}_d$ of $M$. ($\mathbb{Q}_d=$diadic rationals)

Note that $\Phi : R \rightarrow M$ for some faithful normal conditional expectation.

$R = \bigcup_{n=1}^{\infty} R_n^{w^*}$ for some sequence $R_1 \subset R_2 \subset R_3 \subset \ldots$ of von Neumann algebras each of which is finite and is the image of a faithful normal conditional expectation $\Phi_n : R \rightarrow R_n$ for which $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$.

$A \subset M$ maximal subdiagonal $\Rightarrow \hat{A} \subset R$ maximal subdiagonal, where formally $\hat{A} = A \rtimes_{\nu} \mathbb{Q}_d$. Moreover $\Phi$ maps $\hat{A}$ onto $A$, and $\hat{A} \cap \hat{A}^*$ onto $A \cap A^*$[Quanhua Xu, 2005].

In addition the subalgebras $\hat{A}_n = \hat{A} \cap R_n \subset R_n$, are each maximal subdiagonal in $R_n$, with $\bigcup_{n=1}^{\infty} \hat{A}_n$ weak*-dense in $\hat{A}$. 
The Haagerup reduction theorem

Construct the $\sigma$-finite von Neumann super-algebra

$$R = M \rtimes_{\nu} \mathbb{Q}_d$$

of $M$. ($\mathbb{Q}_d$=diadic rationals)

Note that $\Phi : R \to M$ for some faithful normal conditional
expectation.

$$R = \bigcup_{n=1}^{\infty} R_n^{w^*}$$

for some sequence $R_1 \subset R_2 \subset R_3 \subset \ldots$ of von
Neumann algebras each of which is finite and is the image of a
faithful normal conditional expectation $\Phi_n : R \to R_n$ for which

$$\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$$

when $n \geq m$.

$A \subset M$ analytically conditioned $\Rightarrow \hat{A} \subset R$ analytically
conditioned, where formally $\hat{A} = A \rtimes_{\nu} \mathbb{Q}_d$. Moreover $\Phi$ maps $\hat{A}$
on onto $A$, and $\hat{A} \cap \hat{A}^*$ onto $A \cap A^*$ [LL?!?, 2017].

In addition the subalgebras $\hat{A}_n \cap R_n \subset R_n$, are each tracial
subalgebras of $R_n$, with $\bigcup_{n=1}^{\infty} \hat{A}_n$ weak*-dense in $\hat{A}$. 
Invariate subspaces for $H^2$
Wildly noncommutative precepts

The Haagerup reduction theorem

Construct the $\sigma$-finite von Neumann super-algebra

$$R = M \rtimes_\nu \mathbb{Q}_d$$

of $M$. ($\mathbb{Q}_d$=diadic rationals)

Note that $\Phi : R \to M$ for some faithful normal conditional expectation.

$$R = \bigcup_{n=1}^{\infty} R_n^{w*}$$

for some sequence $R_1 \subset R_2 \subset R_3 \subset \ldots$ of von Neumann algebras each of which is finite and is the image of a faithful normal conditional expectation $\Phi_n : R \to R_n$ for which $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$.

$A \subset M$ analytically conditioned $\Rightarrow \hat{A} \subset R$ analytically conditioned, where formally $\hat{A} = A \rtimes_\nu \mathbb{Q}_d$. Moreover $\Phi$ maps $\hat{A}$ onto $A$, and $\hat{A} \cap \hat{A}^*$ onto $A \cap A^*$ [LL?!, 2017].

In addition the subalgebras $\hat{A}_n \cap R_n \subset R_n$, are each tracial subalgebras of $R_n$, with $\bigcup_{n=1}^{\infty} \hat{A}_n$ weak*-dense in $\hat{A}$. 
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Unique state extension property 1

**Lemma**

Let $A$ be an analytically conditioned algebra. If $A$ satisfies the criterion that any $f \in L^1(M)^+$ which is in the annihilator of $A_0$ must belong to $L^1(D)$, then also

- any $f \in L^1(R)^+$ which is in the annihilator of $\hat{A}_0$ must belong to $L^1(\hat{D})$,
- and for any $n$, any $f \in L^1(R_n)^+$ which is in the annihilator of $(\hat{A}_n)_0$, must belong to $L^1(D_n)$. 
Let $A$ be an analytically conditioned algebra. If $A$ satisfies the criterion that any $f \in L^1(M)^+$ which is in the annihilator of $A_0$ must belong to $L^1(D)$, then also

- any $f \in L^1(R)^+$ which is in the annihilator of $\hat{A}_0$ must belong to $L^1(\hat{D})$,
- and for any $n$, any $f \in L^1(R_n)^+$ which is in the annihilator of $(\hat{A}_n)_0$, must belong to $L^1(D_n)$. 
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Unique state extension property 1

Lemma

Let $A$ be an analytically conditioned algebra. If $A$ satisfies the criterion that any $f \in L^1(M)^+$ which is in the annihilator of $A_0$ must belong to $L^1(D)$, then also

- any $f \in L^1(R)^+$ which is in the annihilator of $\hat{A}_0$ must belong to $L^1(\hat{D})$,
- and for any $n$, any $f \in L^1(R_n)^+$ which is in the annihilator of $(\hat{A}_n)_0$, must belong to $L^1(D_n)$. 
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Echoes of Szegö’s formula

**Theorem**

Let $A$ be an analytically conditioned algebra. Then the following are equivalent:

(i) $A$ is maximal subdiagonal, i.e. $A + A^*W^* = M$,

(ii) **Beurling**: For every right $A$-invariant subspace $X$ of $L^2(M)$, the right wandering subspace $W$ of $X$ satisfies $W^*W \subset L^1(D)$, and $W^*(X \ominus [WA]_2) = (0)$.

(iii) **Unique state extension**: The canonical embedding of $A + A_0^*$ into $L^2(M)$ is dense, and any $f \in L^1(M)^+$ which is in the annihilator of $A_0$ belongs to $L^1(D)$.

(iv) **Gleason-Whitney**: There is a unique Hahn-Banach extension to $M$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
Echoes of Szegö’s formula

Theorem

Let $A$ be an analytically conditioned algebra. Then the following are equivalent:

(i) $A$ is maximal subdiagonal, i.e. $A + A^* W^* = M$,

(ii) Beurling: For every right $A$-invariant subspace $X$ of $L^2(M)$, the right wandering subspace $W$ of $X$ satisfies $W^* W \subset L^1(\mathcal{D})$, and $W^* (X \ominus [WA]_2) = (0)$.

(iii) Unique state extension: The canonical embedding of $A + A_0^*$ into $L^2(M)$ is dense, and any $f \in L^1(M)^+$ which is in the annihilator of $A_0$ belongs to $L^1(\mathcal{D})$.

(iv) Gleason-Whitney: There is a unique Hahn-Banach extension to $M$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Echoes of Szegö’s formula

**Theorem**

Let $A$ be an analytically conditioned algebra. Then the following are equivalent:

(i) $A$ is maximal subdiagonal, i.e. $A + A^* w^* = M$,

(ii) **Beurling:** For every right $A$-invariant subspace $X$ of $L^2(M)$, the right wandering subspace $W$ of $X$ satisfies $W^* W \subset L^1(D)$, and $W^*(X \ominus [WA]_2) = (0)$.

(iii) **Unique state extension:** The canonical embedding of $A + A_0^*$ into $L^2(M)$ is dense, and any $f \in L^1(M)^+$ which is in the annihilator of $A_0$ belongs to $L^1(D)$.

(iv) **Gleason-Whitney:** There is a unique Hahn-Banach extension to $M$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
Invariant subspaces for $H^2$

Wildly noncommutative precepts

Echoes of Szegö’s formula

**Theorem**

Let $A$ be an analytically conditioned algebra. Then the following are equivalent:

(i) $A$ is maximal subdiagonal, i.e. $A + A^*W^* = M$,

(ii) **Beurling:** For every right $A$-invariant subspace $X$ of $L^2(M)$, the right wandering subspace $W$ of $X$ satisfies $W^*W \subset L^1(\mathcal{D})$, and $W^*(X \ominus [WA]_2) = (0)$.

(iii) **Unique state extension:** The canonical embedding of $A + A^*_0$ into $L^2(M)$ is dense, and any $f \in L^1(M)^+$ which is in the annihilator of $A_0$ belongs to $L^1(\mathcal{D})$.

(iv) **Gleason-Whitney:** There is a unique Hahn-Banach extension to $M$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
Let $A$ be an analytically conditioned algebra. Then the following are equivalent:

(i) $A$ is maximal subdiagonal, i.e. $A + A^w = M$,

(ii) **Beurling:** For every right $A$-invariant subspace $X$ of $L^2(M)$, the right wandering subspace $W$ of $X$ satisfies $W^* W \subset L^1(D)$, and $W^*(X \ominus [WA]_2) = (0)$.

(iii) **Unique state extension:** The canonical embedding of $A + A^*_0$ into $L^2(M)$ is dense, and any $f \in L^1(M)^+$ which is in the annihilator of $A_0$ belongs to $L^1(D)$.

(iv) **Gleason-Whitney:** There is a unique Hahn-Banach extension to $M$ of any weak* continuous functional on $A$, and this extension is weak* continuous.
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Echoes of Szegö: comments on the proof

* The proofs that $(i) \Rightarrow (ii) \Rightarrow (iii)$ follow by carefully adapting the proofs of the tracial case.
* To prove $(iii) \Rightarrow (i)$, one uses the lemma to conclude that each $\hat{A}_n = \hat{A} \cap R_n$ satisfies the unique state extension property. Now apply the tracial theory to conclude that each $\hat{A}_n$ is maximal subdiagonal. This involves comparing maximal subdiagonality with respect to $\phi|_{R_n}$ to maximal subdiagonality with respect to $\tau_n$. Use the weak* density of $\bigcup \hat{A}_n$ in $\hat{A}$ to conclude that $\hat{A} \subseteq R$ is maximal subdiagonal. From this it follows that $A = \Phi(\hat{A})$ is maximal subdiagonal.
* By a careful modification of the tracial arguments, one can show that maximal subdiagonality is equivalent to a weaker version of the Gleason-Whitney property. (The proof of equivalence to the full Gleason-Whitney property, requires a $\sigma$-finite version of Ueda’s peak set theorem. [BL-2017])
Echoes of Szegö: comments on the proof

- The proofs that \((i) \Rightarrow (ii) \Rightarrow (iii)\) follow by carefully adapting the proofs of the tracial case.

- To prove \((iii) \Rightarrow (i)\), one uses the lemma to conclude that each \(\hat{A}_n = \hat{A} \cap R_n\) satisfies the unique state extension property. Now apply the tracial theory to conclude that each \(\hat{A}_n\) is maximal subdiagonal. This involves comparing maximal subdiagonality with respect to \(\hat{\varphi}|_{R_n}\) to maximal subdiagonality with respect to \(\tau_n\). Use the weak* density of \(\bigcup \hat{A}_n\) in \(\hat{A}\) to conclude that \(\hat{A} \subset R\) is maximal subdiagonal. From this it follows that \(A = \Phi(\hat{A})\) is maximal subdiagonal.

- By a careful modification of the tracial arguments, one can show that maximal subdiagonality is equivalent to a weaker version of the Gleason-Whitney property. (The proof of equivalence to the full Gleason-Whitney property, requires a \(\sigma\)-finite version of Ueda's peak set theorem. [BL-2017])
Invariant subspaces for $H^2$
Wildly noncommutative precepts

Echoes of Szegö: comments on the proof

- The proofs that $(i) \Rightarrow (ii) \Rightarrow (iii)$ follow by carefully adapting the proofs of the tracial case.
- To prove $(iii) \Rightarrow (i)$, one uses the lemma to conclude that each $\hat{A}_n = \hat{A} \cap R_n$ satisfies the unique state extension property. Now apply the tracial theory to conclude that each $\hat{A}_n$ is maximal subdiagonal. This involves comparing maximal subdiagonality with respect to $\hat{\varphi}|_{R_n}$ to maximal subdiagonality with respect to $\tau_n$. Use the weak* density of $\bigcup \hat{A}_n$ in $\hat{A}$ to conclude that $\hat{A} \subset R$ is maximal subdiagonal. From this it follows that $A = \Phi(\hat{A})$ is maximal subdiagonal.
- By a careful modification of the tracial arguments, one can show that maximal subdiagonality is equivalent to a weaker version of the Gleason-Whitney property. (The proof of equivalence to the full Gleason-Whitney property, requires a $\sigma$-finite version of Ueda’s peak set theorem. [BL-2017])
Echoes of Szegö: comments on the proof

- The proofs that $(i) \Rightarrow (ii) \Rightarrow (iii)$ follow by carefully adapting the proofs of the tracial case.
- To prove $(iii) \Rightarrow (i)$, one uses the lemma to conclude that each $\hat{A}_n = \hat{A} \cap R_n$ satisfies the unique state extension property. Now apply the tracial theory to conclude that each $\hat{A}_n$ is maximal subdiagonal. This involves comparing maximal subdiagonality with respect to $\hat{\varphi}|_{R_n}$ to maximal subdiagonality with respect to $\tau_n$. Use the weak* density of $\bigcup \hat{A}_n$ in $\hat{A}$ to conclude that $\hat{A} \subset R$ is maximal subdiagonal. From this it follows that $A = \Phi(\hat{A})$ is maximal subdiagonal.
- By a careful modification of the tracial arguments, one can show that maximal subdiagonality is equivalent to a weaker version of the Gleason-Whitney property. (The proof of equivalence to the full Gleason-Whitney property, requires a $\sigma$-finite version of Ueda’s peak set theorem, [BL-2017]).
Echoes of Szegö: comments on the proof

- The proofs that $(i) \Rightarrow (ii) \Rightarrow (iii)$ follow by carefully adapting the proofs of the tracial case.

To prove $(iii) \Rightarrow (i)$, one uses the lemma to conclude that each $\hat{A}_n = \hat{A} \cap R_n$ satisfies the unique state extension property. Now apply the tracial theory to conclude that each $\hat{A}_n$ is maximal subdiagonal. This involves comparing maximal subdiagonality with respect to $\hat{\varphi} |_{R_n}$ to maximal subdiagonality with respect to $\tau_n$. Use the weak* density of $\bigcup \hat{A}_n$ in $\hat{A}$ to conclude that $\hat{A} \subset R$ is maximal subdiagonal.

From this it follows that $A = \Phi(\hat{A})$ is maximal subdiagonal.

By a careful modification of the tracial arguments, one can show that maximal subdiagonality is equivalent to a weaker version of the Gleason-Whitney property. (The proof of equivalence to the full Gleason-Whitney property, requires a $\sigma$-finite version of Ueda’s peak set theorem. [BL-2017])
Echoes of Szegö: comments on the proof

- The proofs that $(i) \Rightarrow (ii) \Rightarrow (iii)$ follow by carefully adapting the proofs of the tracial case.

- To prove $(iii) \Rightarrow (i)$, one uses the lemma to conclude that each $\hat{A}_n = \hat{A} \cap R_n$ satisfies the unique state extension property. Now apply the tracial theory to conclude that each $\hat{A}_n$ is maximal subdiagonal. This involves comparing maximal subdiagonality with respect to $\hat{\varphi}|_{R_n}$ to maximal subdiagonality with respect to $\tau_n$. Use the weak* density of $\bigcup \hat{A}_n$ in $\hat{A}$ to conclude that $\hat{A} \subset R$ is maximal subdiagonal. From this it follows that $A = \Phi(\hat{A})$ is maximal subdiagonal.

- By a careful modification of the tracial arguments, one can show that maximal subdiagonality is equivalent to a weaker version of the Gleason-Whitney property. (The proof of equivalence to the full Gleason-Whitney property, requires a $\sigma$-finite version of Ueda’s peak set theorem. [BL-2017])
Echoes of Szegö: comments on the proof

- The proofs that \((i) \Rightarrow (ii) \Rightarrow (iii)\) follow by carefully adapting the proofs of the tracial case.
- To prove \((iii) \Rightarrow (i)\), one uses the lemma to conclude that each \(\hat{A}_n = \hat{A} \cap R_n\) satisfies the unique state extension property. Now apply the tracial theory to conclude that each \(\hat{A}_n\) is maximal subdiagonal. This involves comparing maximal subdiagonality with respect to \(\hat{\phi}|_{R_n}\) to maximal subdiagonality with respect to \(\tau_n\). Use the weak* density of \(\bigcup \hat{A}_n\) in \(\hat{A}\) to conclude that \(\hat{A} \subset R\) is maximal subdiagonal. From this it follows that \(A = \Phi(\hat{A})\) is maximal subdiagonal.
- By a careful modification of the tracial arguments, one can show that maximal subdiagonality is equivalent to a weaker version of the Gleason-Whitney property. (The proof of equivalence to the full Gleason-Whitney property, requires a \(\sigma\)-finite version of Ueda’s peak set theorem. [BL-2017]
Invariant subspaces for $H^2$
Wildly noncommutative precepts

**Echoes of Szegö: comments on the proof**

- The proofs that $(i) \Rightarrow (ii) \Rightarrow (iii)$ follow by carefully adapting the proofs of the tracial case.

- To prove $(iii) \Rightarrow (i)$, one uses the lemma to conclude that each $\hat{A}_n = \hat{A} \cap R_n$ satisfies the unique state extension property. Now apply the tracial theory to conclude that each $\hat{A}_n$ is maximal subdiagonal. This involves comparing maximal subdiagonality with respect to $\hat{\varphi}|_{R_n}$ to maximal subdiagonality with respect to $\tau_n$. Use the weak* density of $\cup \hat{A}_n$ in $\hat{A}$ to conclude that $\hat{A} \subset R$ is maximal subdiagonal. From this it follows that $A = \Phi(\hat{A})$ is maximal subdiagonal.

- By a careful modification of the tracial arguments, one can show that maximal subdiagonality is equivalent to a weaker version of the Gleason-Whitney property. (The proof of equivalence to the full Gleason-Whitney property, requires a $\sigma$-finite version of Ueda’s peak set theorem. [BL-2017])
Echoes of Szegö: comments on the proof

- The proofs that (i) ⇒ (ii) ⇒ (iii) follow by carefully adapting the proofs of the tracial case.
- To prove (iii) ⇒ (i), one uses the lemma to conclude that each $\hat{A}_n = \hat{A} \cap R_n$ satisfies the unique state extension property. Now apply the tracial theory to conclude that each $\hat{A}_n$ is maximal subdiagonal. This involves comparing maximal subdiagonality with respect to $\hat{\varphi}|_{R_n}$ to maximal subdiagonality with respect to $\tau_n$. Use the weak* density of $\bigcup \hat{A}_n$ in $\hat{A}$ to conclude that $\hat{A} \subset R$ is maximal subdiagonal. From this it follows that $A = \Phi(\hat{A})$ is maximal subdiagonal.
- By a careful modification of the tracial arguments, one can show that maximal subdiagonality is equivalent to a weaker version of the Gleason-Whitney property. (The proof of equivalence to the full Gleason-Whitney property, requires a $\sigma$-finite version of Ueda’s peak set theorem. [BL-2017])
The proof of the lemma

- Given $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, note that then $\Phi(f) \in L^1(M)^+$ with $\Phi(f) \perp A_0$, and hence that $\Phi(f) \in L^1(D)^+$.
- Show that since $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, the same is true of each of $f_1 = \lambda^*_t f \lambda_t$, $f_2 = (\mathbb{1} + \lambda^*_t) f (\mathbb{1} + \lambda_t)$, and $f_3 = (\mathbb{1} - i \lambda^*_t) f (\mathbb{1} + i \lambda_t)$ ($t \in \mathbb{Q}_d$).
- Conclude that each of $\Phi(f_1)$, $\Phi(f_2)$ $\Phi(f_3)$ belong to $L^1(D)^+$, and use simple arithmetic to conclude that $\Phi(f \lambda_t) \in L^1(D)$ for each $t \in \mathbb{Q}_d$. That is $\tilde{\mathcal{E}}(\Phi(f \lambda_t)) = \Phi(f \lambda_t)$.
- Combine the above fact with the identities $tr_R \circ \Phi = tr_R$, $tr_R \circ \mathcal{E} = tr_R$ and $\mathcal{E} \circ \Phi = \Phi \circ \tilde{\mathcal{E}}$, to see that $tr_R(f \lambda_t b) = tr_R(\tilde{\mathcal{E}}(f) \lambda_t b)$ for all $b \in M$.
- Use the weak* density of $\{\lambda_t b : b \in M, t \in \mathbb{Q}_d\}$ in $R$ to conclude that $tr_R(f a) = tr_R(\tilde{\mathcal{E}}(f) a)$ for all $a \in R$, and hence that $f = \tilde{\mathcal{E}}(f)$ as required.
The proof of the lemma

Given $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, note that then
$\Phi(f) \in L^1(M)^+$ with $\Phi(f) \perp A_0$, and hence that
$\Phi(f) \in L^1(D)^+$.  

Show that since $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, the same is true of
each of $f_1 = \lambda^*_t f \lambda_t$, $f_2 = (1 + \lambda^*_t)f(1 + \lambda_t)$, and
$f_3 = (1 - i\lambda^*_t)f(1 + i\lambda_t)$ ($t \in \mathbb{Q}_d$).

Conclude that each of $\Phi(f_1)$, $\Phi(f_2)$, $\Phi(f_3)$ belong to $L^1(D)^+$,
and use simple arithmetic to conclude that $\Phi(f \lambda_t) \in L^1(D)$
for each $t \in \mathbb{Q}_d$. That is $\hat{E}(\Phi(f \lambda_t)) = \Phi(f \lambda_t)$.

Combine the above fact with the identities $tr_R \circ \Phi = tr_R$,
$tr_R \circ \hat{E} = tr_R$ and $\hat{E} \circ \Phi = \Phi \circ \hat{E}$, to see that
$tr_R(f \lambda_t b) = tr_R(\hat{E}(f) \lambda_t b)$ for all $b \in M$.

Use the weak* density of $\{\lambda_t b : b \in M, t \in \mathbb{Q}_d\}$ in $R$ to
conclude that $tr_R(f a) = tr_R(\hat{E}(f) a)$ for all $a \in R$, and hence
that $f = \hat{E}(f)$ as required.
The proof of the lemma

- Given $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, note that then $\Phi(f) \in L^1(M)^+$ with $\Phi(f) \perp A_0$, and hence that $\Phi(f) \in L^1(D)^+$.
- Show that since $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, the same is true of each of $f_1 = \lambda_t^* f \lambda_t$, $f_2 = (1 + \lambda_t^*) f (1 + \lambda_t)$, and $f_3 = (1 - i\lambda_t^*) f (1 + i\lambda_t)$ ($t \in \mathbb{Q}_d$).
- Conclude that each of $\Phi(f_1), \Phi(f_2), \Phi(f_3)$ belong to $L^1(D)^+$, and use simple arithmetic to conclude that $\Phi(f \lambda_t) \in L^1(D)$ for each $t \in \mathbb{Q}_d$. That is $\hat{E}(\Phi(f \lambda_t)) = \Phi(f \lambda_t)$.
- Combine the above fact with the identities $tr_R \circ \Phi = tr_R$, $tr_R \circ \mathcal{E} = tr_R$ and $\mathcal{E} \circ \Phi = \Phi \circ \hat{E}$, to see that $tr_R(f \lambda_t b) = tr_R(\hat{E}(f) \lambda_t b)$ for all $b \in M$.
- Use the weak* density of $\{\lambda_t b : b \in M, t \in \mathbb{Q}_d\}$ in $R$ to conclude that $tr_R(f a) = tr_R(\hat{E}(f) a)$ for all $a \in R$, and hence that $f = \hat{E}(f)$ as required.
The proof of the lemma

Given $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, note that then $\Phi(f) \in L^1(M)^+$ with $\Phi(f) \perp A_0$, and hence that $\Phi(f) \in L^1(D)^+$.

Show that since $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, the same is true of each of $f_1 = \lambda_t^* f \lambda_t$, $f_2 = (1 + \lambda_t^*) f (1 + \lambda_t)$, and $f_3 = (1 - i \lambda_t^*) f (1 + i \lambda_t)$ ($t \in \mathbb{Q}_d$).

Conclude that each of $\Phi(f_1)$, $\Phi(f_2)$, $\Phi(f_3)$ belong to $L^1(D)^+$, and use simple arithmetic to conclude that $\Phi(f \lambda_t) \in L^1(D)$ for each $t \in \mathbb{Q}_d$. That is $\hat{E}(\Phi(f \lambda_t)) = \Phi(f \lambda_t)$.

Combine the above fact with the identities $tr_R \circ \Phi = tr_R$, $tr_R \circ E = tr_R$ and $E \circ \Phi = \Phi \circ \hat{E}$, to see that $tr_R(f \lambda_t b) = tr_R(\hat{E}(f) \lambda_t b)$ for all $b \in M$.

Use the weak* density of $\{\lambda_t b : b \in M, t \in \mathbb{Q}_d\}$ in $R$ to conclude that $tr_R(f a) = tr_R(\hat{E}(f) a)$ for all $a \in R$, and hence that $f = \hat{E}(f)$ as required.
The proof of the lemma

- Given \( f \in L^1(R)^+ \) with \( f \perp \hat{A}_0 \), note that then \( \Phi(f) \in L^1(M)^+ \) with \( \Phi(f) \perp A_0 \), and hence that \( \Phi(f) \in L^1(D)^+ \).
- Show that since \( f \in L^1(R)^+ \) with \( f \perp \hat{A}_0 \), the same is true of each of \( f_1 = \lambda^*_t f \lambda_t \), \( f_2 = (1 + \lambda^*_t) f(1 + \lambda_t) \), and \( f_3 = (1 - i\lambda^*_t) f(1 + i\lambda_t) \) (\( t \in \mathbb{Q}_d \)).
- Conclude that each of \( \Phi(f_1), \Phi(f_2), \Phi(f_3) \) belong to \( L^1(D)^+ \), and use simple arithmetic to conclude that \( \Phi(f\lambda_t) \in L^1(D) \) for each \( t \in \mathbb{Q}_d \). That is \( \hat{E}(\Phi(f\lambda_t)) = \Phi(f\lambda_t) \).
- Combine the above fact with the identities \( tr_R \circ \Phi = tr_R \), \( tr_R \circ E = tr_R \) and \( E \circ \Phi = \Phi \circ \hat{E} \), to see that \( tr_R(f\lambda_t b) = tr_R(\hat{E}(f)\lambda_t b) \) for all \( b \in M \).
- Use the weak* density of \( \{ \lambda_t b : b \in M, t \in \mathbb{Q}_d \} \) in \( R \) to conclude that \( tr_R(fa) = tr_R(\hat{E}(f)a) \) for all \( a \in R \), and hence that \( f = \hat{E}(f) \) as required.
The proof of the lemma

- Given $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, note that then $\Phi(f) \in L^1(M)^+$ with $\Phi(f) \perp A_0$, and hence that $\Phi(f) \in L^1(D)^+$.
- Show that since $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, the same is true of each of $f_1 = \lambda_t^*f\lambda_t$, $f_2 = (1 + \lambda_t^*)f(1 + \lambda_t)$, and $f_3 = (1 - i\lambda_t^*)f(1 + i\lambda_t)$ ($t \in \mathbb{Q}_d$).
- Conclude that each of $\Phi(f_1)$, $\Phi(f_2)$, $\Phi(f_3)$ belong to $L^1(D)^+$, and use simple arithmetic to conclude that $\Phi(f\lambda_t) \in L^1(D)$ for each $t \in \mathbb{Q}_d$. That is $\hat{\mathcal{E}}(\Phi(f\lambda_t)) = \Phi(f\lambda_t)$.
- Combine the above fact with the identities $tr_R \circ \Phi = tr_R$, $tr_R \circ \mathcal{E} = tr_R$ and $\mathcal{E} \circ \Phi = \Phi \circ \hat{\mathcal{E}}$, to see that $tr_R(f\lambda_t b) = tr_R(\hat{\mathcal{E}}(f)\lambda_t b)$ for all $b \in M$.
- Use the weak* density of $\{\lambda_t b : b \in M, t \in \mathbb{Q}_d\}$ in $R$ to conclude that $tr_R(fa) = tr_R(\hat{\mathcal{E}}(f)a)$ for all $a \in R$, and hence that $f = \hat{\mathcal{E}}(f)$ as required.
The proof of the lemma

- Given \( f \in L^1(R)^+ \) with \( f \perp \hat{A}_0 \), note that then \( \Phi(f) \in L^1(M)^+ \) with \( \Phi(f) \perp A_0 \), and hence that \( \Phi(f) \in L^1(D)^+ \).
- Show that since \( f \in L^1(R)^+ \) with \( f \perp \hat{A}_0 \), the same is true of each of \( f_1 = \lambda_t^*f\lambda_t, f_2 = (1 + \lambda_t^*)f(1 + \lambda_t) \), and \( f_3 = (1 - i\lambda_t^*)f(1 + i\lambda_t) \) (\( t \in \mathbb{Q}_d \)).
- Conclude that each of \( \Phi(f_1), \Phi(f_2) \Phi(f_3) \) belong to \( L^1(D)^+ \), and use simple arithmetic to conclude that \( \Phi(f\lambda_t) \in L^1(D) \) for each \( t \in \mathbb{Q}_d \). That is \( \hat{\mathcal{E}}(\Phi(f\lambda_t)) = \Phi(f\lambda_t) \).
- Combine the above fact with the identities \( tr_R \circ \Phi = tr_R \), \( tr_R \circ \mathcal{E} = tr_R \) and \( \mathcal{E} \circ \Phi = \Phi \circ \hat{\mathcal{E}} \), to see that \( tr_R(f\lambda_t b) = tr_R(\hat{\mathcal{E}}(f)\lambda_t b) \) for all \( b \in M \).
- Use the weak* density of \( \{\lambda_t b : b \in M, t \in \mathbb{Q}_d\} \) in \( R \) to conclude that \( tr_R(fa) = tr_R(\hat{\mathcal{E}}(f)a) \) for all \( a \in R \), and hence that \( f = \hat{\mathcal{E}}(f) \) as required.
The proof of the lemma

- Given \( f \in L^1(R)^+ \) with \( f \perp \hat{A}_0 \), note that then \( \Phi(f) \in L^1(M)^+ \) with \( \Phi(f) \perp A_0 \), and hence that \( \Phi(f) \in L^1(D)^+ \).
- Show that since \( f \in L^1(R)^+ \) with \( f \perp \hat{A}_0 \), the same is true of each of \( f_1 = \lambda_t^* f \lambda_t \), \( f_2 = (1 + \lambda_t^*) f (1 + \lambda_t) \), and \( f_3 = (1 - i\lambda_t^*) f (1 + i\lambda_t) \) (\( t \in \mathbb{Q}_d \)).
- Conclude that each of \( \Phi(f_1) \), \( \Phi(f_2) \) \( \Phi(f_3) \) belong to \( L^1(D)^+ \), and use simple arithmetic to conclude that \( \Phi(f \lambda_t) \in L^1(D) \) for each \( t \in \mathbb{Q}_d \). That is \( \hat{E}(\Phi(f \lambda_t)) = \Phi(f \lambda_t) \).
- Combine the above fact with the identities \( tr_R \circ \Phi = tr_R \), \( tr_R \circ \mathcal{E} = tr_R \) and \( \mathcal{E} \circ \Phi = \Phi \circ \hat{E} \), to see that \( tr_R(f \lambda_t b) = tr_R(\hat{E}(f) \lambda_t b) \) for all \( b \in M \).
- Use the weak* density of \( \{ \lambda_t b : b \in M, t \in \mathbb{Q}_d \} \) in \( R \) to conclude that \( tr_R(f a) = tr_R(\hat{E}(f) a) \) for all \( a \in R \), and hence that \( f = \hat{E}(f) \) as required.
The proof of the lemma

- Given \( f \in L^1(R)^+ \) with \( f \perp \hat{A}_0 \), note that then \( \Phi(f) \in L^1(M)^+ \) with \( \Phi(f) \perp A_0 \), and hence that \( \Phi(f) \in L^1(D)^+ \).
- Show that since \( f \in L^1(R)^+ \) with \( f \perp \hat{A}_0 \), the same is true of each of \( f_1 = \lambda_t^* f \lambda_t \), \( f_2 = (1 + \lambda_t^*) f (1 + \lambda_t) \), and \( f_3 = (1 - i \lambda_t^*) f (1 + i \lambda_t) \) (\( t \in \mathbb{Q}_d \)).
- Conclude that each of \( \Phi(f_1) \), \( \Phi(f_2) \) \( \Phi(f_3) \) belong to \( L^1(D)^+ \), and use simple arithmetic to conclude that \( \Phi(f \lambda_t) \in L^1(D) \) for each \( t \in \mathbb{Q}_d \). That is \( \hat{E}(\Phi(f \lambda_t)) = \Phi(f \lambda_t) \).
- Combine the above fact with the identities \( tr_R \circ \Phi = tr_R \), \( tr_R \circ \mathcal{E} = tr_R \) and \( \mathcal{E} \circ \Phi = \Phi \circ \hat{E} \), to see that \( tr_R(f \lambda_t b) = tr_R(\hat{E}(f) \lambda_t b) \) for all \( b \in M \).
- Use the weak* density of \( \{ \lambda_t b : b \in M, t \in \mathbb{Q}_d \} \) in \( R \) to conclude that \( tr_R(fa) = tr_R(\hat{E}(f)a) \) for all \( a \in R \), and hence that \( f = \hat{E}(f) \) as required.
Invariant subspaces for $H^2$

Wildly noncommutative precepts

The proof of the lemma

- Given $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, note that then $\Phi(f) \in L^1(M)^+$ with $\Phi(f) \perp A_0$, and hence that $\Phi(f) \in L^1(D)^+$.

- Show that since $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, the same is true of each of $f_1 = \lambda^*_t f \lambda_t$, $f_2 = (1 + \lambda^*_t) f(1 + \lambda_t)$, and $f_3 = (1 - i\lambda^*_t) f(1 + i\lambda_t)$ ($t \in \mathbb{Q}_d$).

- Conclude that each of $\Phi(f_1)$, $\Phi(f_2)$ $\Phi(f_3)$ belong to $L^1(D)^+$, and use simple arithmetic to conclude that $\Phi(f \lambda_t) \in L^1(D)$ for each $t \in \mathbb{Q}_d$. That is $\hat{E}(\Phi(f \lambda_t)) = \Phi(f \lambda_t)$.

- Combine the above fact with the identities $tr_R \circ \Phi = tr_R$, $tr_R \circ \mathcal{E} = tr_R$ and $\mathcal{E} \circ \Phi = \Phi \circ \hat{E}$, to see that $tr_R(f \lambda_t b) = tr_R(\hat{E}(f) \lambda_t b)$ for all $b \in M$.

- Use the weak* density of $\{\lambda_t b : b \in M, t \in \mathbb{Q}_d\}$ in $R$ to conclude that $tr_R(f a) = tr_R(\hat{E}(f) a)$ for all $a \in R$, and hence that $f = \hat{E}(f)$ as required.
Invariant subspaces for $H^2$
Wildly noncommutative precepts

The proof of the lemma

- Given $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, note that then $\Phi(f) \in L^1(M)^+$ with $\Phi(f) \perp A_0$, and hence that $\Phi(f) \in L^1(D)^+$.  

- Show that since $f \in L^1(R)^+$ with $f \perp \hat{A}_0$, the same is true of each of $f_1 = \lambda_t^* f \lambda_t$, $f_2 = (1 + \lambda_t^*) f (1 + \lambda_t)$, and $f_3 = (1 - i\lambda_t^*) f (1 + i\lambda_t)$ ($t \in Q_d$).  

- Conclude that each of $\Phi(f_1)$, $\Phi(f_2)$ $\Phi(f_3)$ belong to $L^1(D)^+$, and use simple arithmetic to conclude that $\Phi(f \lambda_t) \in L^1(D)$ for each $t \in Q_d$. That is $\hat{E}(\Phi(f \lambda_t)) = \Phi(f \lambda_t)$.  

- Combine the above fact with the identities $tr_R \circ \Phi = tr_R$, $tr_R \circ E = tr_R$ and $E \circ \Phi = \Phi \circ \hat{E}$, to see that $tr_R(f \lambda_t b) = tr_R(\hat{E}(f) \lambda_t b)$ for all $b \in M$.  

- Use the weak* density of $\{ \lambda_t b : b \in M, t \in Q_d \}$ in $R$ to conclude that $tr_R(fa) = tr_R(\hat{E}(f) a)$ for all $a \in R$, and hence that $f = \hat{E}(f)$ as required.
The proof of the lemma

- Given \( f \in L^1(R)^+ \) with \( f \perp \hat{A}_0 \), note that then \( \Phi(f) \in L^1(M)^+ \) with \( \Phi(f) \perp A_0 \), and hence that \( \Phi(f) \in L^1(D)^+ \).

- Show that since \( f \in L^1(R)^+ \) with \( f \perp \hat{A}_0 \), the same is true of each of \( f_1 = \lambda^*_t f \lambda_t \), \( f_2 = (1 + \lambda^*_t) f (1 + \lambda_t) \), and \( f_3 = (1 - i\lambda^*_t) f (1 + i\lambda_t) \) (\( t \in \mathbb{Q}_d \)).

- Conclude that each of \( \Phi(f_1) \), \( \Phi(f_2) \), \( \Phi(f_3) \) belong to \( L^1(D)^+ \), and use simple arithmetic to conclude that \( \Phi(f \lambda_t) \in L^1(D) \) for each \( t \in \mathbb{Q}_d \). That is \( \hat{E}(\Phi(f \lambda_t)) = \Phi(f \lambda_t) \).

- Combine the above fact with the identities \( tr_R \circ \Phi = tr_R \), \( tr_R \circ \mathcal{E} = tr_R \) and \( \mathcal{E} \circ \Phi = \Phi \circ \hat{E} \), to see that \( tr_R(f \lambda_t b) = tr_R(\hat{E}(f) \lambda_t b) \) for all \( b \in M \).

- Use the weak* density of \( \{ \lambda_t b : b \in M, t \in \mathbb{Q}_d \} \) in \( R \) to conclude that \( tr_R(fa) = tr_R(\hat{E}(f)a) \) for all \( a \in R \), and hence that \( f = \hat{E}(f) \) as required.
Invariant subspaces for $H^2$
A very general Beurling Theorem

Types of invariant subspaces

Let $A$ be maximal subdiagonal and $K \subset L^2(M)$ a closed right-invariant subspace. Given such an invariant subspace, we call

- $W = K \ominus [KA_0]_2$ the right-wandering subspace of $K$
- $K$ type 1 if $[WA]_2 = K$
- $K$ type 2 if $W = \{0\}$
Types of invariant subspaces

Let $A$ be maximal subdiagonal and $K \subset L^2(M)$ a closed right-invariant subspace. Given such an invariant subspace, we call

- $W = K \ominus [KA_0]_2$ the right-wandering subspace of $K$
- $K$ type 1 if $[WA]_2 = K$
- $K$ type 2 if $W = \{0\}$
Types of invariant subspaces

Let $A$ be maximal subdiagonal and $K \subset L^2(M)$ a closed right-invariant subspace. Given such an invariant subspace, we call

- $W = K \ominus [KA_0]_2$ the right-wandering subspace of $K$.
- $K$ type 1 if $[WA]_2 = K$;
- $K$ type 2 if $W = \{0\}$. 


Types of invariant subspaces

Let $A$ be maximal subdiagonal and $K \subset L^2(M)$ a closed right-invariant subspace. Given such an invariant subspace, we call

- $W = K \ominus [KA_0]_2$ the right-wandering subspace of $K$
- $K$ type 1 if $[WA]_2 = K$
- $K$ type 2 if $W = \{0\}$
Invariant subspace theorem


*If A is a maximal subdiagonal subalgebra of M and K is a closed right A-invariant subspace of $L^2(M)$, then:*

1. $K$ may be written uniquely as an (internal) $L^2$-column sum $K_1 \oplus^{\text{col}} K_2$ of a type 1 and a type 2 invariant subspace of $L^2(M)$, respectively.

2. If $K \neq (0)$ then $K$ is type 1 if and only if $K = \bigoplus_i^{\text{col}} u_i H^2$, for $u_i$ partial isometries with mutually orthogonal ranges and $|u_i| \in D$.

3. The right wandering subspace $W$ of $K$ is an $L^2(D)$-module in the sense of Junge and Sherman, and in particular $W^* W \subset L^1(D)$. 
Invariant subspaces for $H^2$
A very general Beurling Theorem

Invariant subspace theorem


If $A$ is a maximal subdiagonal subalgebra of $M$ and $K$ is a closed right $A$-invariant subspace of $L^2(M)$, then:

1. $K$ may be written uniquely as an (internal) $L^2$-column sum $K_1 \oplus^{col} K_2$ of a type 1 and a type 2 invariant subspace of $L^2(M)$, respectively.

2. If $K \neq (0)$ then $K$ is type 1 if and only if $K = \oplus^\text{col} u_i H^2$, for $u_i$ partial isometries with mutually orthogonal ranges and $|u_i| \in \mathcal{D}$.

3. The right wandering subspace $W$ of $K$ is an $L^2(\mathcal{D})$-module in the sense of Junge and Sherman, and in particular $W^* W \subset L^1(\mathcal{D})$. 
Invariant subspaces for $H^2$

A very general Beurling Theorem

**Invariant subspace theorem**


*If $A$ is a maximal subdiagonal subalgebra of $M$ and $K$ is a closed right $A$-invariant subspace of $L^2(M)$, then:*

1. $K$ may be written uniquely as an (internal) $L^2$-column sum $K_1 \oplus^\text{col} K_2$ of a type 1 and a type 2 invariant subspace of $L^2(M)$, respectively.

2. If $K \neq (0)$ then $K$ is type 1 if and only if $K = \oplus_i^\text{col} u_i H^2$, for $u_i$ partial isometries with mutually orthogonal ranges and $|u_i| \in D$.

3. The right wandering subspace $W$ of $K$ is an $L^2(D)$-module in the sense of Junge and Sherman, and in particular $W^* W \subset L^1(D)$. 
Invariant subspaces for $H^2$

A very general Beurling Theorem

**Invariant subspace theorem**


*If $A$ is a maximal subdiagonal subalgebra of $M$ and $K$ is a closed right $A$-invariant subspace of $L^2(M)$, then:*

1. $K$ may be written uniquely as an (internal) $L^2$-column sum $K_1 \oplus^\text{col} K_2$ of a type 1 and a type 2 invariant subspace of $L^2(M)$, respectively.

2. If $K \neq (0)$ then $K$ is type 1 if and only if $K = \bigoplus_i^{\text{col}} u_i H^2$, for $u_i$ partial isometries with mutually orthogonal ranges and $|u_i| \in \mathcal{D}$.

3. The right wandering subspace $W$ of $K$ is an $L^2(\mathcal{D})$-module in the sense of Junge and Sherman, and in particular $W^* W \subset L^1(\mathcal{D})$. 
Invariant subspaces for $H^2$
A very general Beurling Theorem

A Beurling theorem

In the general case we similarly need $W^* W$ to be “all of $D$” in some sense in order to get a decent Beurling theorem.


Let $A$ be a maximal subdiagonal subalgebra of $M$ and $K$ a closed right $A$-invariant subspace of $L^2(M)$.

1. Then $K$ is type 1 if and only if $K = \bigoplus u_i H^2$, for $u_i$ partial isometries with mutually orthogonal ranges and $|u_i| \in D$.

2. $K$ is of the form $uH^2$ for a unitary $u \in M$, if and only if the right wandering subspace of $K$ has a nonzero separating and cyclic vector for the right action of $D$ (i.e. $W \neq \{0\}$ and there exists $\xi \in W$ such that $d \mapsto \xi d$ is one-to-one on $D$, and $\xi D$ dense in $W$).
A Beurling theorem

In the general case we similarly need $W^*W$ to be “all of $D$” in some sense in order to get a decent Beurling theorem.


Let $A$ be a maximal subdiagonal subalgebra of $M$ and $K$ a closed right $A$-invariant subspace of $L^2(M)$.

1. Then $K$ is type 1 if and only if $K = \bigoplus_{i} u_i \, H^2$, for $u_i$ partial isometries with mutually orthogonal ranges and $|u_i| \in \mathcal{D}$.

2. $K$ is of the form $uH^2$ for a unitary $u \in M$, if and only if the right wandering subspace of $K$ has a nonzero separating and cyclic vector for the right action of $\mathcal{D}$ (i.e. $W \neq \{0\}$ and there exists $\xi \in W$ such that $d \mapsto \xi d$ is one-to-one on $\mathcal{D}$, and $\xi D$ dense in $W$).
A Beurling theorem

In the general case we similarly need $W^* W$ to be “all of $D$” in some sense in order to get a decent Beurling theorem.


Let $A$ be a maximal subdiagonal subalgebra of $M$ and $K$ a closed right $A$-invariant subspace of $L^2(M)$.

1. Then $K$ is type 1 if and only if $K = \bigoplus_{i}^{\text{col}} u_i H^2$, for $u_i$ partial isometries with mutually orthogonal ranges and $|u_i| \in D$.
2. $K$ is of the form $uH^2$ for a unitary $u \in M$, if and only if the right wandering subspace of $K$ has a nonzero separating and cyclic vector for the right action of $D$ (i.e. $W \neq \{0\}$ and there exists $\xi \in W$ such that $d \mapsto \xi d$ is one-to-one on $D$, and $\xi D$ dense in $W$).
A Beurling theorem

In the general case we similarly need $W^*W$ to be “all of $D$” in some sense in order to get a decent Beurling theorem.


Let $A$ be a maximal subdiagonal subalgebra of $M$ and $K$ a closed right $A$-invariant subspace of $L^2(M)$.

1. Then $K$ is type 1 if and only if $K = \bigoplus_i^\text{col} u_i H^2$, for $u_i$ partial isometries with mutually orthogonal ranges and $|u_i| \in D$.

2. $K$ is of the form $uH^2$ for a unitary $u \in M$, if and only if the right wandering subspace of $K$ has a nonzero separating and cyclic vector for the right action of $D$ (i.e. $W \neq \{0\}$ and there exists $\xi \in W$ such that $d \mapsto \xi d$ is one-to-one on $D$, and $\xi D$ dense in $W$).
Invariant subspaces for $H^2$
A very general Beurling Theorem

A Beurling theorem

In the general case we similarly need $W^*W$ to be “all of $D$” in some sense in order to get a decent Beurling theorem.


Let $A$ be a maximal subdiagonal subalgebra of $M$ and $K$ a closed right $A$-invariant subspace of $L^2(M)$.

1. Then $K$ is type 1 if and only if $K = \bigoplus_i^\text{col} u_i H^2$, for $u_i$ partial isometries with mutually orthogonal ranges and $|u_i| \in \mathcal{D}$.

2. $K$ is of the form $uH^2$ for a unitary $u \in M$, if and only if the right wandering subspace of $K$ has a nonzero separating and cyclic vector for the right action of $\mathcal{D}$ (i.e. $W \neq \{0\}$ and there exists $\xi \in W$ such that $d \mapsto \xi d$ is one-to-one on $\mathcal{D}$, and $\xi \mathcal{D}$ dense in $W$).
Invariant subspaces for $H^2$

A very general Beurling Theorem

**A Beurling theorem**

In the general case we similarly need $W^*W$ to be "all of $D$" in some sense in order to get a decent Beurling theorem.


Let $A$ be a maximal subdiagonal subalgebra of $M$ and $K$ a closed right $A$-invariant subspace of $L^2(M)$.

1. Then $K$ is type 1 if and only if $K = \bigoplus_i \text{col} u_i H^2$, for $u_i$ partial isometries with mutually orthogonal ranges and $|u_i| \in D$.

2. $K$ is of the form $uH^2$ for a unitary $u \in M$, if and only if the right wandering subspace of $K$ has a nonzero separating and cyclic vector for the right action of $D$ (i.e. $W \neq \{0\}$ and there exists $\xi \in W$ such that $d \mapsto \xi d$ is one-to-one on $D$, and $\xi D$ dense in $W$).