Invariant subspaces for H^2 spaces of σ-finite algebras

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Outline

H^p spaces of the disc

Let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$ and $\mathbb{T}$ the unit circle.

Definition

$f : \mathbb{D} \to \mathbb{C}$ belongs to $H^\infty(\mathbb{D})$ iff f is analytic and bounded on $\mathbb{D}$.

The space $H^\infty(\mathbb{D})$ may be realised as a subspace of $L^\infty(\mathbb{T})$ by the following process:

- By taking radial limits every $f \in H^\infty$ defines a corresponding function $\tilde{f}$ on $\mathbb{T}$;
- By an extension of the Cauchy Integration formulae, f may similarly be recovered from $\tilde{f}$;
- $\sup_{z\in\mathbb{D}} |f(z)| = \operatorname{ess\,sup}_{z\in\mathbb{T}} |\tilde{f}(z)|$.

$H^p(\mathbb{T})$ is then simply the closure in $L^p(\mathbb{T})$ of $H^\infty(\mathbb{T})$.

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Wk* Dirichlet algebras

Let X be a probability space. A weak* closed unital-subalgebra A of $L^\infty(X)$, is called wk* sub-Dirichlet if:

$$\int fg = \int f \int g, \qquad f, g \in A. \tag{1}$$

A exhibits H^∞-like behaviour iff $A + \bar{A}$ is wk* dense in $L^\infty(X)$. Such algebras are called wk* Dirichlet algebras.

- When this condition holds we will write $H^\infty(A)$ for A, and $H^p(A)$ $(1 \le p < \infty)$ for the closure of A in $L^p(X)$. More generally $[S]_p$ will be the norm-closure of $S \subset L^p$ in the p-norm.
- For $A_0 = \{f \in A : \int f = 0\}$, similarly write $H^p_0(A)$ for $[A_0]_p$.

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Fun facts about A

For wk* sub-Dirichlet algebras the following are equivalent:

- $A + \bar{A}$ is wk* dense in $L^\infty(X)$.
- Validity of Szegö's formula: $\forall g \in L^1_+(X)$, $\exp \int \log g = \inf\{\int |1 - f|^2 g : f \in A, \int f = 0\}$.
- Unique state extension: If $g \in L^1(X)$ is nonnegative with $\int fg = \int f$ for all $f \in A$, then $g = 1$ a.e.
- Gleason-Whitney property: there is a unique Hahn-Banach extension to $L^\infty(X)$ of any weak* continuous functional on A, and this extension is weak* continuous.
- Beurling's theorem: every simply A-invariant subspace K of $L^2(X)$, is of the form $u[A]_2$ for some unimodular u.
- Plus about 6 other conditions.

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Quantising $L^\infty(\mathbb{T})$

Context:

- M a (semi)finite von Neumann algebra, equipped with a faithful normal (semi)finite trace $\tau_M = \tau$. (Example: $B(H)$ equipped with Tr.)
- $\widetilde{M}$ the τ_M-measurable operators affiliated to M, i.e. all operators a affiliated to M, such that for every $\varepsilon > 0$ there exists a projection $e \in M$ with $\tau(\mathbb{1} - e) \leq \varepsilon$, and $ae \in M$.

Dictionary: $M = L^\infty(M, \tau)$ and $L^p(M, \tau) = \{a \in \widetilde{M} : \tau(|a|^p) < \infty\}$ for $p > 0$. It turns out that $M_* \equiv L^1(M, \tau)$.

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Tracial subalgebras

Let M be a finite von Neumann algebra equipped with a faithful normal tracial state τ_M.

A tracial subalgebra of M is a wk* closed unital subalgebra A of M for which the trace preserving faithful normal conditional expectation $\mathcal{E} : M \to A \cap A^* = D$ satisfies:

$$\mathcal{E}(a_1 a_2) = \mathcal{E}(a_1)\,\mathcal{E}(a_2), \quad a_1, a_2 \in A. \tag{2}$$

A tracial subalgebra for which $A + A^*$ is weak* dense in M, is maximal as a tracial subalgebra (Exel, 1988). The tracial subalgebras satisfying this weak* density criterion are said to be *finite maximal subdiagonal subalgebras* These are our noncommutative H^∞'s.

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Fun facts about tracial subalgebras

For any tracial subalgebra A of M, the following are equivalent (BL):

- $\overline{A + A^*}^{w*} = M$.
- A satisfies a Szegö-like formula formulated in terms of the Fuglede-Kadison determinant.
- Unique state extension: $A + A_0^*$ is a dense subspace of $L^2(M)$, and any $g \in L^1(M)_+$ in the annihilator of A_0, is in $L^1(D)$.
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For any tracial subalgebra A of M, the following are equivalent (BL):

- $\overline{A + A^*}^{w*} = M$.
- A satisfies a Szegö-like formula formulated in terms of the Fuglede-Kadison determinant.
- Unique state extension: $A + A_0^*$ is a dense subspace of $L^2(M)$, and any $g \in L^1(M)_+$ in the annihilator of A_0, is in $L^1(D)$.
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A von Neumann algebra M equipped with an *fns* weight ν, M embeds into a semifinite von Neumann algebra $N = M \rtimes_\nu \mathbb{R}$ admitting an operator valued weight $T : \widehat{N} \to \widehat{M}$ and a one-parameter group of *-automorphisms $\{\theta_s\}$ ($s \in \mathbb{R}$) for which

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H^∞-spaces : general case 1

Let $D \subset M$ be a von Neumann subalgebra of a σ-finite von Neumann algebra M equipped with a faithful normal state ν, which admits a weak*-continuous contractive projection $\mathcal{E}$ onto D satisfying $\nu \circ \mathcal{E} = \nu$. (The canonical conditional expectation from M onto D.)

As before we say that a unital weak* closed subalgebra $A \subset M$ is subdiagonal with respect to $\mathcal{E}$ if

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Theorem (Xu 2005; Ji–Ohwada–Saito 1998)

Let M be σ-finite, and $A \subset M$ subdiagonal. Then A is maximal subdiagonal if and only if $\sigma_t^\nu(A) = A$ for all $t \in \mathbb{R}$.

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Analytically conditioned subalgebras

Based on the maximality criteria for the σ-finite case, we say that a weak*-closed unital subalgebra $A \subset M$ is an analytically conditioned subalgebra if

(1) $A = \sigma_t^\nu(A)$ for each $t \in \mathbb{R}$

(2) and if the conditional expectation $\mathcal{E}$ onto $A \cap A^* = D$ leaving ν invariant, is multiplicative on A.

Is it possible for the earlier equivalences to survive the transition to the σ-finite case?

Problem: Type III σ-finite algebras necessarily do not admit a Fuglede-Kadison determinant. So there is no analogue of the Szegö formula in this setting!!!

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The Haagerup reduction theorem

Construct the σ-finite von Neumann super-algebra $R = M \rtimes_\nu \mathbb{Q}_d$ of M. ($\mathbb{Q}_d$=diadic rationals)

Note that $\Phi : R \to M$ for some faithful normal conditional expectation.

$R = \overline{\cup_{n=1}^{\infty} R_n}^{w*}$ for some sequence $R_1 \subset R_2 \subset R_3 \subset \dots$ of von Neumann algebras each of which is finite and is the image of a faithful normal conditional expectation $\Phi_n : R \to R_n$ for which $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$.

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$R = \overline{\cup_{n=1}^{\infty} R_n}^{w*}$ for some sequence $R_1 \subset R_2 \subset R_3 \subset \ldots$ of von Neumann algebras each of which is finite and is the image of a faithful normal conditional expectation $\Phi_n : R \to R_n$ for which $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$.

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Unique state extension property 1

Lemma

Let A be an analytically conditioned algebra. If A satisfies the criterion that any $f \in L^1(M)^+$ which is in the annihilator of A_0 must belong to $L^1(\mathcal{D})$, then also

- *any $f \in L^1(R)^+$ which is in the annihilator of $\widehat{A}_0$ must belong to $L^1(\widehat{\mathcal{D}})$,*
- *and for any n, any $f \in L^1(R_n)^+$ which is in the annihilator of $(\widehat{A}_n)_0$, must belong to $L^1(\mathcal{D}_n)$.*

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Echoes of Szegö's formula

Theorem

Let A be an analytically conditioned algebra. Then the following are equivalent:

(i) *A is maximal subdiagonal, i.e.* $\overline{A + A^*}^{w*} = M$,

(ii) *Beurling: For every right A-invariant subspace X of* $L^2(M)$, *the right wandering subspace W of X satisfies* $W^*W \subset L^1(\mathcal{D})$, *and* $W^*(X \ominus [WA]_2) = (0)$.

(iii) *Unique state extension: The canonical embedding of* $A + A_0^*$ *into* $L^2(M)$ *is dense, and any* $f \in L^1(M)^+$ *which is in the annihilator of* A_0 *belongs to* $L^1(\mathcal{D})$.

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- The proofs that $(i) \Rightarrow (ii) \Rightarrow (iii)$ follow by carefully adapting the proofs of the tracial case.
- To prove $(iii) \Rightarrow (i)$, one uses the lemma to conclude that each $\widehat{A}_n = \widehat{A} \cap R_n$ satisfies the unique state extension property. Now apply the tracial theory to conclude that each $\widehat{A}_n$ is maximal subdiagonal. This involves comparing maximal subdiagonality with respect to $\widehat{\varphi}|_{R_n}$ to maximal subdiagonality with respect to τ_n. Use the weak* density of $\cup \widehat{A}_n$ in $\widehat{A}$ to conclude that $\widehat{A} \subset R$ is maximal subdiagonal. From this it follows that $A = \Phi(\widehat{A})$ is maximal subdiagonal.
- By a careful modification of the tracial arguments, one can show that maximal subdiagonality is equivalent to a weaker version of the Gleason-Whitney property. (The proof of equivalence to the full Gleason-Whitney property, requires a σ-finite version of Ueda's peak set theorem. [BL-2017])

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The proof of the lemma

- Given $f \in L^1(R)^+$ with $f \perp \widehat{A}_0$, note that then $\Phi(f) \in L^1(M)^+$ with $\Phi(f) \perp A_0$, and hence that $\Phi(f) \in L^1(D)^+$.
- Show that since $f \in L^1(R)^+$ with $f \perp \widehat{A}_0$, the same is true of each of $f_1 = \lambda_t^* f \lambda_t$, $f_2 = (\mathbb{1} + \lambda_t^*) f (\mathbb{1} + \lambda_t)$, and $f_3 = (\mathbb{1} - i\lambda_t^*) f (\mathbb{1} + i\lambda_t)$ $(t \in \mathbb{Q}_d)$.
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- Combine the above fact with the identities $tr_R \circ \Phi = tr_R$, $tr_R \circ \mathcal{E} = tr_R$ and $\mathcal{E} \circ \Phi = \Phi \circ \widehat{\mathcal{E}}$, to see that $tr_R(f\lambda_t b) = tr_R(\widehat{\mathcal{E}}(f)\lambda_t b)$ for all $b \in M$.
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Let A be maximal subdiagonal and $K \subset L^2(M)$ a closed right-invariant subspace. Given such an invariant subspace, we call

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