Temperley-Lieb Channels

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Quantum Channels

$\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_E$: finite dimensional Hilbert spaces.

- (**Def**) A quantum state $\rho \in B(\mathcal{H}_A)$ is a positive matrix with trace 1. For a quantum state we define the von Neumann entropy $H(\rho)$ by

$$H(\rho) := -\mathrm{Tr}(\rho \log \rho).$$

- $\mathcal{H}_A$, $\mathcal{H}_B$: finite dimensional Hilbert spaces.
  (**Def**) A trace-preserving CP map $\Phi : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is called a quantum channel.
- The Stinespring theorem says that there is an isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ such that

$$\Phi(\rho) = (id \otimes \mathrm{Tr})(V\rho V^*).$$

For this fixed isometry V we define the complementary channel $\tilde{\Phi}$ by

$$\tilde{\Phi}(\rho) := (\mathrm{Tr} \otimes id)(V\rho V^*).$$

Information quantities

- (**Def**) The Holevo capacity $\chi(\Phi)$ of a quantum channel $\Phi : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ is given by

$$\chi(\Phi) := \sup \left\{ H(\Phi(\sum_x p_x \rho_x)) - \sum_x p_x H(\Phi(\rho_x)) \right\},$$

  where the supremum is taken among all possible choice of ensembles $(p_x, \rho_x)_{x \in I}$ (i.e. $p_x \geq 0$, $\sum_x p_x = 1$, ρ_x quantum states in $B(\mathcal{H}_A)$)
- (**Def**) The classical capacity $C(\Phi)$ is given by the regularization of $\chi(\Phi)$, i.e.

$$C(\Phi) := \lim_{n \to \infty} \frac{\chi(\Phi^{\otimes})}{n}.$$

Information quantities: continued

- (**Def**) The "one-shot" quantum capacity $Q^{(1)}(\Phi)$ is given by

$$Q^{(1)}(\Phi) := \sup_{\rho \in \mathcal{D}(\mathcal{H}_A)} \left\{ H(\Phi(\rho)) - H(\tilde{\Phi}(\rho) \right\},$$

  where $\mathcal{D}(\mathcal{H}_A)$ is the set of all quantum states in $B(\mathcal{H}_A)$.
- (**Def**) The quantum capacity $Q(\Phi)$ is given by the regularization of $Q^{(1)}(\Phi)$, i.e.

$$Q(\Phi) := \lim_{n\to\infty} \frac{Q^{(1)}(\Phi^{\otimes})}{n}.$$

- In general we have

$$Q^{(1)}(\Phi) \leq \chi(\Phi) \text{ and } Q(\Phi) \leq C(\Phi).$$

Information quantities: continued 2

- (**Def**) The minimum output entropy $H_{\min}(\Phi)$ is given by

$$H_{\min}(\Phi) := \min H(\Phi(\rho)),$$

  where the minimum is taken among all possible choice of quantum state $\rho \in B(\mathcal{H}_A)$.
- The above quantity is closely related to $\chi(\Phi)$.

Questions

- (**Q1**) Can we compute or estimate $C(\Phi)$ and $Q(\Phi)$?
- (**Q2, Additivity of** χ) For any two Q-ch's Φ and Ψ do we have
$$\chi(\Phi \otimes \Psi) \stackrel{?}{=} \chi(\Phi) + \chi(\Psi).$$
- (**Q3, Additivity of** H_{min}) For any two Q-ch's Φ and Ψ do we have
$$H_{\text{min}}(\Phi \otimes \Psi) \stackrel{?}{=} H_{\text{min}}(\Phi) + H_{\text{min}}(\Psi).$$
- (**Shor, '04**) (Q1) is true $\Leftrightarrow$ (Q2) is true.

Properties of Q-channels and consequences

- (**Def**) A Q-channel $\Phi : B(H_A) \to B(\mathcal{H}_B)$ with the Stinespring isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$ is called degradable if there exists a Q-channel $\Phi : B(H_B) \to B(\mathcal{H}_E)$ such that

$$\tilde{\Phi} = \Psi \circ \Phi.$$

  Moreover, Φ is called anti-degradable if there exists a Q-channel $\Phi' : B(H_E) \to B(\mathcal{H}_B)$ such that $\Phi = \Psi' \circ \tilde{\Phi}$, i.e. if $\tilde{\Phi}$ is degradable.

- (**Rem**)
  Φ: degradable $\Rightarrow Q(\Phi) = Q^{(1)}(\Phi)$.
  Φ: anti-degradable $\Rightarrow Q(\Phi) = Q^{(1)}(\Phi) = 0$.

Properties of Q-channels and consequences: continued

- (**Def**) A Q-channel $\Phi : B(H_A) \to B(\mathcal{H}_B)$ is called entanglement-breaking (shortly, EBT) if its Choi matrix

$$C_\Phi = \sum_{i,j=1}^{d_A} \Phi(e_{ij}^A) \otimes e_{ij}^A$$

  is separable in $B(\mathcal{H}_A \otimes \mathcal{H}_B)$ (after the normalization). Φ is called PPT if C_Φ is PPT in $B(\mathcal{H}_A \otimes \mathcal{H}_B)$.
- (**Rem**) Φ: EBT $\Leftrightarrow$ Φ: a quantum-classical-quantum channel $\Rightarrow$ Φ: anti-degradable.
  Φ: EBT $\Rightarrow C(\Phi) = \chi(\Phi)$.
  Φ: PPT $\Rightarrow Q(\Phi) = Q^{(1)}(\Phi) = 0$.

Answering the questions and models of Q-channels

- For (Q1) we would like to exibit computable Q-channels with a higly non-trivial structure.
- (**Hastings, '09**) The questions (Q2)&(Q3) have negative answers.
- In the above we do not have a deterministic counter-example for (Q2)&(Q3). $\Rightarrow$ More models of Q-channels for possible candidates of a deterministic counter-example.

Clebsch-Gordan channels

- $\mathbb{G}$: a compact quantum group with $(C(\mathbb{G}), \Delta)$.
- A unitary $u \in B(\mathcal{H}_u) \otimes C(\mathbb{G})$ is called a representation if $(id \otimes \Delta)(u) = u_{12}u_{13}$.
- For two unitary rep'ns $u \in B(\mathcal{H}_u) \otimes C(\mathbb{G})$ and $v \in B(\mathcal{H}_v) \otimes C(\mathbb{G})$ we define the direct sum $u \oplus v$ and the tensor product $u \mathbin{\textcircled{\top}} v := u_{13}v_{23}$, which clarifies irreducibility.
- $\mathrm{Irr}(\mathbb{G})$: a complete collection of irred. unitary rep'ns. For $u, v, w \in \mathrm{Irr}(\mathbb{G})$ we write $u \subseteq v \mathbin{\textcircled{\top}} w$ if u is a subrep'n $v \mathbin{\textcircled{\top}} w$, which gives us an isometry $\alpha_u^{v,w} : \mathcal{H}_u \to \mathcal{H}_v \otimes \mathcal{H}_w$ such that $(u \mathbin{\textcircled{\top}} w)(\alpha_u^{v,w} \otimes 1) = (\alpha_u^{v,w} \otimes 1)(u)$.
- The above isometry can be described by

$$\alpha_u^{v,w}(e_i^u) = \sum_{j=1}^{n_v} \sum_{k=1}^{n_w} C(v, w, u; j, k, i) e_j^v \otimes e_k^w,$$

where the constants $C(v, w, u; j, k, i)$ are called the Clebsch-Gordan coefficients.

Clebsch-Gordan channels: continued

- (**Def**) The quantum channel $\Phi_u^{\bar{v},w} : B(\mathcal{H}_u) \to B(\mathcal{H}_w)$ with the Stinespring isometry $\alpha_u^{v,w}$ is called a Clebsch-Gordan channel (shortly, CG-channel) , i.e.

$$\Phi_u^{\bar{v},w}(\rho) = (\mathrm{Tr}_v \otimes id)(\alpha_u^{v,w} \rho (\alpha_u^{v,w})^*).$$

Similarly, we consider its complementary channel $\Phi_u^{v,\bar{w}} : B(\mathcal{H}_u) \to B(\mathcal{H}_v)$

$$\Phi_u^{v,\bar{w}}(\rho) = (id \otimes \mathrm{Tr}_w)(\alpha_u^{v,w} \rho (\alpha_u^{v,w})^*).$$

Temperley-Lieb Channels

- When $\mathbb{G} = SU(2)$ or O_N^+, $N \geq 2$ we know that $\mathrm{Irr}(\mathbb{G})(\cong \mathbb{N}) = \{u^{(n)} : n \in \mathbb{N}\}$ with the fusion rule

$$u^{(n)} \textcircled{\scriptsize T} u^{(m)} \cong u^{(|n-m|)} \oplus \cdots \oplus u^{(n+m)},\ n, m \in \mathbb{N}.$$

- We simply write $\Phi_{u^{(k)}}^{\bar{u^{(l)}}, u^{(m)}}$ by $\Phi_k^{\bar{l},m}$ and similarly we write $\Phi_k^{l,\bar{m}}$.
- The representation category for the above $\mathbb{G}$ can be understood as Temperley-Lieb category, which suggests us the name Temperley-Lieb channels (shortly, TL-channels) for $\Phi_k^{\bar{l},m}$ and $\Phi_{k,\mathbb{G}}^{l,\bar{m}}$.
- TL-channels have been studied by Al Nuwairan '13 (for $\mathbb{G} = SU(2)$), Brannan/Collins, '16 (for $\mathbb{G} = O_N^+$).

EBT/PPT/(anti-)degradability for $\mathbb{G} = SU(2)$

- In this case CG-coefficients are known by a complicated formula and we have $\Phi_k^{\bar{l},m} = \Phi_k^{m,\bar{l}}$, which allows us to assume that $l \geq m$.
- (**Thm, Brannan/Collins/L./Youn, in progress**)
$$\begin{cases} \Phi_k^{\bar{l},m} \text{ is EBT} \Leftrightarrow k = 0 \Leftrightarrow \Phi_k^{\bar{l},m} \text{ is PPT.} \\ \Phi_k^{l,\bar{m}} \text{ is EBT} \Leftrightarrow k = l - m \Leftrightarrow \Phi_k^{l,\bar{m}} \text{ is PPT.} \end{cases}$$
- (**Thm, BCLY**, beyond "EBT $\Rightarrow$ anti-degradable")
$$\begin{cases} \Phi_k^{\bar{l},l}\text{: degradable and anti-degradable.} \\ \Phi_{l+m}^{l,\bar{m}} \text{ are degradable with } \Phi_l^{\overline{l-m},m} \circ \Phi_{l+m}^{l,\bar{m}} = \Phi_{l+m}^{\bar{l},m}. \\ \Phi_k^{\bar{l},m},\ l > m\text{: not degradable.} \\ \Phi_3^{3,\bar{2}},\ \Phi_3^{4,\bar{3}} \text{ and } \Phi_5^{4,\bar{3}} \text{ are not degradable.} \end{cases}$$
- In summary, TL-channels have "non-trivial structure" except extremal cases.

EBT/PPT/(anti-)degradability for $\mathbb{G} = O_N^+$

- In this case CG-coefficients are not known and we have in general $\Phi_k^{\bar{l},m} \neq \Phi_k^{m,\bar{l}}$, which forces us to be more careful on which side being traced out.
- (**Thm, BCLY**) We have
$$\begin{cases} \Phi_k^{\bar{l},m} \text{ is EBT} \Rightarrow k = l - m. \\ \Phi_k^{l,\bar{m}} \text{ is EBT} \Rightarrow k = m - l. \\ \Phi_k^{\bar{l},m} \text{ is PPT} \Rightarrow k = l - m \text{ for big enough } N. \\ \Phi_k^{l,\bar{m}} \text{ is PPT} \Rightarrow k = m - l \text{ for big enough } N. \end{cases}$$
- (**Thm, BCLY**) $\Phi_k^{\bar{l},m}$, $\Phi_k^{l,\bar{m}}$, $k > |l - m|$, are neither degradable nor anti-degradable for big enough N.

Key ingredients 1: bistochastic channels and capacities

- For $u \in \mathrm{Irr}(\mathbb{G})$ we can consider a $\mathbb{G}$-action on $B(\mathcal{H}_u)$ by

$$\beta_u : B(\mathcal{H}_u) \to C(\mathbb{G}) \otimes B(\mathcal{H}_u),\ X \mapsto u(X \otimes 1)u^*.$$

- (**Def, BCLY**) For $u, w \in \mathrm{Irr}(\mathbb{G})$ we say that a Q-channel $\Phi : B(\mathcal{H}_u) \to B(\mathcal{H}_w)$ is (u, w)-$\mathbb{G}$-equivariant if

$$(id \otimes \Phi)(\beta_u(\rho)) = \beta_w(\Phi(\rho)).$$

- (**Prop**) A (u, w)-$\mathbb{G}$-equivariant channel Φ is bistochastic, i.e.

$$\Phi(\frac{1_u}{n_u}) = \frac{1_w}{n_w}.$$

- (**Prop**) When $\mathbb{G}$ is of Kac type, the channels $\Phi_u^{\bar{v},w}$ and $\Phi_u^{v,\bar{w}}$ are (u, w)-$\mathbb{G}$-equivariant and (u^*, w^*)-$\mathbb{G}^{\mathrm{op}}$-equivariant, respectively, so that they are both bistochastic.

Bistochastic channels and capacities: continued

- (**Prop**) For a bistochastic Q-channel $\Phi : B(\mathcal{H}_A) \to B(\mathcal{H}_B)$ with the Stinespring isometry $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_E$, we have

$$\log(\frac{d_B}{d_E}) \leq Q^{(1)}(\Phi) \leq C(\Phi) \leq \min\{\log d_A, \log d_B, \log(\frac{d_A d_B}{d_E})\}$$

- For $l > m$ we have $Q^{(1)}(\Phi_k^{l,\bar{m}}) > 0$, so that $\Phi_k^{l,\bar{m}}$ is neither anti-degradable nor PPT.

Key ingredients 2: Choi matrices

- (**Thm, Brannan/Collins**) The Choi matrices are given by

$$C_{\Phi_k^{\bar{l},m}} = \frac{[k+1]_q}{[l+1]_q} \alpha_l^{m,k} (\alpha_l^{m,k})^*, \; C_{\Phi_k^{l,\bar{m}}} = \frac{[k+1]_q}{[m+1]_q} \alpha_m^{k,l} (\alpha_m^{k,l})^*.$$

  Here, $[\cdot]_q$ is the q-integer with $q = q(N) = \frac{2}{N(1+\sqrt{1-4/N^2})}$.
- The above can be proved by diagrammatic presentation of Temperley-Lieb category.
- When $\mathbb{G} = SU(2)$ we can check the above directly by using CG-coefficients.
- (**Consequences**) The channels $\Phi_k^{\bar{l},m}$, $k \neq l - m$ are not EBT and so on.

Key ingredients 3: Jones-Wenzl projections and $\alpha_k^{l,m}$

- For O_N^+ we have projections p_k on $\mathcal{H}_1^{\otimes k}$ determined by the recursion

$$\begin{aligned} p_1 &= id_{\mathcal{H}_1}, \\ p_k &= id_{\mathcal{H}_1} \otimes p_{k-1} \\ &\quad - \frac{[k-1]_q}{[k]_q}(id_{\mathcal{H}_1} \otimes p_{k-1})(T_1 T_1^* \otimes id_{\mathcal{H}_1^{\otimes(k-2)}})(id_{\mathcal{H}_1} \otimes p_{k-1}), \end{aligned}$$

  where $T_1 = \sum_{i=1}^N e_i \otimes e_i$.

- Diagrammatic presentation of Temperley-Lieb category allows us to establish

$$\alpha_k^{l,m} = C(l,m,k) \cdot (p_l \otimes p_m)(id_{\mathcal{H}_1^{\otimes(l-r)}} \otimes T_r \otimes id_{\mathcal{H}_1^{\otimes(m-r)}})p_k,$$

  where T_r is given recursively by $T_r = (id_{\mathcal{H}_1} \otimes T_1 \otimes id_{\mathcal{H}_1})T_{r-1}$.

Jones-Wenzl projections and $\alpha_k^{l,m}$: continued

- Precise description of p_k's and consequently of $\alpha_k^{l,m}$ is still very difficult, but at least we know

  $$\{|\mathbf{x}\rangle = e_{j_1} \otimes \cdots \otimes e_{j_k} : j_p \neq j_{p+1}, 1 \leq p \leq k-1\} \subseteq \mathrm{Ran}(p_k).$$

  For the above specific vectors we have a better understanding of $\Phi_k^{\bar{l},m}(|\mathbf{x}\rangle\langle\mathbf{x}|)$.

Asymptotic estimates of MOE and capacities

- (**Thm, Brannan/Collins, to appear in CMP**) We have

$$H_{\min}(\Phi_k^{\bar{l},m}) \geq \frac{l+m-k}{2}\log N - C(N)$$

and

$$\chi(\Phi_k^{\bar{l},m}) \leq \frac{m+k-l}{2}\log N + D(N)$$

with $C(N), D(N) \to 0$ as $N \to \infty$.

- (**Thm, BCLY**) We actually have

$$\lim_{N\to\infty} |H_{\min}(\Phi_k^{\bar{l},m}) - \frac{l+m-k}{2}\log N| = 0$$

and

$$\lim_{N\to\infty} |Q^{(1)}(\Phi_k^{\bar{l},m}) - \frac{m+k-l}{2}\log N| = 0.$$

Moreover, we may replace $Q^{(1)}(\cdot)$ with $\chi(\cdot)$ in the above.

Comparison with modified TRO channels

- (**Def, Gao/Junge/Laracuente**)
  We begin with a TRO (ternary ring of operators) $X = \bigoplus_{i\in I} M_{n_i,m_i} \otimes 1_{l_i} \subseteq B(\mathcal{H}_E, \mathcal{H}_B)$ with $\mathcal{H}_E = \bigoplus_{i\in I} \mathbb{C}^{m_i} \otimes \mathbb{C}^{l_i}$ and $\mathcal{H}_B = \bigoplus_{i\in I} \mathbb{C}^{n_i} \otimes \mathbb{C}^{l_i}$. We set $\mathcal{H}_A = (X, \langle\cdot,\cdot\rangle_A)$, where $\langle x, y\rangle_A := \mathrm{Tr}_E(y^* x)$. Then the Q-channel

  $$\mathcal{N} : B(\mathcal{H}_A) \to B(\mathcal{H}_B), \ \ |x\rangle\langle y| \mapsto xy^*$$

  is called a TRO-channel. Moreover, for a normalized density $f \in B(\mathcal{H}_E)$ (i.e. positive and $\frac{\mathrm{Tr}(f)}{d_E} = 1$) "strongly independent from $\mathcal{R}(X)$" the Q-channel

  $$\mathcal{N}_f : B(\mathcal{H}_A) \to B(\mathcal{H}_B), \ \ |x\rangle\langle y| \mapsto xfy^*$$

  is called a modified TRO-channel.

Comparison with modified TRO channels: continued

- (**Thm, Gao/Junge/Laracuente**) We have

$$\log\Big(\sum_i n_i\Big) = C(\mathcal{N}) \le C(\mathcal{N}_f) \le C(\mathcal{N}) + \frac{\mathrm{Tr}(f \log f)}{d_E}$$

and

$$\log(\max_i n_i) = Q(\mathcal{N}) \le Q(\mathcal{N}_f) \le Q(\mathcal{N}) + \frac{\mathrm{Tr}(f \log f)}{d_E}.$$

Comparison with modified TRO channels: continued 2

- (**Prop, BCLY**) For $\mathbb{G} = SU(2)$ the Q-channel $\Phi_1^{\bar{2},1} : M_2 \to M_2$ is not equivalent to any modified TRO-channels in the sense that there is no $\begin{cases} \mathcal{N}_f : M_2 \to M_n, \\ \text{a unitary } U : \mathbb{C}^2 \to \mathbb{C}^2 \\ \text{an isometry } V : \mathbb{C}^2 \to \mathbb{C}^n \end{cases}$ such that

$$V\Phi_1^{\bar{2},1}(U\rho U^*)V^* = \mathcal{N}_f(\rho).$$

Thank you for your attention