Fourier Multipliers on Free Groups associated with the First Segment

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Fourier Multipliers on Free groups

$\mathbb{F}_n$: group of n free generators, $1 \leq n \leq \infty$.
$\lambda_s : \delta_t \in \ell_2(\mathbb{F}_n) \mapsto \delta_{st} \in \ell_2(\mathbb{F}_n)$

Given $x = \sum_s c_s \lambda_s$, set
$\tau x = c_e$;

$$L^p(\hat{\mathbb{F}}_n) = \{x; \|x\|_{L^p} = (\tau|x|^p)^{\frac{1}{p}} < \infty\}$$

Given $m \in \ell_\infty(\mathbb{F}_n, \mathbb{C})$, when does

$$\|T_m : \sum_s c_s \lambda_s \mapsto \sum_s m(s) c_s \lambda_s\|_{c.b.L^p \mapsto L^p} < \infty?$$

- $p = 2$; $\|T_m\| = \|m\|_{\ell_\infty}$.
- $p = 1, \infty$; iff $m(s^{-1}t) = \langle Q(s), R(t)\rangle_H$ with $\sup_{s,t} \|Q(s)\|_H \|R(t)\|_H < C$.
- $1 < p \neq 2 < \infty$; less understood, radial multipliers, $m(s) = |s|^i$ H^∞-calculus Junge-Le Merdy-Xu....Riesz transfroms, BMO...

Fourier Multipliers associated with the first segment

- $1 < p \neq 2 < \infty$ and non-radial multipliers

$$T_m : \lambda_s \mapsto m(k_1(s))\lambda_s; \quad m \in \ell_\infty(\mathbb{Z}, \mathbb{C})$$

with $k_1(s)$ the first power index in the reduced word,

$$s = g_{i_1}^{k_1} g_{i_2}^{k_2} ... g_{i_j}^{k_j} .., i_v \neq i_{v+1}, k_v \in \pm\mathbb{N}$$

A Motivating Example
Let $m = \chi_{\mathbb{F}_2^+}$, with $\mathbb{F}_2^+ = \{s \in \mathbb{F}_2; k_1(s) > 0\}$

$$T_m : \lambda_s \mapsto \chi_{\mathbb{F}_2^+} \lambda_s.$$

Biane/Pisier; Pimsner/Voiculescu; Ozawa....

Remark By taking the group homomorphism $\pi : \mathbb{F}_\infty \mapsto \mathbb{F}_2$ sending generators g_k to $a^k b a^k$, we see that $k_1(\pi(s)) = k$ iff $\pi(s) \geq g_k$. So T_m catches the information on the starting letter through π.

Free Hilbert transforms

Set $\mathbb{L}_0 = \{e\} \subset \mathbb{F}_\infty$;

$$\begin{aligned}
\mathbb{L}_j &= \{s; s \geq g_j\} = \{s; i_1(s) = j, k_1(s) > 0\}; j \in \mathbb{N} \\
\mathbb{L}_j &= \{s; s \geq g_j^{-1}\} = \{s; i_1(s) = j, k_1(s) < 0\}; j \in -\mathbb{N} \\
L_j &: \lambda_s \mapsto \chi_{\mathbb{L}_j} \lambda_s; \qquad H_\varepsilon = \sum_{j \in \mathbb{Z}} \varepsilon_j L_j, \quad \varepsilon_j = \pm 1.
\end{aligned}$$

Theorem (M-Ricard, 17') For any $x \in C_c(\mathbb{F}_\infty)$ and $1 < p < \infty$,

$$\|H_\varepsilon x\|_{L^p(\hat{\mathbb{F}}_\infty)} \simeq \|x\|_{L^p(\hat{\mathbb{F}}_\infty)}.$$

True for

almagamated free products;

replacing L_j by L_j^d, the projection to reduced words having g_j as their d-th letter.

replacing L_j by L_h^d, the projection to reduced words starting with $h, |h| = d$. But the c.b version for $\mathbb{F}_\infty$ is not true for L_h^d if $d \geq 2$.

Question Is this a lucky case, or there is a general result for all Fourier multipliers associated with the first segment ?

Fourier multipliers associated with the first segment

Theorem 1 (M-Xu) Given $1 < p < \infty$, then

$$T_m : \lambda_s \mapsto m(k_1(s))\lambda_s$$

extends to a completely bounded operator on $L^p(\hat{\mathbb{F}}_\infty)$ if its restriction on $\mathbb{F}_1$

$$T_m|_{\mathbb{F}_1} : \lambda_{g_1^k} \mapsto m(k)\lambda_{g_1^k}$$

is completely bounded.

Remark No chance for the case of $p = \infty$. Take $m(k) = \chi_{[-2,2]}(k)$, which is the symbol of a c.b multiplier on $L^\infty(\mathbb{F}_1)$. Then the multiplier T_m is the projection onto the set $\{s; |k_1(s)| \leq 2\}$ of $L^\infty(\mathbb{F}_\infty)$. To see this, let $x = \sum_{-N<k<N} c_k(g_1 g_2^3)^k g_1$ and note $T_m(x) = \sum_{0\leq k<N} c_k(g_1 g_2^3)^k g_1$.

A Littlewood-Paley Theory on Free groups

Example Let $A_0 = \{0\}$

$$A_j = [2^{j-1}, 2^j), j \in \mathbb{N}; A_j = -A_{-j}, j \in -\mathbb{N}.$$

Let $\chi_j = \chi_{A_j}$ for $j \in \mathbb{Z}$. Let

$$m = \sum_{j \in \mathbb{Z}} \varepsilon_j \chi_j$$

for any $\varepsilon_j = \pm$.

Corollary For $2 \leq p < \infty$,

$$\|x\|_{L^p} \simeq \|(\sum_j |T_{\chi_j}(x)|^2)^{\frac{1}{2}}\|_{L^p} + \|(\sum_j |T_{\chi_j}(x)^*|^2)^{\frac{1}{2}}\|_{L^p}.$$

The Mikhin condition

Corollary If m satisfies the Mikhlin condition e.g.

$$\sup_{k\in\mathbb{Z}}\{|m(k)|, k|m(k)-m(k-1)|\} < C.$$

Then

$$T_m : \lambda_s \mapsto m(k_1(s))\lambda_s$$

extends to a completely bounded linear operator on $L^p(\hat{\mathbb{F}}_\infty)$ for all $1 < p < \infty$.

Example $m(k) = k^{ti}$.

Ingredient: A Dirac operator on the free group

Let $D_\varepsilon(\lambda_e) = 0$ and

$$D_\varepsilon(\lambda_s) = \varepsilon_{i_1} k_1 \lambda_s$$

for

$$s = g_{i_1}^{k_1} g_{i_2}^{k_2} \cdots g_{i_m}^{k_m},$$

and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n ...)$ a sequence of $|\varepsilon_k| = 1$.

Theorem 2 (M-Xu) For $1 < p < \infty$,

$$S_t = e^{itD_\varepsilon}, -\infty < t < \infty$$

extends to a uniformly c.b group of operators on $L^p(\hat{\mathbb{F}}_\infty)$.

Transference $\Rightarrow$ Theorem 1 that $m(D_\varepsilon)$ is c. bounded on $L^p(\hat{\mathbb{F}}_n)$ for m being the symbol of a c.b. Fourier multiplier on $L^p(\hat{\mathbb{Z}})$, e.g.

$$sign(D) = H_\varepsilon.$$

Fourier multipliers along a geodesic path?

Proposition (Chua)
Fix a geodesic path $\mathbb{P}$ on the Cayley graph of $\mathbb{F}_\infty$, then

$$T_m : \lambda_s \mapsto m(|E_{\mathbb{P}} s|)\lambda_s$$

extends to a completely bounded linear operator on $L^p(\hat{\mathbb{F}}_\infty)$ for all $1 < p < \infty$ if m has a bounded variation, i.e.

$$\sum_{k\in\mathbb{N}} |m(k-1) - m(k)| < C.$$

Remark $|E_{\mathbb{P}} s| = |k_1(s)|$ for $s \geq g_j$ if $\mathbb{P} = \{g_j^k, k > 0\}$. The Propsition is a consequence of the boundedness of the free Hilbert transforms H_ε.