Aut($\mathbb{F}_5$) has Kazhdan's property (T)

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M. Kaluba, P. Nowak, and N. Ozawa; Aut($\mathbb{F}_5$) has property (T). Preprint. arXiv:1712.07167

Introduction: mathematics as an experimental science

$\mathrm{Aut}(\mathbb{F}_d)$ is a noncommutative analogue of $\mathrm{GL}(d,\mathbb{Z})$. It's known $\mathrm{GL}(d,\mathbb{Z})$ has property (T) iff $d \geq 3$, but $\mathrm{Aut}(\mathbb{F}_3)$ does not (McCool 1989).

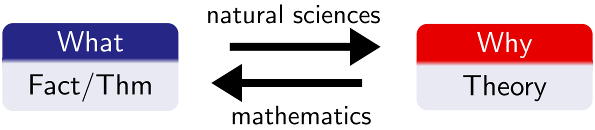
Problem (. . ., Lubotzky's book 1994, . . .)

Does $\mathrm{Aut}(\mathbb{F}_d)$ have Kazhdan's property (T)?

Theorem (KNO 2017)

$\mathrm{Aut}(\mathbb{F}_5)$ has Kazhdan's property (T).

- This is proved by a computer (but **it's rigorous!**).
- This leaves the cases of $d = 4$ and $d \geq 6$ unsettled.
- The outcome was somewhat unexpected in two respects: it has property (T) and a proof is found in the computer's range.
- We were encouraged by the previous successful implementations by Netzer–Thom, Fujiwara–Kabaya, and Kaluba–Nowak on groups such as $\mathrm{SL}(d \geq 3,\mathbb{Z})$ for which (T) were already known. Their results have greatly improved known estimates of Kazhdan constants.

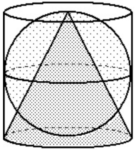


As computer and algorithm get stronger, even in math, we can sometimes observe true facts before knowing why.  ⋯ ⋯

Conjecture

Aut($\mathbb{F}_d$) has Kazhdan's property (T) for (at least) $d \geq 5$.

From *Archimedes* (by E. Jan Dijksterhuis, Princeton University Press)



⋯ I am convinced that this (= mechanical experiments for mathematical questions) is no less useful for finding the proofs of these same theorems. ⋯ For it is easier to supply the proof when we have previously acquired, by the method, some knowledge of the questions than it is to find it without any previous knowledge.

Kazhdan's property (T)

Theorem (Kazhdan 1967)

Any simple Lie group G of real rank ≥ 2 (e.g., $G = \mathrm{SL}(d \geq 3, \mathbb{R})$) and its lattice Γ (e.g., $\Gamma = \mathrm{SL}(d \geq 3, \mathbb{Z})$) have property (T).
In particular, Γ is finitely generated and has finite abelianization.

Definition (for a discrete group Γ)

Γ has (T) $\overset{\text{def}}{\Longleftrightarrow}$ $\exists S \Subset \Gamma\ \exists \kappa > 0$ s.t. $\forall (\pi, \mathcal{H})$ unitary rep'n and $\forall v \in \mathcal{H}$

$$d(v, \mathcal{H}^\Gamma) \leq \kappa^{-1} \max_{s \in S} \|v - \pi_s v\|.$$

$\Longleftrightarrow$ Γ is f.g. & $\forall S \Subset \Gamma$ generating $\exists \kappa = \kappa(S) > 0$ s.t. $\cdots$

The optimal $\kappa(S)$ is called the Kazhdan constant for (Γ, S).

- Property (T) inherits to finite-index subgroups and quotient groups.
- Property (T) + Amenability = Finite.
  $\rightsquigarrow$ Any finite-index subgroup of a (T) group has finite abelianization.
- Tons of applications in many branches of math and computer sciences.

An application of property (T): Expander graphs

Explicit construction of expanders (Margulis 1973)

$\Gamma = \langle S \rangle$, X a finite set, and $\Gamma \curvearrowright X$ transitively

$\rightsquigarrow$ Schreier graph: Vertices $= X$ and Edges $= \{\{x, sx\} : x \in X,\ s \in S\}$ is a $(|S|, \frac{\kappa(S)^2}{2})$-expander. Namely, for $\forall A \subset X$ one has

$$|\partial A| \geq \frac{\kappa(S)^2}{2}|A|(1 - \frac{|A|}{|X|}).$$

Product Replacement Algorithm (Celler et al., Lubotzky–Pak 2001)

$\mathrm{SAut}(\mathbb{F}_d) = \langle R_{i,j}^{\pm}, L_{i,j}^{\pm} \rangle \leq_{\text{index } 2} \mathrm{Aut}(\mathbb{F}_d)$, where $\mathbb{F}_d = \langle g_1, \ldots, g_d \rangle$ and

$$R_{i,j}^{\pm} \colon (g_1, \ldots, g_d) \mapsto (g_1, \ldots, g_{i-1}, g_i g_j^{\pm}, g_{i+1}, \ldots, g_d),$$
$$L_{i,j}^{\pm} \colon (g_1, \ldots, g_d) \mapsto (g_1, \ldots, g_{i-1}, g_j^{\pm} g_i, g_{i+1} \ldots, g_d).$$

PRA is a practical algorithm to obtain "random" elements in a given finite group $\Gamma = \langle h_1, \ldots, h_d \rangle$ via the action $\mathrm{SAut}(\mathbb{F}_d) \curvearrowright \mathrm{Epi}(\mathbb{F}_d, \Gamma)$.
Start with $\pi_0 \colon g_i \mapsto h_i$ and apply $R_{i,j}^{\pm}$ and $L_{i,j}^{\pm}$ randomly for n times to obtain π_n and then choose $\pi_n(g_i) \in \Gamma$, with $i \in \{1, \ldots, d\}$ random.
If $\mathrm{SAut}(\mathbb{F}_d)$ has (T), then π_n is considered "random" at $n = O(\log|\Gamma|)$.

Algebraic characterization of property (T)

$\mathbb{R}[\Gamma]$ real group algebra with the involution $(\sum_t \alpha_t t)^* = \sum_t \alpha_t t^{-1}$.

$\Sigma^2\mathbb{R}[\Gamma] = \{\sum_i f_i^* f_i\} = \{\sum_{x,y} P_{x,y} x^{-1} y : P \in \mathbb{M}_\Gamma^+\}$ where $\mathbb{M}_\Gamma$ finite matrices

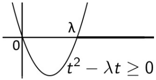
For $\Gamma = \langle S \rangle$ with $S = S^{-1}$ finite, $\Delta := \frac{1}{2}\sum_{s\in S}(1-s)^*(1-s) = |S| - \sum_{s\in S} s$.

Theorem (O 2013)

Γ has (T) $\iff \exists\lambda > 0$ such that $\Delta^2 - \lambda\Delta \in \Sigma^2\mathbb{R}[\Gamma]$ (or $\Sigma^2\mathbb{Q}[\Gamma]$)

Proof of the easier direction ($\Leftarrow$):

$$\Delta^2 - \lambda\Delta \in \Sigma^2\mathbb{R}[\Gamma] \quad\Rightarrow\quad \pi(\Delta)^2 - \lambda\pi(\Delta) \geq 0 \text{ for any } (\pi, \mathcal{H})$$



$$\Rightarrow\quad \mathrm{Sp}(\pi(\Delta)) \subset \{0\} \cup [\lambda, \infty) \text{ for any } (\pi, \mathcal{H})$$

$$\Rightarrow\quad \Gamma \text{ has property (T) with } \kappa(S) \geq \sqrt{2\lambda/|S|}$$

because $\mathcal{H}^\Gamma$ is the eigenspace of $\pi(\Delta)$ corresp. to eigenvalue 0. □

Stability (Netzer–Thom): In fact it suffices if $\|\Delta^2 - \lambda\Delta - \sum_i f_i^* f_i\|_1 \ll \lambda$.

Semidefinite Programming (SDP)

Γ has (T) $\iff \exists E \Subset \Gamma\ \exists\lambda > 0$ s.t. $\Delta^2 - \lambda\Delta \in \{\sum_{x,y} P_{x,y}x^{-1}y : P \in \mathbb{M}_E^+\}$
By fixing $E \Subset \Gamma$, we arrive at the SDP:

$$\begin{array}{ll} \texttt{minimize} & -\lambda \\ \texttt{subject to} & \Delta^2 - \lambda\Delta = \sum_{x,y\in E} P_{x,y}x^{-1}y, \quad P \in \mathbb{M}_E^+ \end{array}$$

- SDP can be solved "effectively" by computer.
- Due to computer capacity limitation, we almost always take
  $$E := \mathrm{Ball}(2) = \{e\} \cup S \cup S^2 = \mathrm{supp}\,\Delta \cup \mathrm{supp}\,\Delta^2.$$
  This may appear too small, but has proved valid in many cases.
- Size of SDP: dimension $|E| \times |E|$ and constraints $|E^{-1}E| = |\,\mathrm{Ball}(4)|$.

Certification Procedure:

Suppose (λ_0, P_0) is a hypothetical solution obtained by a computer.
Find $P_0 \approx Q^{\mathrm{T}}Q$ with $Q\mathbf{1} = 0$ and calculate with guaranteed accuracy

$$r := \|\Delta^2 - \lambda_0\Delta - \textstyle\sum_{x,y}(Q^{\mathrm{T}}Q)_{x,y}x^{-1}y\|_1 \ll \lambda_0.$$

$\rightsquigarrow$ Γ has (T) with $\lambda = \lambda_0 - 2r$ (in the case of $E = \mathrm{Ball}(2)$).

Finding a solution is computationally hard, but certification is relatively easy.

Previous implementation results

Netzer–Thom (2014), Fujiwara–Kabaya (2017), and Kaluba–Nowak (2017)

- $\Gamma = \mathrm{SL}(d,\mathbb{Z})$ and $S = \{I \pm E_{i,j} : i \neq j\}$.
  $|S| = 2d(d-1) = 12, 24, 40, 60, \ldots$ for $d = 3, 4, 5, 6, \ldots$.
  $\lambda(\Gamma, S) > 0.27,\ 1.3,\ 2.6$ for $d = 3, 4, 5$. No results for $d \geq 6$.
- $\Gamma = \mathrm{SAut}(\mathbb{F}_d)$ and $S = \{L_{i,j}^{\pm}, R_{i,j}^{\pm}\}$. $|S| = 4d(d-1)$. No results at all.
  (To identify elements in $E^{-1}E$, apply them to the standard free basis.)
- Some other groups which are already known to have property (T).

Invariant SDP ($\cdots$, KNO 2017)

$$\Sigma := \{\sigma \in \mathrm{Aut}(\Gamma) : \sigma(S) = S\} \quad \text{a finite subgroup of } \mathrm{Aut}(\Gamma)$$

$$\cong \mathfrak{S}_d \ltimes (\mathbb{Z}/2)^{\oplus d} \text{ for } \Gamma = \mathrm{SL}(d,\mathbb{Z}) \text{ and } \mathrm{SAut}(\mathbb{F}_d),\ |\Sigma| = 2^d d!.$$

The previous SDP reduces to the Σ-invariant SDP:

$$\begin{array}{ll} \texttt{minimize} & -\lambda \\ \texttt{subject to} & (\Delta^2 - \lambda\Delta)_t = \sum_{\substack{x,y \in E \\ x^{-1}y = t}} P_{x,y},\ \forall t \in E^{-1}E/\Sigma, \quad P \in (\mathbb{M}_E^{\Sigma})^+ \end{array}$$

Invariant SDP

$$\begin{array}{ll} \texttt{minimize} & -\lambda \\ \texttt{subject to} & (\Delta^2-\lambda\Delta)_t = \sum_{x^{-1}y=t} P_{x,y},\ \forall t\in E^{-1}E/\Sigma, \quad P\in(\mathbb{M}_E^\Sigma)^+ \end{array}$$

To implement the Σ-invariant SDP, we have calculated the explicit descriptions of $E^{-1}E/\Sigma$ and of isomorphisms

$$\rho\colon \Sigma \curvearrowright \ell_2(E) \cong \bigoplus_{\pi\in\hat{\Sigma}} m_\pi \pi,$$
$$\rho(\mathbb{R}[\Sigma]) \cong \bigoplus_{\pi\in\hat{\Sigma}} \mathbb{M}_{\dim\pi}\otimes\mathbb{R}1_{m_\pi},$$
$$\mathbb{M}_E^\Sigma = \rho(\mathbb{R}[\Sigma])' \cong \bigoplus_{\pi\in\hat{\Sigma}} \mathbb{R}1_{\dim\pi}\otimes\mathbb{M}_{m_\pi}.$$

It amounts to find a system $\{p_\pi\}_{\pi\in\hat{\Sigma}}$ of minimal projections in $\mathrm{C}^*(\Sigma)$ and an orthonormal basis for each ran $\rho(p_\pi)$. Luckily, this isn't too difficult (mathematically) for $\Sigma=\mathfrak{S}_d \ltimes (\mathbb{Z}/2)^{\oplus d}$. It complicates coeff matrices, but greatly facilitates the SDP to make it feasible by an ordinary computer: For $\Gamma=\mathrm{SAut}(\mathbb{F}_5)$, one has $\dim\mathbb{M}_{\mathrm{Ball}(2)} = 4641^2$ and $|\mathrm{Ball}(4)| = 11\,154\,301$, but $\dim\mathbb{M}^\Sigma_{\mathrm{Ball}(2)} = 13\,232$ with 36 blocks and $|\mathrm{Ball}(4)/\Sigma| = 7\,229$.

Outcome: $E = \mathrm{Ball}(2)$

- $\mathrm{SAut}(\mathbb{F}_3)$: Does not have property (T). (McCool 1989)
- $\mathrm{SAut}(\mathbb{F}_4)$: No result. ⋯☹ ☹ ☹
- $\mathrm{SAut}(\mathbb{F}_5)$: Property (T) with $\lambda > 1.19$. !☺人☺人☺! **Eureka!**
- $\mathrm{SL}(3, \mathbb{Z}[X])$: No result, even though it's known to have property (T). (Shalom & Vaserstein, Ershov–Jaikin-Zapirain 2010)
- $\mathrm{SL}(4, \mathbb{Z}[X])$: Property (T) with $\lambda > 2$. (Shalom 2006)
- The estimated spectral gap for $\mathrm{SL}(d, \mathbb{Z})$, $d = 3$, was considerably worse than $d = 4, 5, 6$, suggesting $\mathrm{Ball}(2)$ is not really adequate for $d = 3$.

We didn't reach a conclusion on $\mathrm{SAut}(\mathbb{F}_4)$, because either

1. it does not have property (T);
2. it has property (T), but $E = \mathrm{Ball}(2)$ is too small to observe it;
3. it has property (T), but $\lambda > 0$ is too small to recognize, etc.

$\mathrm{SAut}(\mathbb{F}_d)$, $d \geq 6$? There seems no "ladder" connecting $d = 5$ to higher:

Conjecture? (Bridson and Vogtmann)

$\mathrm{SAut}(\mathbb{F}_{d+1})$ has an infinite quotient Γ in which $\mathrm{SAut}(\mathbb{F}_d)$ has finite image.