The Hörmander-Mikhlin theorem for high rank semisimple Lie groups

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Oberwolfach — May 7-11, 2018

Three basic goals...

Let us fix $\mathrm{G} = \mathrm{SL}_n(\mathbf{R}), \widehat{\mathrm{G}} = \mathrm{vNA}(\mathrm{SL}_n(\mathbf{R}))$ and $m : \mathrm{G} \to \mathbf{C}$...

Fourier L_p-multipliers. Let

$$T_m\Big(\int_{\mathrm{G}} \widehat{f}(g)\lambda(g)\, d\mu(g)\Big) = \int_{\mathrm{G}} m(g)\widehat{f}(g)\lambda(g)\, d\mu(g).$$

PROBLEM. For which symbols m do we find L_p-bounded Fourier multipliers?

Literature so far is limited to **rigidity results/necessary conditions**.

We are also interested in *what can be done* and *how far from optimal* is it.

THREE GOALS...

- First **sufficient conditions** in $\mathrm{SL}_n(\mathbf{R})$ for L_p-boundedness.
- **Quantitatively**: How far are these conditions from being **optimal**?
- **Qualitatively**: How far are these conditions from being **necessary**?

Our main results along this talk generalize to other semisimple Lie groups

First rigidity results

- **Harish-Chandra function**

$$\Xi(g) = \int_{\mathrm{K}} \Delta(gk)^{-\frac{1}{2}} dk \in L_q(\mathrm{G}) \setminus L_2(\mathrm{G}) \quad \text{for all} \quad q > 2$$

where $\mathrm{G} = \mathrm{KP}$ (Iwasawa) and $\Delta =$ Left K-invariant modular function on P.

Theorem [Cowling/Haagerup/Howe '88]

Let $\pi_f(g) = \langle \pi(g) f, f \rangle$. Then

i) $|\pi_f(g)| \leq \Xi(g) \|f\|_2^2$ for all left K-invariant f and $\pi \prec \lambda$.

ii) $\pi \prec \lambda \Leftrightarrow \pi_f \in L_q(\mathrm{G})$ for all left K-invariant f and $q > 2$.

It spotlights a dramatic difference between abelian/semisimple harmonic analysis.

- G is **weakly amenable** when there exists $m_j \in \mathcal{C}_c(\mathrm{G})$ s.t.
  - $\lim_{j \to \infty} m_j = 1$ uniformly on compact sets
  - $\sup_{j \geq 1} \big\| T_{m_j} : L_\infty(\widehat{\mathrm{G}}) \to L_\infty(\widehat{\mathrm{G}}) \big\|_{\mathrm{cb}} < \infty$

  } Fourier L_∞-summability

Theorem [Haagerup '86] + [Cowling/Haagerup '89]

$\mathrm{SL}_n(\mathbf{R})$ fails weak amenability for $n \geq 3$. / $\mathrm{SL}_n(\mathbf{R})$ is weakly amenable for $n = 2$.

Lafforgue/de la Salle's rigidity theorem

Theorem [Lafforgue/de la Salle '11] + [de Laat/de la Salle '18]

Let $n \geq 3$ and $1 \leq p \leq \infty$. Then

$$\Big| \frac{1}{2} - \frac{1}{p} \Big| > \frac{1}{2(\lfloor \frac{n}{2} \rfloor + 1)} \quad \Rightarrow \quad L_p(\widehat{\mathrm{SL}_n(\mathbf{Z})}) \text{ fails the CBAP.}$$

Same for other $\mathbf{R}$-semisimple Lie groups, lattices and nonarchimidean local fields.

Harmonic analysis terminology...

Fourier L_p-summability fails in $\widehat{\mathrm{SL}_n(\mathbf{R})}$ when $|1/2 - 1/p|$ is large in terms of rank.

A key point in the proof

Given any SO_3-binvariant $m \in \mathcal{C}_0(\mathrm{SL}_3(\mathbf{R}))$ and $|1/2 - 1/p|$ large...

$$\left| m \begin{pmatrix} e^s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-s} \end{pmatrix} \right| \leq C_\varepsilon e^{-\varepsilon s} \big\| T_m : L_p(\widehat{\mathrm{SL}_n(\mathbf{R})}) \to L_p(\widehat{\mathrm{SL}_n(\mathbf{R})}) \big\|_{\mathrm{cb}}.$$

REMARK. Again **asymptotic decay** phenomenon / What about **local structure**?

Hörmander-Mikhlin smoothness criterion

The relation between **smoothness**/L_p-**boundedness** of Fourier multipliers is central in Euclidean harmonic analysis, with applications in differential geometry and PDEs. This topic orbits around...

Hörmander-Mikhlin theorem

Let $1 < p < \infty$ and $m : \mathbf{R}^n \to \mathbf{C}$. Then

$$\big\| T_m : L_p(\mathbf{R}^n) \to L_p(\mathbf{R}^n) \big\| \le C_{p,n} \max_{0 \le |\gamma| \le [\frac{n}{2}]+1} \Big\| |\xi|^{|\gamma|} \big| \partial_\xi^\gamma m(\xi) \big| \Big\|_\infty .$$

This imposes m to be a bounded smooth function over $\mathbf{R}^n \setminus \{0\}$:

- **Local behavior**
  Admissible a singularity at 0 with mild control of derivatives around it.
- **Asymptotic behavior**
  Derivatives decay to 0 at a polynomial rate given by differentiation order.

- **Optimal formulation**
  Optimal local/asymptotic exponents $|\xi|^{|\gamma|}$.
  Optimal order of classical derivatives: Up to $[n/2] + 1$.
  Optimal order of Sobolev-fractional derivatives: Up to $n/2 + \varepsilon$.

Historical interlude – Nonabelian HM criteria

- **Nilpotent groups**

  Christ, Cowling, Müller, Ricci, Stein...

- **Riemannian symmetric spaces**

  $\mathrm{G} = \mathbf{R}$-semisimple Lie group / $\mathrm{K} \subset \mathrm{G}$ maximal compact
  G/K with its G-invariant measure under left multiplication
  Multipliers in $L_2(\mathrm{G}/\mathrm{K})$ = Weyl-group-invariant elements in $L_\infty(\mathfrak{a}^*)$

  Hörmander-Mikhlin: [Clerc/Stein '74] + [Stanton/Tomas '78] + [Anker '90]

- **Group von Neumann algebras**

  G unimodular locally compact group
  $\beta : \mathrm{G} \to \mathcal{H}$ (injective) finite-dimensional orthogonal cocycle
  Fourier symbols $m : \mathrm{G} \to \mathbf{C}$ are lifted via $m = \dot{m} \circ \beta \rightsquigarrow$ CZ, Riesz transforms...

  Hörmander-Mikhlin: [Junge/Mei/Parcet '14 '18] + [González/Junge/Parcet '17]

REMARK. None of the Hörmander-Mikhlin criteria above apply in our framework...

Main obstructions towards HM in $\mathrm{SL}_n(\mathbf{R})$

Decay phenomenon
Sufficient conditions must incorporate rigidity known results

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Kazhdan's property (T)
Euclidean geometry mirrors $\mathrm{SL}_n(\mathbf{R})$ *via nonorthogonal actions*

(Rank 1: No finite-dimensional orthogonal representations)

$||$

No similar conditions for Hörmander-Mikhlin
Substantially and necessarily different HM *formulation*

New sufficient conditions

- Given $g \in \mathrm{SL}_n(\mathbf{R})$, set
  - $|g| = \min\{1, \mathrm{dist}(g,e)\}$,
  - $L(g) = \max\{\|g\|, \|g^{-1}\|\}$,
  - $\Theta_\varepsilon(g) = L(g)^{-\frac{1}{2}[\frac{n^2}{2}]-\varepsilon} \rightsquigarrow \Theta_0 \in L_q(\mathrm{SL}_n(\mathbf{R})) \setminus L_2(\mathrm{SL}_n(\mathbf{R}))$.
- Given $\gamma = (\mathrm{X}_{j_1}, \mathrm{X}_{j_2}, \ldots, \mathrm{X}_{j_{|\gamma|}})$ in $\mathfrak{sl}_n(\mathbf{R})$

$$\partial_{\mathrm{X}_j} f(g) = \frac{d}{ds}\Big|_{s=0} f\big(g \exp(s\mathrm{X}_j)\big),$$

$$d_g^\gamma f(g) = \partial_{\mathrm{X}_{j_1}} \partial_{\mathrm{X}_{j_2}} \cdots \partial_{\mathrm{X}_{j_{|\gamma|}}} f(g) = \Big(\overrightarrow{\prod_{1 \le k \le |\gamma|}} \partial_{\mathrm{X}_{j_k}} \Big) f(g).$$

HM criterion in $\mathrm{SL}_n(\mathbf{R})$ [PRS '18]

Let $\sigma_n = [n^2/2] + 1$ and $m \in \mathcal{C}^{\sigma_n}(\mathrm{SL}_n(\mathbf{R}) \setminus \{e\})$ such that

$$|g|^{|\gamma|} \big| d_g^\gamma m(g) \big| \le \Theta_\varepsilon^2(g) \quad \textit{for all} \quad |\gamma| \le \sigma_n \quad \textit{and some} \quad \varepsilon > 0.$$

Then, we get $\big\| T_m \colon L_p(\widehat{\mathrm{SL}_n(\mathbf{R})}) \to L_p(\widehat{\mathrm{SL}_n(\mathbf{R})}) \big\|_{\mathrm{cb}} \le C_{p,n}^\varepsilon$ for all $1 < p < \infty$.

New sufficient conditions

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Then, we get $\big\| T_m \colon L_p(\widehat{\mathrm{SL}_n(\mathbf{R})}) \to L_p(\widehat{\mathrm{SL}_n(\mathbf{R})}) \big\|_{\mathrm{cb}} \le C_{p,n}^\varepsilon$ *for all* $1 < p < \infty$.

- HM sufficient condition
- **Local analysis** – $\|g\|^{|\gamma|} \big| d_g^\gamma m(g) \big| \le 1$
  - Optimal Euclidean order for even ranks
  - We loose up to one derivative for odd ranks
  - We also get weaker *Sobolev condition* of order $n^2/2 + \varepsilon$

  Admissible local singularities with expected regularity around
- **Asymptotic analysis** – $\big| d_g^\gamma m(g) \big| \le \Theta_\varepsilon^2(g)$
  - σ_n-th Lie derivatives determine the rest $\rightsquigarrow$ Decay
  - Later we shall give further remarks on asymptotic behavior

New rigidity conditions

Given an open interval $J \subset \mathbf{R}$ and $\alpha > 0$

$$\mathcal{C}^{\alpha}(J) = \Big\{ f \in \mathcal{C}^{[\alpha]}(J) \colon \ d^{[\alpha]} f \text{ is } (\alpha - [\alpha])\text{-Hölder on compact sets of } J \Big\}.$$

We shall also write $\mathcal{C}^{\alpha^-}(J)$ for the intersection space $\bigcap_{\beta<\alpha} \mathcal{C}^{\beta}(J)$ with $0 < \beta < \alpha$.

New rigidity conditions

Rigidity criterion for radial multipliers in $\mathrm{SL}_n(\mathbf{R})$ [PRS '18]

Let $n \geq 3$, consider the radial $\mathrm{SL}_n(\mathbf{R})$-symbol $m(g) = \varphi\big(\|g\|_{\mathrm{HS}}/\sqrt{n}\big)$ for certain generating function $\varphi : \mathbf{R} \to \mathbf{C}$. If $S_m(g,h) = m(g^{-1}h)$ defines an S_p-bounded Schur multiplier for some $p > 2 + \frac{2}{n-2}$, then

$$\varphi \in \mathcal{C}^{\alpha_0^-}\big((1,\infty)\big) \quad \textit{for} \quad \alpha_0 = \frac{n-2}{2} - \frac{n-1}{p} > 0.$$

Moreover, the following decay estimates hold for φ:

- $\lim_{\xi \to \infty} \varphi(\xi) = \varphi_\infty$ *and*

$$\big|\varphi(\xi) - \varphi_\infty\big| \leq C_{p,n} \frac{\|S_m\|_{\mathcal{B}(S_p(L_2(\mathrm{G})))}}{\xi^{c_0}} \quad \textit{for} \quad c_0 = \frac{n}{\big[\frac{3}{1-\frac{2}{p}}\big]}\delta_{\alpha_0>1} + \frac{\alpha_0 n}{n-2}\delta_{\alpha_0<1}.$$

- *Given $\xi > 1$ and an integer $1 \leq k < \alpha_0$*

$$\Big|\frac{d^k}{d\xi^k}\varphi(\xi)\Big| \leq C_{p,n}\|S_m\|_{\mathcal{B}(S_p(L_2(\mathrm{G})))} \frac{1}{(\xi-1)^k \xi^{c_k}} \quad \textit{with} \quad c_k = \frac{n}{\big[\frac{2k+1}{1-\frac{2}{p}}\big]}.$$

New rigidity conditions

Rigidity criterion for radial multipliers in $\mathrm{SL}_n(\mathbf{R})$ [PRS '18]

Let $n \geq 3$ *and* $m(g) = \varphi\big(\|g\|_{\mathrm{HS}}/\sqrt{n}\big)$. *If* S_m *is an* S_p*-bounded Schur multiplier for some* $p > 2 + \frac{2}{n-2}$, *then* $\varphi \in \mathcal{C}^{\alpha_0^-}\big((1,\infty)\big)$ *with* $\alpha_0 = \frac{n-2}{2} - \frac{n-1}{p}$ *and the following estimates hold for* $0 \leq k < \alpha_0(p,n)$

$$\Big| \frac{d^k}{d\xi^k} \big(\varphi(\xi) - \varphi_\infty\big) \Big| \leq C_{p,n} \frac{\|S_m\|_{\mathcal{B}(S_p(L_2(\mathrm{G})))}}{(\xi-1)^k \xi^{c_k}} \quad \textit{with} \quad c_k \sim \frac{n}{[\frac{2k+1}{1-\frac{2}{p}}]}.$$

Ok for Fourier multipliers

- **Lafforgue/de la Salle rigidity...**
  Stronger estimates for this class / Partial results for K-biinvariant
- **Hörmander-Mikhlin thm in $\mathrm{SL}_n(\mathbf{R})$...**
  Qualitatively – Similar regularity/decay up to $\alpha_0 \sim$ rank $<$ dim$/2$
- **Euclidean radial L_p-multipliers...**
  70's: Similar prior to HNS-thm / Local: Same for rank! / Asympt: Extra c_k's...
- **Different multipliers in terms of the rank...**
  Certain φ fulfill HM conditions in rank n and fail rigidity ones in rank $m >> n$.

Local measurement of nonamenability

- Cotlar, Calderón, Coifman/Weiss... $\rightsquigarrow$ *Amenability*
- Neuwirth/Ricard, Caspers/de la Salle $\rightsquigarrow$ *Amenable group algebras*
- Junge/Mei/Parcet $\rightsquigarrow$ *Transference through BMO via orthogonal cocycles*

Our groups: Not amenable/do not admit finite-dimensional orthogonal cocycles

Let us measure how amenability is distorted over relatively compact neighborhoods of identity

Local measurement of nonamenability

Define

$$\delta_\Sigma(\Omega) = \inf_{\substack{\|\phi\|_2=1 \\ \mathrm{supp}\phi \subset \Sigma}} \sup_{g\in\Omega} \frac{1}{2}\int_{\mathrm{G}} \big|\phi(gh)-\phi(h)\big|^2 d\mu(h)$$

for any pair Ω, Σ of relatively compact open sets in a given unimodular group G.

Then, the following dichotomy holds

i) If G is amenable, then $\lim_{\Sigma\to\mathrm{G}} \delta_\Sigma(\Omega) = 0$ for all Ω.

ii) If G is nonamenable, then we get $\lim_{\Omega\to\mathrm{G}} \lim_{\Sigma\to\mathrm{G}} \delta_\Sigma(\Omega) = 1$.

Of course, $\delta_\Sigma(\Omega) \approx 0$ for Ω small and $\delta_\Sigma(\Omega) \approx 1$ for Ω large and G nonamenable.
Thus $\delta_\Sigma(\Omega)$ measures the 'amenability distortion over Ω' when G is nonamenable.

Precise Følner estimates in the parabolic part of $\mathrm{SL}_n(\mathbf{R})$ yield...

Distortion constants for $\mathrm{SL}_n(\mathbf{R})$ [PRS '18]

Let $\Omega \times \Sigma = B_R \times B_{2nR}$ and $q > 2$. Then

$$C_q \exp\Big(\frac{1}{q}\sigma_n R\Big) \le \frac{1}{1-\delta_\Sigma(\Omega)} \le C_0 \exp\Big(\frac{1}{2}\sigma_n R\Big).$$

Local Fourier-Schur transference

Given ϕ as above, let

$$j_p\Big(\int_{\mathrm{G}} \widehat{f}(g)\lambda(g)d\mu(g)\Big) = \int_{\mathrm{G}\times\mathrm{G}} \phi(g)^{\frac{2}{p}} \widehat{f}(gh^{-1})e_{g,h}d\mu(g)d\mu(h).$$

Key Estimate [PRS '18]

Let $p \geq 2$, then:

- $j_p : L_p(\widehat{\mathrm{G}}) \to S_p(L_2(\mathrm{G}))$ is a complete contraction.
- If $\widehat{f}(g) = 0$ for all $g \notin \Omega$, then $\|f\|_p \leq_{\mathrm{cb}} \dfrac{1}{1-\delta_\Sigma(\Omega)}\|j_p(f)\|_p$.

Given $m : \mathrm{G} \to \mathbf{C}$, let S_m/T_m be the Schur/Fourier multipliers associated to it...

Local Fourier-Schur transference [PRS '18]

If $\mathrm{supp}\, m \subset \Omega$

$$\big\|S_m\big\|_{\mathrm{cb}(S_p(L_2(\mathrm{G})))} \leq \big\|T_m\big\|_{\mathrm{cb}(L_p(\widehat{\mathrm{G}}))} \leq \frac{1}{1-\delta_\Sigma(\Omega)}\big\|S_m\big\|_{\mathrm{cb}(S_p(L_2(\Sigma)))}.$$

Twisted Fourier multipliers

Let

$$\mathcal{R} = L_\infty(\mathbf{R}^{n^2}) \bar{\otimes} \mathcal{B}(L_2(\Sigma)) = \Big\{ \Sigma \times \Sigma \text{ matrix-valued functions} \Big\}.$$

Let $M_g(\xi) = M(g\xi)$ for some $M : \mathbf{R}^{n^2} \to \mathbf{C}$ whose restriction to $\mathrm{SL}_n(\mathbf{R})$ coincides with the symbol m. The **twisted Fourier multiplier** $\widetilde{T}_M$ is the linear map defined by

$$\Big(f_{gh} \Big)_{g,h\in\Sigma} \mapsto \Big(T_{M_g}(f_{gh}) \Big)_{g,h\in\Sigma}$$

Schur-Twisted Fourier transference [PRS '18]

If $2 \leq p \leq \infty$ and M is continuous

$$\Big\| S_m : S_p(L_2(\Sigma)) \to S_p(L_2(\Sigma)) \Big\|_{\mathrm{cb}} \leq \Big\| \widetilde{T}_M : L_p(\mathcal{R}) \to L_p(\mathcal{R}) \Big\|_{\mathrm{cb}}.$$

REMARK. Transference $\neq$ [JMP] / Ok for unimodular G with nonorthogonal β

The goal is then to study twisted multipliers associated to nonorthogonal actions

HM condition – Sketch of proof I

- Twisted Hilbert transforms fail L_p-boundedness!! [Parcet/Rogers '16]
- All previous approaches appear to be inefficient with nonorthogonal actions twists

A key tool [JMP '18]

If $M \in \mathrm{HM}(\mathrm{dim}/2 + \varepsilon)$, then

$$T_M = \sum_{k \geq 1} \Lambda_k^2 \circ R_{u_k}$$

is a *Littlewood-Paley average of ε-fractional laplacian Riesz transforms* R_{u_k}.

- It does not apply directly in $\widehat{\mathrm{SL}_n(\mathbf{R})}$ for lack of orthogonality
- Twisted form in $\mathcal{R}$ using the inf-dim (orthogonal) cocycles of $(-\Delta)^{\varepsilon}$.

$$\widetilde{T}_M = \sum\nolimits_k \Lambda_k^2 \circ \widetilde{R}_{u_k}$$

- A duality argument: It suffices to prove an RC_p-inequality for the maps $\widetilde{R}_{u_k}$.
- Twisted Riesz transforms: Highly asymmetric $\rightsquigarrow$ **Row/Col drastically different**.

HM condition – Sketch of proof II

The row case

- It seems to fail in general over $L_p(\mathcal{R})$.
- Keeping track of local transference, we live in $\Lambda_p(\mathcal{R}) \subset L_p(\mathcal{R})$...
- Inverting transference: Row-ineq becomes a Col-ineq by group inversion
- Group inversion is locally smooth in $\mathrm{SL}_n(\mathbf{R})$: The HM constants are comparable

The column case

- $\widetilde{R}_{u_k} = R_{\widetilde{u}_k} \circ \widetilde{H} \rightsquigarrow$ Multidirectional Riesz + Twisted homogeneous
- $\mathbf{k} = n^2 + 2 + \delta_{n \text{ even}}$, $\varepsilon = 1/2 \rightsquigarrow \mathbf{k}/2 + \varepsilon \in 2\mathbf{Z} \rightsquigarrow$ Almost orthogonal u_k's
- Almost orthogonality + [JMP '18] $\Rightarrow L_p(C_p)$-bdness of multidirectional Riesz
- Operator-valued CZ + Schur estimates $\Rightarrow L_p$-bdness of twisted homogeneous

$\Downarrow$

Weak HM with $[\mathbf{k}/2] + 1$ derivatives

$\Downarrow$

Local form of Littlewood-Paley theorem

$\Downarrow$

Strong/local form of HM theorem with σ_n derivatives

Asymptotically... Bad constants for large Ω / $\Theta_\varepsilon^2 \in L_{1-\delta}(\mathrm{SL}_n(\mathbf{R})) \rightsquigarrow$ Patching

Asymptotic behavior...

Let $\beta > 0$ be the optimal value for the asymptotic HM condition

$$\big|d_g^\gamma m(g)\big| \le \Theta_\varepsilon^\beta(g) \quad \text{for all} \quad |\gamma| \le \sigma_n \quad \text{and some} \quad \varepsilon > 0.$$

Recall that $\Xi \le \lim_{\varepsilon \to 0} \Theta_\varepsilon = \Theta \in L_q \setminus L_2$. Thus Θ is a good approximation of Ξ.

- We have proved $1/n \lesssim \beta \le 2$...
- If we fix γ with $|\gamma| = \sigma_n$, the above condition is

$$L(g)^{\sigma_n} \big|d_g^\gamma m(g)\big| \le \frac{L(g)^{1-\beta\varepsilon}}{\Theta(g)^{2-\beta}}.$$

  Note: The behavior for $|\gamma| = \sigma_n$ determines the whole HM condition.
- CZ methods away from e + Junge's inequality imply

$$\sup_{g \in \Sigma} \sup_{j \in \mathbf{Z}} \Big(\int_{\frac12 < |\xi| < 2} 2^{2j} \big|\partial^\gamma M_g(2^j \xi)\big|^2 d\xi \Big)^{\frac12} \rightsquigarrow \Sigma\text{-distortion, worse than } \Theta_\varepsilon^2$$

  In particular, **Euclidean HA methods** appear to be **inefficient** bellow $\beta = 2$.
- $\beta < 2$ is conceivable, but it would require new ideas. Local transference $\rightsquigarrow \beta \ge 1$.
- **Rank 1.** $\mathrm{SL}_2(\mathbf{R})$ weakly amenable / Asymptotics for K-biinvariant = Radial case?

References

SEMISIMPLE LIE GROUPS

Tempered representations	Harish-Chandra	Amer J Math 80, 1958
	Harish-Chandra	Acta Math 116, 1966
	Cowling/Haagerup/Howe	Crelle's J 387, 1988
	Oh	Duke 113, 2002
Fourier multipliers – Rank 1	Canniére/Haagerup	Amer J Math 107, 1985
	Cowling / Haagerup	Inventiones 96, 1989
Fourier multipliers – High ranks	Haagerup – 1986	J Lie Th 26, 2016
	Lafforgue/de la Salle	Duke 160, 2011
	T. de Laat/de la Salle	Crelle's J 737, 2018

HÖRMANDER-MIKHLIN CRITERIA

Euclidean	Mikhlin	Doklady 109, 1956
	Hörmander	Acta Math 104, 1960
	Heo/Nazarov/Seeger	Acta Math 206, 2011
$\mathrm{SL}_n(\mathbf{R})/\mathrm{SO}_n$	Clerc/Stein	PNAS 71, 1974
	Stanton/Tomas	Acta Math 140, 1978
	Anker	Ann of Math 132, 1990
	Ionescu	Duke 114, 2002
Group von Neumann algebras	Junge/Mei/Parcet	GAFA 24, 2014
	Junge/Mei/Parcet	JEMS 20, 2018
	González-Pérez/Junge/Parcet	Ann. Sci. ENS 50, 2017

RELATED RESULTS IN GROUP ALGEBRAS

Transference	Neuwirth/Ricard	Canadian J 63, 2011
	Caspers/de la Salle	TAMS 367, 2015
	Caspers/Parcet/Perrin/Ricard	Forum Math Σ 3, 2015
More on multipliers and APs	Haagerup	Inventiones 50, 1979
	Junge/Ruan	Duke 117, 2003
	Cowling et al	Duke 127, 2005
	Parcet/Rogers	Crelle's J 710, 2016
	Mei/Ricard	Duke 166, 2017