# Return time and entropy estimates

Marius Junge

L.Gao & N. LaRacuente

May 7, 2018

# Plan

- Semigroups
- Return time
- Varopoulos dimension
- Kernel estimates
- Gradients and Fisher Information
- Main result
- Generators on matrix algebras
- Additional operator space features.

# Setup for dynamical systems

- $N \subset B(H)$ is a von Neumann algebra, i.e. closed a subalgebra closed under $*$, and the weak operator topology.
- $\tau$ is normal faithful tracial state.
- $(T_t)$ is a semigroup of completely positive selfadjoint maps on $N$ , i.e. $T_{t+s} = T_t T_s$, for all $x$, there exists $y$ such that $id \otimes T_t(x^*x) = y^*y$, and $\tau(T_t(x)^*y) = \tau(x^* T_t(y))$.
- The fixpoint algebra $M = \{x | T_t(x) = x\}$ does not depend on $t$, and there is a conditional expectation $E : N \to M$, i.e. $\tau(xy) = \tau(xE(y))$ for $x \in M, y \in N$.
- $\|x\|_p = \tau(|x|^p)^{1/p}$

# Problem

❋ Problem: (Return to equilibrium) Find $t_0$ such that for $t$

$$\| T_t(x) - E(x)\|_p \;\le\; \frac{1}{2}\|x\|_p ?\,.$$

❋ $p = 2$: Spectral gap:

$$\lambda_{min} \;=\; \inf_{E(x)=0} \frac{\|Ax\|_2}{\|x\|_2}\,.$$

Then

$$\| T_t(x) - E(x)\|_2 \;=\; \| T_t(x - Ex)\|_2 \;\le\; e^{-t\lambda_{\min}}\|x - E(x)\|_2$$

This $t \sim \log \lambda_{\min}^{-1}$.

❋ Works great for $1 < p < \infty$, but not for $p = 1, \infty$!

# Saloff-Coste

Let $T_p(M,) = \inf\{t : \| T_t - id : L_p \to L_p\| \le \frac{1}{2}\}$

**Theorem**

*(Saloff-Coste) Let G be a compact connected simple Lie group of dimension n . Then*

$$T_p(G) \sim \log n$$

Remark: For the sphere $T_p(S^n) \sim \frac{\log n}{2n}$. He also proves more precise upper and lower bounds (representation theory).

# Varopoulos dimension

- $A$ has Varopoulos dimension $n$ if $\|T_t : L_1(N) \to N\| \leq Ct^{-n/2}$.
- $A$ has complete Varopoulos dimension $n$ (for small $t$) if $\|T_t : L_1(N) \to N\|_{cb} \leq Ct^{-n/2}$ (holds for $0 < t \leq 1$).
- Free groups with word length: $\mathrm{Vdim} = 3/2$ (for small $t$).
- Quantum Euclidean Spaces generated by one parameter groups $u_k(t)u_j(s) = e^{\theta_{kj}ts}u_j(s)u_k(t)$ have $\mathrm{CVdim} = n$.

# Kernel

- $L_1(N, \tau)$ is an operator space. With respect to the linear bracket $(x, y) = \tau(xy)$ the the operator space dual of $L_1(N, \tau)$ is $N^{op}$, $N$ equipped with the opposite multiplication.
- The anti-linear dual is $N$.
- Main tool: Effros-Ruan

$$N^{op}\bar{\otimes}N \;=\; CB(L_1(N), N)\;.$$

- Conclusion: The heat kernel is an element $w_t \in N^{op}\bar{\otimes}N$ such that

$$\|T_t\|_{cb} \;=\; \|w_t\|_{N^{op}\bar{\otimes}N}$$

  and $T_t(f) = (\tau \otimes id)(w_t(f \otimes 1))$.
- The kernel for $E(f) = \tau(f)1$ is $1 \otimes 1$.

# NC-Varopoulos Theorem

**Theorem**

*(J-Zhao)* $T_t = e^{-tA}$ *sa subunital. TFAE*

i) $\|T_t : L_1(N) \to L_\infty\|_{cb} \le ct^{-n/2}$;

ii) *(Sobolev Ineq)* $\|A^{1/2} : L_2^0 \to L_{\frac{2n}{n-1}}\|_{cb} < \infty$

iii) *(Nash)* $\|x\|_2^{2+4/n} \le C\,(x, id \otimes A(x)) \quad \|x\|_{L_2(M_m,L_1)}^{4/n}$ *for matrices in* $L_2(M_m \otimes N)$

Remark: a) Minyu Zhao (Facebook) corrected an assertion from [JM] in the non-cb setting.
b) The proof of $i) \Rightarrow ii)$ uses the full power of [JP] embedding theory of NC-$L_p$ and BMO estimates.

# Short proof

Let $\frac{1}{2} = \frac{1-\theta}{1} + \frac{(2n-1)\theta}{n}$.
Let $x \in M_m(M)$ with $\|x\|_{L_2(M_m,L_1)} \leq 1$. Then we differentiate $f(t) = \|T_t x\|^2_{L_2(M_m \otimes M)}$ and get

$$-f'(t) \;=\; (T_t \otimes id(x), AT_t \otimes x)$$

and

$$\begin{aligned} f(t) &= \|T_t x\|^2_{L_2(M_m,L_2(M))} \\ &\leq \|T_t x\|^{(2(1-\theta)}_{L_2(L_1)} \|T_t x\|^{2\theta}_{L_2(L_{\frac{2n}{n-1}}(M))} \\ &\leq \|x\|^{1-\theta}_{L_2(L_1)} \|A^{1/2} T_t(x)\|^{2\theta}_{L_2(M_m,L_2(M))} \\ &\leq C^{2\theta}(-f'(t))^{2\theta} \,. \end{aligned}$$

Grönwall's Lemma ...

# Saloff-Coste kernels estimates

- Let us assume for a moment that $T_t$ is ergodic, acting on a finite vNa, with finite *cb*-dimension.
- $S_t(x) = T_t(x) - \tau(x)$ admits a kernel $w_t$, and we can define

$$\alpha_{p,q}(t) \ = \ \|S_t : L_p(N) \to L_q(N)\|_{cb}$$

- Lemma: $\alpha_{1,2}(t)^2 = \alpha_{1,\infty}(2t)$
- Theorem: $T_t$ has spectral gap and finite cb-V-dimension, then

$$\|w_t - 1 \otimes 1\|_{N^{op}\bar{\otimes}N} \ \leq \ \frac{1}{4}$$

for $t \ \geq \ t_0$.

# Example: Quantum Euclidean Spaces

- For quantum euclidean space, we have a normal *-homomorphism $\pi : L_\infty(\mathbb{R}^n) \to R_\theta^{op} \bar{\otimes} R_\theta$ given by

$$\pi(e^{i(\xi,\cdot)} \;=\; u_n(-\xi_n)\cdots u_1(-\xi_1) \otimes u_1(\xi_1)\cdots u_n(\xi_n)\ .$$

- $k_t = \pi(\frac{e^{|x|^2}4t}{(4\pi t)^{n/2})})$ is the nc heat kernel, i.e.

$$T_r(f) \;=\; tr_\theta(K_t f \otimes 1)\ .$$

- Hence the classical heat group estimates transfer to the quantum euclidean setting, and give cb-estimates, so far only known in hyperfinite setting.

# Gradient Form

- Let $T_t = e^{-tA}$. Then the gradient form is given by

$$2\Gamma_A(x,y) \; = \; A(x^*)y + x^*A(y) - A(x^*y) \, .$$

- Example: $T$ cp sa. Then $A = id - T$ is a generator and

$$2\Gamma(x,y) \; = \; x^*y - T(x^*)y - x^*T(y) + T(x^*y) \, .$$

- Lemma: $S \leq_{cp} CT$ implies $\Gamma_{I-S} \leq C\Gamma_{I-T}$ as bilinear forms for matrices, i.e. $\Gamma_{I-S}(x,x) \; \leq \; C\Gamma_{I-T}(x,x)$ for matrices.
- Corollary: Spectral gap + cb V dim. Then there exitst a $t_0$ such that

$$\Gamma_{I-E} \; \leq \; 4\Gamma_{I-T_t}$$

for some $t \; \geq \; t_0$,because the kernel of the conditional expectation is $1 \otimes 1$.

# Subordinated semigroup

- For very $\theta$, $A^\theta$ is still a generator of a semigroup.
- For $\theta = 1/2$, we have

$$P_t = e^{-tA^{1/2}} = \frac{t}{2\sqrt{2}} \int_0^\infty e^{-t/s^2} T_s s^{-3/2} ds \, .$$

- $\frac{P_t}{t} \leq_{cp} \frac{P_s}{s}$ (Mei)
- Prop: Spectral gap + cb V dim. Then $\Gamma_{I-E} \leq c_0 \Gamma_{A^{1/2}}$. Indeed,

$$c(t_0)\, \Gamma_{I-E} \leq \frac{\Gamma_{I-P_{t_0}}}{t_0} \leq \lim_{s \to 0} \frac{\Gamma_{I-P_s}}{s} = \Gamma_{A^{1/2}}$$

- Remains true for $A^\theta$, $0 < \theta < 1$.

# Derivations

- **Theorem:** (JRS) If $\Gamma(x,x) \in L_1(N,\tau)$ for $x \in \mathrm{dom}(A^{1/2})$, then there exists a finite von Neumann algebra $M \supset N$, and a derivation $\delta : \mathrm{dom}(A^{1/2}) \to L_1(M)$ such that

$$\Gamma(x,y) \;=\; E_N(\delta(x)^*\delta(y))$$

and $\|\delta(x)\|_p \sim \max\{\|\Gamma(x,x)^{1/2}\|_p, \|\Gamma(x^*,x^*)^{1/2}\|_p\}$.
- The Fisher information with respect to $A$ is given by $I_A(\rho) = \tau(A(\rho)\ln\rho)$.
- Let $A, B$ be generators of semigroups with the same fixpoint algebra. Then $\Gamma_B \le C\Gamma_A$ implies

$$I_B(\rho) \;\le\; CI_A(\rho)\,.$$

# Relative entropy

- For two positive operator $\rho, \sigma$, the relative entropy is given by

$$D(\rho|\sigma) \ = \ \tau(\rho \ln \rho) - \tau(\rho \ln \sigma) \ .$$

- $D(\rho|\sigma) = \lim_{p\to 1} \frac{\|\sigma^{-1/2p'}\rho\sigma^{-1/2p'}\|_p - 1}{p-1}$ is positive if $\rho, \sigma$ are states. The expression for fixed $p$ is called Sandwiched Entropy.
- Using vector-valued $L_1(L_p)$ spaces (in the sense of [JP]) we can prove the well-known data processing inequality $D(\Phi(\rho)|\Phi(\sigma)) \ \leq \ D(\rho|\sigma)$ for subtracial completely positive maps for von Neumann algebras.

# From Otto-Villani to Carlen Maas

- Following Otto-Villani we say that $A$ satisfies a logarithmic Sobolev inequality (LSI)
$$c_A D(\rho|E(\rho)) \ \le \ I_A(\rho)$$
where $E$ is the conditional expectation onto the fixed point algebra (allow non-ergodic systems).
- We say that $A$ satisfies a complete logarithmic inequality, CLSI, if $A \otimes id$ satisfies LSI with a uniform constant $c_{cb}(A)$.
- In contrast to LSI, complete version CLSI is stable under tensor products, i.e. $c_{cb}(A \otimes 1 + 1 \otimes B) \ \ge \ \min\{c_{cb}(A), c_{cb}(B)\}$-Thanks to the data-processing inequality.
- Carlen-Maas constructed a sub-Riemanian metric on the states space such that $LSI$ implies
$$d_g(\rho, E(\rho)) \ \le \ \sqrt{\frac{2D(\rho|\mathcal{E}(\rho))}{c_A}} \ .$$

# Talagrand type inequality

- The key: Entropy decay $D(T_t(\rho)|E(\rho)) \ \leq \ e^{-c_A t} D(\rho|\sigma)$.
- Let $d_\delta(\rho,\sigma) = \sup_{\|\delta(x)\|\leq 1} |\tau((\rho-\sigma)x)|$ the Rieffel distance. Then

$$d_\delta(\rho|E(\rho) \ \leq \ 2d_g(\rho, E(\rho)) \, .$$

- Corollary: LSI implies

$$d_{\delta_A}(\rho_1,\rho_2) \ \leq \ C\sqrt{D(\rho_1|E(\rho_1))} + C\sqrt{D(\rho_2|E(\rho_2))} \, .$$

# The example

Bardet et al:

$$\begin{aligned} I_{I-E} &= \tau((I-E)(\rho)\ln\rho) \\ &= \tau(\rho\ln\rho) - \tau(E(\rho)\ln\rho) \\ &= \tau(\rho\ln\rho) - \tau(E(\rho)\ln E(\rho)) + \tau(E(\rho)\ln E(\rho)) - \tau(E(\rho)\ln\rho) \\ &= D(\rho|E(\rho)) + D(E(\rho)|\rho) \\ &\geq\ D(\rho|E(\rho)) \end{aligned}$$

satisfies *CLSI* with constant 1.

# Main result

## Theorem

*Let* $(T_t = e^{-tA})$ *be semigroup of selfadjpoint completely positive maps with finite cb-Varopoulos dimension, spectral gap and* $0 < \theta < 1$. *Then* $T_t^\theta = e^{-tA^\theta}$ *satisfies*

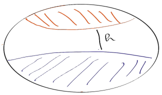
i) $d_A(\theta)\Gamma_{I-E} \ \leq \ \Gamma_{A^\theta}$;

ii) $A^\theta$ *satisfies CLSI;*

iii) $\|T_t^\theta(x) - E(x)\|_p \leq e^{-d(\theta)t}\|x\|_p$;

iv) *For two projections* $e, f$ $\ d_{\delta_\theta}(\frac{e}{\tau(e)}, \frac{f}{\tau(f)}) \ \geq \ h$ *implies Talagrand's inequality*

$$\tau(e)\tau(f) \ \leq \ e^{-\frac{h^2}{c'}}$$



# Proof

cb-V-dim+ spectral gap implies $\Gamma_{I-E} \ \leq \ C(\theta)\Gamma_{A^\theta}$ and hence

$$D(\rho|E(\rho)) \ \leq \ I_{I-E} \ \leq \ c(\theta)^{-1} I_{A^\theta} \ .$$

Remark: This new property *$L\Gamma E$*

$$\Gamma_{I-E} \ \leq \ C_A \Gamma_A$$

automatically transfers to *cb*-setting, and implies the norm decay for $p = 1$, $p = \infty$. However, $L\Gamma E$ is <u>not</u> stable under tensor products.

# Stein-Rothaus-Zegarlinsky

Using the deep work on commutators in compact Lie groups (local estimates!) we have

**Theorem (JGL)**

*Let $G$ be a Lie compact group, $\{X_1, ..., X_d\}$ be a Hörmander system (vector fields generating the Lie algebra), and $\Delta = \sum_j X_j^2$ the corresponding Laplacian. Then $A = |\Delta|^\theta$ satisfies $CLSI$, and return time estimates, and also every tensor product $(e^{-tA})^{\otimes_n}$.*

# Transfer

- Let $G$ be a Lie group acting on $M_m$, and $X_1, ..., X_k$ elements in the tangent space, i.e. the Lie algebra. Then the is a derived map $\overline{\pi} : \mathrm{g} \to iM_m^{sa}$.
- Let $\pi(X_k) = ia_k$ and $\mathcal{L}(x) = \sum_k a_k^2 x + x a_k^2 - 2a_k x a_k$. Then

$$\delta(x) \; = \; ([a_k, x])$$

  is the induced derivation.
- Let $\sigma : M_m \to L_\infty(G, M_m)$ be given by $\sigma(x)(g) = \pi(g)x\pi(g)^{-1}$. Then we have a commutative diagram

$$\begin{array}{ccc} L_\infty(G, M_m) & \overset{T_t \otimes id}{\to} & L_\infty(G, M_m) \\ \uparrow_\sigma & & \uparrow_\sigma \\ M_m & \overset{e^{-t\mathcal{L}}}{\to} & M_m \end{array}$$

- Hence kernels estimates for $\Delta_X$ transfer to $\mathcal{L}$.

# Application

**Theorem**

The set of selfadjoint generators satisfying CLSI on $M_m$ is dense.

Remark: These fractional powers can now be used replacing the classical two point inequality on $\ell^2_\infty$, because we can take tensor products and 'central limits'.

# Operator Spaces?

- For ergodic $T_t$ it suffices to consider cb-estimates from $L_1 \to L_\infty$.
- For non-ergodic semigroups we need cb-estimates of the form

$$\| T_t : L_1(M) \to L_1^\infty(N \subset M)\|_{cb} \ \leq \ ct^{-n/2}$$

- Here $L_1^\infty(N \subset M)$ is a conditioned $L_1(L_\infty)$ space so that

$$\|x\|_{L_1(M_m, L_1^\infty(N \subset M))} \ = \ \inf_{x=ayb} \|a\|_{L_2(M_m(N))}\|y\|_{M_m(M)}\|b\|_{L_2(M_m(N))}$$

  introduced by Pisier, studied in [JP], but considered by Holevo in seventies.
- A key tool in the proof are double operator integrals and the module $M \otimes_\Gamma M$ implemented in the JRS extension via $\delta$!

# Thanks

Thanks for coming and listening