ON THE COHOMOLOGICAL DIMENSION OF SOME PRO-p-EXTENSIONS ABOVE THE CYCLOTOMIC $\mathbb{Z}_p$-EXTENSION OF A NUMBER FIELD

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Abstract. Let $\widetilde{\mathrm{K}}_S^T$ be the maximal pro-p-extension of the cyclotomic $\mathbb{Z}_p$-extension K^{cyc} of a number field K, unramified outside the places above S and totally split at the places above T. Let $\widetilde{\mathrm{G}}_S^T = \mathrm{Gal}(\widetilde{\mathrm{K}}_S^T/\mathrm{K})$.

In this work we adapt the methods developed by Schmidt in [Sch3] in order to show that the group $\widetilde{\mathrm{G}}_S^T = \mathrm{Gal}(\widetilde{\mathrm{K}}_S^T/\mathrm{K})$ is of cohomological dimension 2 provided the finite set S is well chosen. This group $\widetilde{\mathrm{G}}_S^T$ is in fact *mild* in the sense of Labute [La]. We compute its Euler characteristic, by studying the Galois cohomology groups $H^i(\widetilde{\mathrm{G}}_S^T, \mathbb{F}_p)$, $i = 1, 2$. Finally, we provide new situations where the group $\widetilde{\mathrm{G}}_S^T$ is a free pro-p-group.

1. Introduction

Fix a prime $p > 2$ and an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$.

Let K be a number field with signature (r_1, r_2) and let K^{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of K. Let $\Gamma = \mathrm{Gal}(\mathrm{K}^{cyc}/\mathrm{K}) \simeq \mathbb{Z}_p$. In our work, we study pro-p-extensions L of K^{cyc} with restricted ramification where some given places split.

Let S and T be two disjoint finite sets of finite places of K. Denote by $\widetilde{\mathrm{K}}_S^T$ the maximal pro-p-extension of K^{cyc}, unramified outside the places above S and totally split at those above T (unramified outside S, T-split). Put $\widetilde{\mathrm{G}}_S^T = \mathrm{Gal}(\widetilde{\mathrm{K}}_S^T/\mathrm{K})$. It is a quotient of $\mathrm{G}_{S\cup \mathrm{Pl}_p} = \mathrm{Gal}(\mathrm{K}_{S\cup\mathrm{Pl}_p}/\mathrm{K})$, where $\mathrm{K}_{S\cup\mathrm{Pl}_p}$ is the maximal pro-p-extension of K unramified outside $S \cup \mathrm{Pl}_p$. It follows that $\widetilde{\mathrm{G}}_S^T$ is a finitely generated pro-p-group. We later show that it admits a finite presentation.

We remark that if v is a place of K not dividing p which is not ramified in $\mathrm{L}/\mathrm{K}^{cyc}$ then $\mathrm{L}_v = \mathrm{K}_v^{cyc}$. Thus it is sufficient to consider finite subsets S of places of K and T of p-adic places Pl_p of K such that $S \cap T = \emptyset$.

The case where T is empty and S contains the p-adic places of K is already well-known. The pro-p-extension $\widetilde{\mathrm{K}}_S^T$ is indeed the maximal pro-p-extension of K unramified outside S, and the group $\widetilde{\mathrm{G}}_S := \mathrm{Gal}(\mathrm{K}_S/\mathrm{K})$ has cohomological dimension 2 (see [NSW]). Moreover, if S does not contain all the places above p, Schmidt proved in the series of papers [Sch1], [Sch2], [Sch3] that one can achieve the cohomological dimension 2 property for G_S by adding to S a finite set of tame places. He also computed its Euler characteristic.

In this article, we show:

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Theorem 1.1. *Let* K *be a number field not containing the* p^{th} *roots of unity.*

Let S' *and* T *be two finite sets of places of* K *with* $S' \cap T = \emptyset$. *Then there exists a finite set* S *of finite places of* K *such that*

(i) $S' \subset S$ *and* $(S - S') \cap \mathrm{Pl}_p = \emptyset$*;*

(ii) the pro-p-group $\widetilde{\mathrm{G}}_S^T$ *has cohomological dimension at most* 2*;*

(iii) $\chi(\widetilde{\mathrm{G}}_S^T) = r_1 + r_2 - \sum_{v \in S_p} [\mathrm{K}_v : \mathbb{Q}_p]$ *where* $\chi(\widetilde{\mathrm{G}}_S^T)$ *stands for the Euler characteristic associated to the cohomology groups* $H^*(\widetilde{\mathrm{G}}_S^T, \mathbb{F}_p)$.

Remark 1.2. *As noticed by Schmidt in* [Sch3]*, one can be more precise and ask* S *to avoid a fixed set of places of* K *with zero Dirichlet density. For instance, one can choose the places of* S *in the set of degree one places of* $\mathrm{K}/\mathbb{Q}$.

Let $\mathcal{H}_S^T = \mathrm{Gal}(\widetilde{\mathrm{K}}_S^T/\mathrm{K}^{cyc})$ and $\mathcal{X}_S^T = \mathcal{H}_S^T/[\mathcal{H}_S^T, \mathcal{H}_S^T]$. In the case where $T = \emptyset$, we denote $\mathcal{H}_S := \mathcal{H}_S^{\emptyset}$ and $\mathcal{X}_S := \mathcal{H}_S/[\mathcal{H}_S, \mathcal{H}_S]$. The $\mathbb{Z}_p[[\Gamma]]$-module $\mathcal{X}_S$ corresponds to the inverse limit of the p-S-class group along $\mathrm{K}^{cyc}/\mathrm{K}$ and the homology groups $H_i(\mathcal{H}_S, \mathbb{F}_p)$ are $\mathbb{F}_p[[\Gamma]]$-modules. Let r_S denote the $\mathbb{F}_p[[\Gamma]]$-rank of $\mathcal{X}_S$. Recall that $r_S = 0$ conjecturally (the Ferrero-Washington theorem assures that it is true for abelian extensions of $\mathbb{Q}$ and the works of Schneps [Schn] and Gillard [Gi] guarantee that it holds also for abelian extensions of imaginary quadratic fields).

Using the Cebotarev density theorem, one can prove that by adding a finite set S of well-chosen tame finite places, the group $\mathcal{H}_S$ becomes a free $\mathbb{F}_p[[\Gamma]]$-module.

Corollary 1.3. *There is a finite set* S *of places of* K *such that* $S \cap \mathrm{Pl}_p = \emptyset$ *and such that* $H_2(\mathcal{H}_S, \mathbb{F}_p) \simeq \mathbb{F}_p[[\Gamma]]^{r_1+r_2+r_S}$.

Now we suppose that $T = \mathrm{Pl}_p$ and we consider $\mathcal{H}'_S = \mathrm{Gal}(\widetilde{\mathrm{K}}_S^T/\mathrm{K}^{cyc})$. When $S = \emptyset$, the $\mathbb{Z}_p[[\mathrm{Gal}(\mathrm{K}^{cyc}/\mathrm{K})]]$-module $\mathcal{X}' := \mathcal{H}'_{\emptyset}/[\mathcal{H}'_{\emptyset}, \mathcal{H}'_{\emptyset}]$ is well-known. It is the inverse limit of the p-groups of p-classes along $\mathrm{K}^{cyc}/\mathrm{K}$. Conjecturally again, the coinvariants $\mathcal{X}'_{\Gamma}$ are finite: it is a conjecture of Gross (see [?]).

Again, adding well-chosen primes, we obtain the following.

Corollary 1.4. *There is a finite set* S *of places of* K*, satisfying* $S \cap \mathrm{Pl}_p = \emptyset$*, such that* $H_2(\mathcal{H}'_S, \mathbb{F}_p) \simeq \mathbb{F}_p[[\Gamma]]^{r_1+r_2+r}$.

The proof of Theorem 1.1 requires information on the relations of the group $\widetilde{\mathrm{G}}_S^T$ and this can be done by considering the cup product (see [NSW]§III.9)

$$H^1(\widetilde{\mathrm{G}}_S^T, \mathbb{F}_p) \times H^1(\widetilde{\mathrm{G}}_S^T, \mathbb{F}_p) \to H^2(\widetilde{\mathrm{G}}_S^T, \mathbb{F}_p).$$

We first need to compute $H^1(\widetilde{\mathrm{G}}_S^T, \mathbb{F}_p)$. This is classical and is done through class field theory. Then we need to estimate the second cohomology group $H^2(\widetilde{\mathrm{G}}_S^T, \mathbb{F}_p)$. This is done by considering the Shafarevich kernel $Ш(\widetilde{\mathrm{G}}_S^T)$ which mesures the obstruction of the relations of $H^2(\widetilde{\mathrm{G}}_S^T, \mathbb{F}_p)$ to being local, and one gets control of this group with the use of a Kummer group $\widetilde{V}_S^T/\mathcal{K}^{\times p}$ (to be defined in §3.1). These estimations which are the core of our work are gathered in the following theorem.

Theorem 1.5. *Let* S, T *be disjoint sets of places of a number field* K *such that* $T \subset Pl_p$ *and* K_v *contains the* p^{th} *roots of unity for any* $v \in S - Pl_p$. *Then we have* *(i)* $d_p(\widetilde{\mathrm{G}}_S^T) = -(r_1+r_2+|T|-1+\theta(\mathrm{K}))+|\mathrm{Pl}_p-S_p|+\sum_{v\in S_p}([\mathrm{K}_v:\mathbb{Q}_p]+\theta(\mathrm{K}_v))+|S_0|+d_p(\widetilde{V}_S^T/\mathcal{K}^{\times p})$;

(ii) $d_p Ш(\widetilde{\mathrm{G}}_S^T) \leq d_p(\widetilde{V}_S^T/\mathcal{K}^{\times p})$*;*

(iii) $d_p H^2(\widetilde{\mathrm{G}}_S^T) \leq d_p(\widetilde{V}_S^T/\mathcal{K}^{\times p})+|S_0|+|\mathrm{Pl}_p-(S_p\cup T)|+\sum_{v\in S_p}\theta(\mathrm{K}_v)-\theta(\mathrm{K})+\theta(\mathrm{K},S)$;

where $\theta(K,S),\theta(K_v),\theta(K)\in\{0,1\}$ *(see* §2.1 *for a precise definition).*

As a consequence of these computations, we obtain new situations where the group $\widetilde{\mathrm{G}}_S^T$ is a free pro-p-group.

Corollary 1.6. *Suppose the following three conditions hold:*

(i) K *contains the* p^{th} *roots of unity,*

(ii) $|S|=1$ *and* $S\cup T = Pl_p$*;*

(iii) There is no non-trivial p*-extension of* K*, unramified outside* T *and totally split at* S*.*

Then $\widetilde{\mathrm{G}}_S^T$ *is a free pro-p-group with* $d(\widetilde{\mathrm{G}}_S^T) = 1-(r_1+r_2)+[\mathrm{K}_{v_0}:\mathbb{Q}_p]$ *generators, where* v_0 *is the place of* S*.*

Remark 1.7. *A field* K *satisfying the conditions of Corollary 1.6 for the place* v_0 *is* v_0*-rational according to the work of Jaulent and Sauzet* [JSa].

Corollary 1.8. *Suppose the three conditions are true:*

(i) The fields K *and* K_v *do not contain the* p^{th}*-roots of unity;*

(ii) $S\cup T = Pl_p$;

(iii) there is no non-trivial unramified outside T *and* S*-split* p*-extension of* $\mathrm{K}'=\mathrm{K}(\zeta_p)$.

Then $\widetilde{\mathrm{G}}_S^T$ *is a free pro-p-group with* $d(\widetilde{\mathrm{G}}_S^T) = 1-(r_1+r_2)+\sum_{v\in S}[\mathrm{K}_v:\mathbb{Q}_p]$ *generators.*

The paper is organised as follow. In section 2 we make precise the notation and the context of our work. Section 3 is devoted to the estimations of the groups $H^i(\widetilde{\mathrm{G}}_S^T,\mathbb{F}_p)$, for $i=1$ and 2, and of a certain Shafarevich kernel $Ш(\widetilde{\mathrm{G}}_S^T)$. For that purpose, we study a Kummer group $\widetilde{V}_S^T$. Section 4 contains technical prerequisites for the proof of our main theorem 1.1, contained in Section 5.

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2. The arithmetic context

2.1. **First notation.** Fix a prime number $p>2$ and a number field K, an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Put $\overline{\mathrm{G}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathrm{K})$.

- μ_p is the set of p^{th} roots of unity in $\overline{\mathbb{Q}}$, ζ_p is a generator of μ_p.
- $\mu(\mathrm{K})$ denotes the set of p^∞-th roots of unity contained in K.
- Let $\mathcal{K}^\times$ denote $\mathbb{Z}_p\otimes\mathrm{K}^\times$.
- Unless otherwise stated, all the considered sets of places are made of finite places.

- Let M/L be a field extension, Q a set of places of L, R a set of places of M. If no confusion is possible, Q denotes also the set of places Q_M of M lying above the places of Q, and R denotes also the set R_L of places of L lying under R.
- $\mathrm{Pl}_p = \{v, v|p\}$ is the set of p-adic places of K and, if S is a set of places of K, put $S_p := S \cap \mathrm{Pl}_p$ and $S_0 := S - S_p$.
- If S is a set of places of a field L, let $\theta(\mathrm{L}, S)$ be 1 if $S = \emptyset$ and L contains the p^{th} roots of unity, and 0 otherwise. Put $\theta(\mathrm{L}) := \theta(\mathrm{L}, \emptyset)$.
- If M is a $\mathbb{Z}_p$-module, denote by ${}_pM = M/pM$ the cokernel of $M \xrightarrow{p} M$, by $M[p]$ its kernel and by $d_pM := \dim_{\mathbb{F}_p}(M/pM)$ the p-rank of M.
- If G is a pro-p-group and if $i \geq 0$, let us denote by $H^i(\mathrm{G})$ the cohomology group $H^i(\mathrm{G}, \mathbb{F}_p)$. Moreover, if G admits a finite presentation, put $d(\mathrm{G}) := d_pH^1(\mathrm{G}, \mathbb{F}_p)$, $r(G) = d_pH^2(\mathrm{G}, \mathbb{F}_p)$, $\chi_2(\mathrm{G}) = 1 - d(\mathrm{G}) + r(\mathrm{G})$ and (as soon as it makes sense) $\chi(\mathrm{G}) = \sum_{i \geq 0} (-1)^i \mathrm{d}_pH^i(\mathrm{G}, \mathbb{F}_p)$.
- If G is an abelian group, we put $\mathrm{G}^* = \mathrm{Hom}(\mathrm{G}, \mathbb{Q}/\mathbb{Z})$.
- If S and T are sets of places of K, let K_S^T denote the maximal pro-p-extension of K unramified outside S and where the places of T split completely. We put $\mathrm{K}_S = \mathrm{K}_S^\emptyset$ and $\mathrm{K}^T = \mathrm{K}_\emptyset^T$. The Galois group $\mathrm{Gal}(\mathrm{K}_S^T/\mathrm{K})$ is denoted by G_S^T and we omit in the same way $\emptyset$ in the notation.
- If L/K is a Galois extension (possibly infinite) of Galois group G and v is a place of K, we denote by G_v the decomposition group inside G of a fixed place of L above v.

2.2. **Local setting.** Fix a finite place v of K and let $\overline{\mathrm{K}_v}$ be the maximal pro-p-extension of K_v. Let $\overline{\mathrm{G}}_v := \mathrm{Gal}(\overline{\mathrm{K}_v}/\mathrm{K}_v)$ be the absolute pro-p Galois group of K_v and let $\overline{I_v}$ be its inertia subgroup.

Denote by

- Frob_v the arithmetic Frobenius of $\overline{\mathrm{G}}_v/\overline{I_v}$,
- $\mathrm{K}_v^{nr} := \overline{\mathrm{K}_v}^{\overline{I_v}}$ the maximal unramified pro-p-extension of K_v,
- K_v^{cyc} the cyclotomic $\mathbb{Z}_p$-extension of K_v (in particular, if $v \nmid p$, $\mathrm{K}_v^{cyc} = \mathrm{K}_v^{nr}$),
- $\mathrm{K}_v^{cr} := \mathrm{K}_v^{nr}\mathrm{K}_v^{cyc}$, the maximal cyclotomically ramified pro-p-extension of K_v,
- $\widetilde{I}_v = \mathrm{Gal}(\overline{\mathrm{K}_v}/\mathrm{K}_v^{cyc}\mathrm{K}_v^{nr})$, $\widetilde{H}_v = \mathrm{Gal}(\overline{\mathrm{K}_v}/\mathrm{K}_v^{cyc})$, $\mathrm{G}_v^{cr} = \mathrm{Gal}(\mathrm{K}_v^{cr}/\mathrm{K}_v)$ and $\mathrm{G}_v^{cyc} = \mathrm{Gal}(\mathrm{K}_v^{cyc}/\mathrm{K}_v)$.

Definition 2.1. *A pro-p-extension* $\mathrm{L}_v/\mathrm{K}_v$ *is said to be*

- *locally cyclotomic if* $\mathrm{L}_v \subset \mathrm{K}_v^{cyc}$*;*
- *cyclotomically ramified if* $\mathrm{L}_v \subset \mathrm{K}_v^{cr}$*.*

For $v|p$, remark that if $\mathrm{L}_v/\mathrm{K}_v$ is cyclotomically ramified, the Galois group $\mathrm{Gal}(\mathrm{L}_v/\mathrm{K}_v)$ is a quotient of $\mathrm{G}_v^{cr} \simeq \mathbb{Z}_p^2$.

Lemma 2.2. *The extension* $\mathrm{K}_v^{cr}/\mathrm{K}_v$ *is contained in a pro-p-free extension* $\mathrm{F}_v/\mathrm{K}_v$

Proof. If $v \nmid p$, it is obvious because $\mathrm{K}_v^{cyc} = \mathrm{K}_v^{nr}$ and $\mathrm{G}_v^{cr} \simeq \mathbb{Z}_p$.

For $v|p$, we take F_v to be the extension $\mathrm{K}_v\overline{\mathbb{Q}_p}/\mathrm{K}_v$ (here $p > 2$, so that $\overline{\mathbb{Q}_p}/\mathbb{Q}_p$ is pro-p-free). □

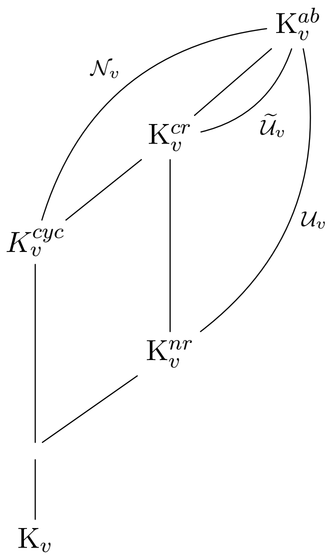
The normic local groups. Let K_v^{ab} be the maximal abelian pro-p-extension of K_v and let U_v be the group of units of K_v.

Denote by $\mathcal{K}_v^\times := \varprojlim_n \mathrm{K}_v^\times/(\mathrm{K}_v^\times)^{p^n}$ the pro-p-completion of $\mathrm{K}_v^\times$ and define $\mathcal{U}_v = \mathbb{Z}_p \otimes U_v$.

The extension $\mathrm{K}_v^{ab}/\mathrm{K}_v^{nr}$ is generated, via the Artin map, by the local units $\mathcal{U}_v$ and the extension $\mathrm{K}_v^{ab}/\mathrm{K}_v^{cyc}$ by the elements $\mathcal{N}_v$ of $\mathcal{K}_v$ which are norms in $\mathrm{K}_v^{cyc}/\mathrm{K}_v$. The extension $\mathrm{K}_v^{ab}\mathrm{K}_v^{nr}/\mathrm{K}_v^{cyc}$ is therefore generated by the subgroup $\widetilde{\mathcal{U}}_v := \mathcal{N}_v \cap \mathcal{U}_v$ of the group of units: they are the units which are norms in $\mathrm{K}_v^{cyc}/\mathrm{K}_v$. It is the group of the locally cyclotomic units.

Thus:

- $\mathcal{U}_v \simeq \mu(\mathrm{K}_v)$ if $v \nmid p$, otherwise $\mathcal{U}_v \simeq \mu(\mathrm{K}_v) \times \mathbb{Z}_p^{[\mathrm{K}_v:\mathbb{Q}_p]}$;
- $\mathcal{N}_v = \mathcal{U}_v$ if $v \nmid p$, otherwise $\mathcal{N}_v \simeq \mu(\mathrm{K}_v) \times \mathbb{Z}_p^{[\mathrm{K}_v:\mathbb{Q}_p]}$;
- $\widetilde{\mathcal{U}}_v = \mathcal{U}_v \simeq \mu(\mathrm{K}_v)$ if $v \nmid p$, otherwise, $\widetilde{\mathcal{U}}_v \simeq \mu(\mathrm{K}_v) \times \mathbb{Z}_p^{[\mathrm{K}_v:\mathbb{Q}_p]-1}$.



2.3. **Global setting.** From now on, we fix S and T two finite sets of finite places of K satisfying:

- For any place v of S not dividing p, the complete field K_v contains the p^{th} roots of unity;
- $S \cap T = \emptyset$;
- $T \subset \mathrm{Pl}_p$.

Recall that $S_p = S \cap \mathrm{Pl}_p$ and $S_0 = S - S_p$.

Definition 2.3. *We denote by* $\widetilde{\mathrm{K}}_S^T$ *the pro-p-extension of* K, *maximal for the following conditions:*

- $\widetilde{\mathrm{K}}_S^T/\mathrm{K}$ *is cyclotomically ramified outside* S*;*
- $\widetilde{\mathrm{K}}_S^T/\mathrm{K}$ *is locally cyclotomic at all places* v *of* T.

Put $\widetilde{\mathrm{G}}_S^T = \mathrm{Gal}(\widetilde{\mathrm{K}}_S^T/\mathrm{K})$.

Remark 2.4. $\widetilde{\mathrm{K}}_S^T/\mathrm{K}$ *is a subextension of the maximal pro-p-extension* K_Σ *of* K *unramified outside* $\Sigma = \mathrm{Pl}_p \cup S_0$.

Remark 2.5. *The field* $\widetilde{\mathrm{K}}_S^T$ *contains the cyclotomic* $\mathbb{Z}_p$*-extension* K^{cyc} *of* K *and* $\widetilde{\mathrm{K}}_S^T$ *is the maximal pro-*p*-extension of* K^{cyc} *unramified outside the places above* S *and totally split at the places above* T.

One can describe the abelianization $\widetilde{\mathrm{G}}_S^{T,ab}$ of $\widetilde{\mathrm{G}}_S^T$ by using the Artin map. Indeed, it induces an isomorphism $\widetilde{\mathrm{G}}_S^{T,ab} \simeq \mathcal{J}/\mathcal{K}^\times \widetilde{\mathcal{U}}_S^T$, where $\mathcal{J} = \mathcal{J}_{\mathrm{K}}$ is the restricted product over all places v of K of the groups $\mathcal{K}_v^\times$ with respect to the groups $\mathcal{U}_v$ and

$$\widetilde{\mathcal{U}}_S^T = \left(\prod_{v \notin S_0 \cup \mathrm{Pl}_p} \mathcal{U}_v\right)\left(\prod_{v \in T} \mathcal{N}_v\right)\left(\prod_{v \in \mathrm{Pl}_p - (S_p \cup T)} \widetilde{\mathcal{U}}_v\right) = \prod_{v \notin S \cup T} \widetilde{\mathcal{U}}_v \prod_{v \in T} \mathcal{N}_v.$$

2.4. **The Labute and Schmidt works.** In [La], Labute gave for the first time examples of groups G_S of cohomological dimension 2 with $S \cap \mathrm{Pl}_p = \emptyset$.

These results come from the study of the Lie algebra obtained considering the descending central p-series of G_S. In certain good situations, the relations in this algebra can be deduced from the first terms of the relations of G_S which were due to Koch [Ko]. In this case, the group G_S is said to be mild.

In the series of articles [Sch2], [Sch3], Schmidt gives interpretations of Labute's work, introducing the $k(\pi,1)$ property for G_S (and G_S^T) groups. He shows in particular that, after enlarging the set S, the groups G_S have finite cohomological dimension, and he proves a Riemann existence theorem. In our setting, we use the following result (already partially present in [La]).

Theorem 2.6 (Schmidt [Sch2])**.** *Let* G *be a pro-*p*-group of finite type, not pro-*p*-free. Suppose that* $H^1(\mathrm{G}) = U \oplus V$ *with*

(i) $\forall \chi_1, \chi_2 \in V$, $\chi_1 \cup \chi_2 = 0 \in H^2(\mathrm{G})$;

(ii) $H^1(\mathrm{G}) \otimes V \overset{\cup}{\twoheadrightarrow} H^2(\mathrm{G})$.

Then $cd(\mathrm{G}) = 2$.

Remark 2.7. *When the previous hypotheses are satisfied, the group* G *is mild, and the Poincaré series* $P(t)$ *associated to* G *is* $(1 - d(G)t + r(G)t^2)^{-1}$.

We are going to apply Theorem 2.6 to the groups $\widetilde{\mathrm{G}}_S^T$.

2.5. **The pro-p-groups in the context of Iwasawa theory.** Let G be a pro-p-group, $\mathcal{H}$ a closed normal subgroup of G such that $\mathrm{G}/\mathcal{H} = \Gamma \simeq \mathbb{Z}_p$. Denote by $\Lambda = \mathbb{Z}_p[[\Gamma]]$ the Iwasawa algebra associated with the pro-p-group Γ and by Ω the integral local ring $\Omega = \mathbb{F}_p[[\Gamma]] := \varprojlim_U \mathbb{F}_p[\Gamma/U]$.

If M is a Λ-module of finite type, denote by $\rho_\Lambda(M)$ its Λ-rank and by ρ_Ω its Ω-rank.

We recall that the module M is torsion if, for example, the set M_Γ of coinvariants is a torsion $\mathbb{Z}_p$-module.

2.5.1. *The case where* G *is pro-*p*-free.* In this case, the normal subgroup $\mathcal{H}$ of G is pro-p-free too. Put $\mathcal{X} = \mathcal{X}_\emptyset$. The following result is well known:

Proposition 2.8. *If* G *is pro-*p*-free, then* $\mathcal{X}$ *is a free* Λ*-module of rank* $d(\mathrm{G}) - 1$.

Proof. See [NSW], Proposition 5.6.11 and Corollary 5.6.12. □

2.5.2. *The case where* G *has cohomological dimension* 2. The case where G has cohomological dimension at most 2 has been also widely studied, certainly because the group G_S has cohomological dimension 2 as soon as S contains the p-adic places. We recall the following fact:

Proposition 2.9. *Suppose that* G *has cohomological dimension at most* 2. *Put* $r = \rho_\Omega(\mathcal{X})$. *Then*

(i) *the* Λ*-module* $H_2(\mathcal{H}, \mathbb{Z}_p)$ *is free of rank* t*, with* $t = \rho(\mathcal{X}) + \chi(\mathrm{G})$*;*

(ii) *the* Ω*-module* $H_2(\mathcal{H}, \mathbb{F}_p)$ *is free of rank* t'*, with* $t' = t + r$.

Proof. This comes from Proposition 5.6.7 of [NSW] for the module $H_2(\mathcal{H}, \mathbb{F}_p)$. Notice that Ω has projective dimension 1.

□

2.6. **Known results.** The groups $\widetilde{\mathrm{G}}^T := \widetilde{\mathrm{G}}^T_\emptyset$ have been studied by several authors: Jaulent and Soriano [JSo], Jaulent and Maire [JM2], Assim [As], et al.

The groups $\widetilde{\mathrm{G}}_S := \widetilde{\mathrm{G}}^\emptyset_S$ have been studied by Salle in [Sa].

In these situation, it turns out that the groups coming into play admit a finite presentation. An estimation of the Euler characteristic is also given. It has to be noted that this estimation gives rise to situations where $\widetilde{\mathrm{G}}^T_S$ is not analytic (see [JM1] and [JM3]).

Theorem 2.10 (Salle [Sa], Jaulent-Maire [JM1])**.** *We still suppose* $p > 2$.

(i) $d(\widetilde{\mathrm{G}}_S)$ *and* $r(\widetilde{\mathrm{G}}_S)$ *are finite and*

$$\chi_2(\widetilde{\mathrm{G}}_S) \leq r_1 + r_2 + \theta(\mathrm{K}, S) - \sum_{v \in S_p} [\mathrm{K}_v : \mathbb{Q}_p],$$

where $\theta(\mathrm{K}, S) = 1$ *if* S *is empty and* K *contains the* p^{th} *roots of unity,* $\theta(\mathrm{K}, S) = 0$ *otherwise.*

(ii) *If* $T = \mathrm{Pl}_p$*,* $d(\widetilde{\mathrm{G}}^T)$ *and* $r(\widetilde{\mathrm{G}}^T)$ *are finite and*

$$\chi_2(\widetilde{\mathrm{G}}^T) \leq r_1 + r_2 + \theta(\mathrm{K}),$$

where $\theta(\mathrm{K}) = 1$ *if* K *contains the* p^{th} *roots of unity,* 0 *otherwise.*

3. The groups $H^i(\widetilde{\mathrm{G}}^T_S)$

3.1. **Kummer groups and Shafarevich kernel.** Throughout this paragraph, S and T are again two finite sets of finite places of K satisfying the conditions of §2.3.

Definition 3.1. *We consider* $\widetilde{V}^T_S = \{x \in \mathcal{K}^{\times\times}, x \in \mathcal{J}^p \prod_{v \notin S \cup T} \widetilde{\mathcal{U}}_v \prod_{v \in T} \mathcal{N}_v\}$.

This Kummer group will play a central role in the following. The next proposition is important, because it shows that, taking a suitable S, one can ensure the triviality of $\widetilde{V}^T_S/\mathcal{K}^{\times p}$.

Proposition 3.2. *(i) The group* $\widetilde{V}^T_S/\mathcal{K}^{\times p}$ *is finite.*

(ii) If $S \subset S'$*, then* $\widetilde{V}^T_{S'}/\mathcal{K}^{\times p} \hookrightarrow \widetilde{V}^T_S/\mathcal{K}^{\times p}$.

(iii) If $\mathrm{K}(\zeta_p)$ *has no non-trivial* p*-extension unramified outside* Pl_p *and totally split at* S *then* $\widetilde{V}^T_S/\mathcal{K}^{\times p} = \{1\}$.

(iv) There is a finite set S *of finite places of* K *not dividing* p *such that* $\widetilde{V}^T_S/\mathcal{K}^{\times p} = \{1\}$ *for any* $T \subset Pl_p$.

Proof. Points (i) and (ii) are clear.

(iii) follows from Kummer theory. Let $x \in \widetilde{V}_S^T$. Suppose that K contains the p^{th}-roots of unity. Then the extension $\mathrm{K}(\sqrt[p]{x})/\mathrm{K}$ is cyclic of degree dividing p, unramified outside Pl_p and totally split at S. Therefore $\mathrm{K}(\sqrt[p]{x})/\mathrm{K}$ is trivial, thus $x \in \mathrm{K}^p$.

If now $\zeta_p \notin \mathrm{K}$, the element x is for the same reason a p^{th} power, but this time in $\mathrm{K}(\zeta_p)$. As the degree of $\mathrm{K}(\zeta_p)/\mathrm{K}$ is prime to p, we obtain by taking the norm that x is also a p^{th} power in K.

(iv) Using Cebotarev density theorem, one can consider a finite set S of places of K containing places whose Frobenius morphisms generate the maximal abelian p-elementary extension of $\mathrm{K}(\zeta_p)$ unramified outside Pl_p. By (iii), such a set S satisfies the desired property. $\square$

Let us state the following result which will be used later in Section 3.

Corollary 3.3. *Let S and T be two finite sets of places satisfying the conditions of §2.3. Then there is a set Σ of tame places of* K *such that, for any place $\mathfrak{p} \in S \cup \Sigma$,*

$$\widetilde{V}^T_{S\cup\Sigma-\{\mathfrak{p}\}}/\mathcal{K}^{\times p} = \{1\}.$$

Proof. We take two disjoint finite sets S_1 and S_2 of tame places satisfying (iv) (using Cebotarev density theorem). By (ii) one deduces the desired result with $\Sigma = S_1 \cup S_2$. $\square$

Before stating the first important result of this work, consider the following Shafarevich kernel.

Definition 3.4.
Let

$$Ш(\widetilde{\mathrm{G}}_S^T) = \ker\left(H^2(\widetilde{\mathrm{G}}_S^T) \to \bigoplus_{v\in S}^{*} H^2(\overline{\mathrm{G}}_v) \oplus \bigoplus_{v\in \mathrm{Pl}_p-(S_p\cup T)} H^2(\mathrm{G}_v^{cr})\right),$$

where the first direct sum is taken over the set S, deprived of a place in the case where K *contains the p^{th} roots of unity and S is not empty.*

3.2. **The main result.**

Theorem 3.5. *Let S, T be disjoint sets of places of a number field* K *such that $T \subset Pl_p$ and K_v contains the p^{th} roots of unity for any $v \in S - Pl_p$. Then we have*

(i) $d_p(\widetilde{\mathrm{G}}_S^T) = -(r_1+r_2+|T|-1+\theta(\mathrm{K}))+|\mathrm{Pl}_p-S_p|+\sum_{v\in S_p}([\mathrm{K}_v:\mathbb{Q}_p]+\theta(\mathrm{K}_v))+|S_0|+d_p(\widetilde{V}_S^T/\mathrm{K}^p)$;

(ii) $d_p Ш(\widetilde{\mathrm{G}}_S^T) \le d_p(\widetilde{V}_S^T/\mathcal{K}^{\times p})$;

(iii) $d_p H^2(\widetilde{\mathrm{G}}_S^T) \le d_p(\widetilde{V}_S^T/\mathcal{K}^{\times p})+|S_0|+|\mathrm{Pl}_p-(S_p\cup T)|+\sum_{v\in S_p}\theta(\mathrm{K}_v)-\theta(\mathrm{K})+\theta(\mathrm{K},S)$.

We remark that, under these conditions, the dimension of each of the $H^2(\overline{\mathrm{G}}_v)$ and $H^2(\mathrm{G}_v^{cr})$ appearing in Definition 3.4 is smaller than 1.

One deduces immediately the following estimation:

Corollary 3.6.

$$\chi_2(\widetilde{\mathrm{G}}_S^T) \leq r_1 + r_2 - \sum_{v\in S_p} [\mathrm{K}_v : \mathbb{Q}_p] + \theta(\mathrm{K}, S).$$

Proof. (of Theorem 3.5)

(i) - This is a classical computation. We start with the following exact sequence coming from class field theory:

$$\begin{tikzcd} 1 \arrow[r] & \widetilde{V}_S^T/\mathcal{K}^{\times p} \arrow[r] & V^T/\mathcal{K}^{\times p} \arrow[r] & \prod_{v\notin T}\mathcal{U}_v \prod_{v\in T}\mathcal{K}_v^\times/(\widetilde{\mathcal{U}}_S^T \prod_v \mathcal{U}_v^p) \arrow[r] & {}_p\widetilde{\mathrm{G}}_S^{T,ab} \arrow[d] \\ & & & & {}_p\mathrm{G}^{T,ab} \end{tikzcd}$$

where $V^T = \left(\prod_{v\notin T}\mathcal{U}_v \prod_{v\in T}\mathcal{K}_v^\times \mathcal{J}^p\right) \cap \mathcal{K}^\times$. We then compute the dimensions using the exact sequence:

$$\begin{tikzcd} 1 \arrow[r] & E_{\mathrm{K}}^T/(E_{\mathrm{K}}^T)^p \arrow[r] & V^T/\mathcal{K}^{\times p} \arrow[r] & \mathrm{G}^{T,ab}[p] \arrow[r] & 1 \end{tikzcd},$$

where E_{K}^T is the group of T-units of K.

(ii) and (iii) - In the commutative diagramm (see in particular Paragraph 2.1 and Defintion 3.4 for notation)

$$\begin{tikzcd} H^2(\widetilde{\mathrm{G}}_S^T) \arrow[r] \arrow[d] \arrow[dr, "\varphi=\oplus_v\varphi_v"] & H^2(\overline{\mathrm{G}}) \arrow[d, hook] \\ \bigoplus_v H^2(\widetilde{\mathrm{G}}_{S,v}^T) \arrow[r] & \bigoplus_v^* H^2(\overline{\mathrm{G}}_v) \end{tikzcd}$$

the morphism φ factors as follow:

- for $v \in T$, $\varphi_v : H^2(\widetilde{\mathrm{G}}_{S,v}^T) \to H^2(\mathrm{G}_v^{cyc}) \to H^2(\overline{\mathrm{G}}_v)$. As $\mathrm{G}_v^{cyc} \simeq \mathbb{Z}_p$, φ_v is zero.
- for $v \notin S \cup T$, $\varphi_v : H^2(\widetilde{\mathrm{G}}_{S,v}^T) \to H^2(\mathrm{G}_v^{cr}) \to H^2(\overline{\mathrm{G}}_v)$. As G_v^{cr} is a subextension of a pro-p-free extension (see Lemma 2.2), φ_v has also a null image.

Therefore, it follows that

$$\ker\left(H^2(\widetilde{\mathrm{G}}_S^T) \to H^2(\overline{\mathrm{G}})\right) = \ker\left(H^2(\widetilde{\mathrm{G}}_S^T) \to \bigoplus_{v\in S}^* H^2(\overline{\mathrm{G}}_v)\right)$$

and thus $\text{Ш}(\widetilde{\mathrm{G}}_S^T)$ is a subgroup of $\ker\left(H^2(\widetilde{\mathrm{G}}_S^T) \to H^2(\overline{\mathrm{G}})\right)$.

One deduces (ii) from the following lemma.

Lemma 3.7. *We have*

$$\text{Ш}(\widetilde{\mathrm{G}}_S^T) \hookrightarrow \left(\widetilde{V}_S^T/\mathcal{K}^{\times p}\right)^* .$$

The local and global inflation-restriction sequences give rise to the commutative diagramm:

$$\begin{tikzcd}
H^1(\overline{\mathrm{G}})/H^1(\widetilde{\mathrm{G}}_S^T) \arrow[r, hook] \arrow[d, "\phi"] & H^1(Gal(\overline{\mathbb{Q}}/\widetilde{\mathrm{K}}_S^T))^{\widetilde{\mathrm{G}}_S^T} \arrow[r, two heads] \arrow[d, hook] & \ker\left(H^2(\widetilde{\mathrm{G}}_S^T) \to H^2(\overline{\mathrm{G}})\right) \arrow[d] \\
\Gamma(S,T) \arrow[r, hook] & \Delta(S,T) \arrow[r, two heads] & \bigoplus_{v\notin S\cup T} H^2(\mathrm{G}_v^{cr})
\end{tikzcd}$$

where

$$\Gamma(S,T) = \bigoplus_{v\notin S\cup T} H^1(\overline{\mathrm{G}}_v)/H^1(\mathrm{G}_v^{cr}) \oplus \bigoplus_{v\in T} H^1(\overline{\mathrm{G}}_v)/H^1(\mathrm{G}_v^{cyc}),$$

and where

$$\Delta(S,T) = \bigoplus_{v\notin S\cup T} H^1(\mathrm{G}_v^{cr})^{\overline{\mathrm{G}}_v} \oplus \bigoplus_{v\in T} H^1(\mathrm{G}_v^{cyc})^{\overline{\mathrm{G}}_v}.$$

Notice that the bottom right morphism is indeed surjective because for any $v \notin S\cup T$, the morphism $H^2(\mathrm{G}_v^{cr}) \to H^2(\overline{\mathrm{G}}_v)$ has a null image (again, $\mathrm{K}_v^{cr}/\mathrm{K}_v$ is a subextension of pro-p-free extension, cf. lemma 2.2).

Then we have

$$\bigoplus_{v\notin S\cup T} H^2(\mathrm{G}_v^{cr}) = \bigoplus_{v\in \mathrm{Pl}_p-(S_p\cup T)} H^2(\mathrm{G}_v^{cr}).$$

Thus the kernel of the right vertical arrow is exactly $\text{Ш}(\widetilde{\mathrm{G}}_S^T)$. By the snake lemma, we obtain

$$\text{Ш}(\widetilde{\mathrm{G}}_S^T) \hookrightarrow \mathrm{coker}\phi\ .$$

In order to relate it with $\left(\widetilde{V}_S^T/\mathcal{K}^{\times p}\right)^*$, we need the following result.

Proposition 3.8 (Salle [Sa]).

$$H^1(\overline{\mathrm{G}}_v)/H^1(\mathrm{G}_v^{cr}) \simeq (\widetilde{I}_v/\widetilde{I}_v^p[\overline{\mathrm{G}}_v,\overline{\mathrm{G}}_v])^*.$$

It comes from the fact that $\overline{\mathrm{G}}_v/\widetilde{I}_v \simeq \mathbb{Z}_p^2$ together with the following lemma:

Lemma 3.9. *Let* G *be a pro-p-group of finite type and* H *a closed normal subgroup of* G *such that* $\mathrm{G}/H \simeq \mathbb{Z}_p^t$. *Then* $\mathrm{G}^p \cap H \subset H^p[\mathrm{G},\mathrm{G}]$.

Proof. Recall that G^p denotes the closure in G of the subgroup of G generated by the elements g^p, $g \in \mathrm{G}$. Let $x \in \mathrm{G}^p \cap H$. As $[\mathrm{G},\mathrm{G}]$ is open in G, there exists $y \in \mathrm{G}$ such that $x \equiv y^p \mod [\mathrm{G},\mathrm{G}]$, in particular $x \equiv y^p \mod H$ and $y^p = 1$ in G/H. As this last quotient is p-torsion free, we have $y \in H$, that is, $x \in H^p[\mathrm{G},\mathrm{G}]$. $\square$

We come back to the proof of Lemma 3.7. Taking the dual, we obtain

$$\ker\left(\prod_{v\notin S\cup T} \widetilde{I}_v/\widetilde{I}_v^p[\overline{\mathrm{G}}_v,\overline{\mathrm{G}}_v] \prod_{v\in T} \widetilde{H}_v/\widetilde{H}_v^p[\overline{\mathrm{G}}_v,\overline{\mathrm{G}}_v] \to \ker\left({}_p\overline{\mathrm{G}}^{ab} \to {}_p\widetilde{\mathrm{G}}_S^{T,ab}\right)\right) \twoheadrightarrow \text{Ш}(\widetilde{\mathrm{G}}_S^T)^*.$$

By class field theory, one has

$$\ker\left({}_p\overline{\mathrm{G}}^{ab} \to {}_p\widetilde{\mathrm{G}}_S^{T,ab}\right) \simeq \mathcal{K}^\times \mathcal{J}^p \widetilde{\mathcal{U}}_S^T/\mathcal{K}^\times \mathcal{J}^p,$$

and

$$\left(\prod_{v\notin S\cup T} \widetilde{I}_v/\widetilde{I}_v^p[\overline{\mathrm{G}}_v,\overline{\mathrm{G}}_v]\right)\left(\prod_{v\in T} \widetilde{H}_v/\widetilde{H}_v^p[\overline{\mathrm{G}}_v,\overline{\mathrm{G}}_v]\right) \simeq \widetilde{\mathcal{U}}_S^T/(\widetilde{\mathcal{U}}_S^T)^p.$$

We obtain that $(\mathrm{coker}\phi)^* \simeq \widetilde{\mathcal{U}}_S^T \cap (\mathcal{K}^\times \mathcal{J}^p)/(\widetilde{\mathcal{U}}_S^T)^p \simeq \widetilde{V}_S^T/\mathcal{K}^{\times p}$, the last isomorphism coming from a local-global principle for p^{th} powers together with the fact that $\widetilde{\mathcal{U}}_S^T \cap \mathcal{J}^p = (\widetilde{\mathcal{U}}_S^T)^p$. Lemma 3.7 follows.

(iii) of Theorem 3.5 follows from the following inequalities:

$$\begin{aligned} r(\widetilde{\mathrm{G}}_S^T) &\leq \sum_{v\in S}^{*} \theta(\mathrm{K}_v) + d_p\left(\ker\left(H^2(\widetilde{\mathrm{G}}_S^T) \to H^2(\overline{\mathrm{G}})\right)\right) \\ &\leq \sum_{v\in S}^{*} \theta(\mathrm{K}_v) + d_p Ш(\widetilde{\mathrm{G}}_S^T) + |\mathrm{Pl}_p - (S_p \cup T)| \end{aligned}$$

(the $*$ meaning that the sum is deprived of the contribution of one place in the case where $\mu_p \subset \mathrm{K}$ and S is not empty), the last inequality coming from the exact sequence

$$0 \longrightarrow Ш(\widetilde{\mathrm{G}}_S^T) \longrightarrow \ker\left(H^2(\widetilde{\mathrm{G}}_S^T) \to H^2(\overline{\mathrm{G}})\right) \longrightarrow \bigoplus_{v\in \mathrm{Pl}_p-(S_p\cup T)} H^2(\mathrm{G}_v^{cr}).$$

Then Lemma 3.7 and

$$\sum_{v\in S}^{*} \theta(\mathrm{K}_v) = |S_0| + \sum_{v\in S_p} \theta(\mathrm{K}_v) - \theta(\mathrm{K}) + \theta(\mathrm{K}, S)$$

give the desired result. □

3.3. **Some consequences.**

3.3.1. *On the freeness of* $\widetilde{\mathrm{G}}_S^T$. Thanks to Theorem 3.5, we see that the triviality of $\widetilde{V}_S^T/\mathcal{K}^{\times p}$ implies that the global relations of $\widetilde{\mathrm{G}}_S^T$ are indeed local, i.e. that we have an injection:

$$H^2(\widetilde{\mathrm{G}}_S^T) \hookrightarrow \bigoplus_{v\in S_0}^{*} H^2(\mathrm{G}_v) \bigoplus \bigoplus_{v\in \mathrm{Pl}_p-(S_p\cup T)} H^2(\mathrm{G}_v^{cr}).$$

It allows us to give new situations where the group $\widetilde{\mathrm{G}}_S^T$ is free (other than $\mathbb{Z}_p$).

We thus obtain Corollaries 1.6 and 1.8.

Example 3.10. *Let $a \equiv 1 \mod 3$ be an integer and θ a root of $X^3 + aX + 1$. Let $\mathrm{K} = \mathbb{Q}(\theta, \zeta_3)$ (here $p = 3$). There is a place v_0 of K above 3 with residual degree 4. Let v_1 be the second place of K above 3. Put $S = \{v_0\}$ and $T = \{v_1\}$.*

It can be checked (with GP PARI [PA] *for instance) that, for some values of a, there is no non trivial 3-extension of K unramified outside T (e.g. take $a = 1$, $a = 4$...) In that cases, $\widetilde{\mathrm{G}}_S^T$ is isomorphic to the free pro-3-group with 2 generators.*

3.3.2. *On the analytic structure of the groups* $\widetilde{\mathrm{G}}_S^T$. Let G be a pro-$p$-group with rank $d(\mathrm{G})$ at least 3. If G is p-adic analytic, we have $r(\mathrm{G}) > d(\mathrm{G})^2/4$ (see [Se], Annexe 3).

Thus, we immediately obtain the following.

Corollary 3.11. *Suppose that $d(\widetilde{\mathrm{G}}_S^T) \geq 3$ and that*

$$\alpha := r_1 + r_2 + \theta(\mathrm{K}, S) - \sum_{v\in S_p} [\mathrm{K}_v : \mathbb{Q}_p] \geq 0.$$

If $d(\widetilde{\mathrm{G}}_S^T) \geq 2 + 2\sqrt{\alpha}$, then the pro-$p$-group $\widetilde{\mathrm{G}}_S^T$ is not p-adic analytic.

3.3.3. *The splitting of* $\widetilde{\mathrm{G}}_S^{T,ab}$.

Definition 3.12. *Let* $\widetilde{\mathrm{K}}_S^{T,el}/\mathrm{K}$ *denote the maximal abelian p-elementary pro-p-extension of* K *contained in* $\widetilde{\mathrm{K}}_S^{T}/\mathrm{K}$. *The p-extension* $\widetilde{\mathrm{K}}_S^{T,el}/\mathrm{K}$ *corresponds by Galois theory to the quotient* ${}_p\widetilde{\mathrm{G}}_S^{T,ab} = \widetilde{\mathrm{G}}_S^{T,ab}/(\widetilde{\mathrm{G}}_S^{T,ab})^p$.

Corollary 3.13. *Suppose that* $\widetilde{V}_S^T/\mathcal{K}^{\times p}$ *is trivial. Let* Σ *be a set of finite places of* K *with norm congruent to* $1 \mod p$. *Then* $d(\widetilde{\mathrm{G}}_{S\cup\Sigma}^T) = d(\widetilde{\mathrm{G}}_S^T) + |\Sigma|$ *and all the places of* Σ *are ramified in* $\widetilde{\mathrm{K}}_\Sigma^{T,el}/\widetilde{\mathrm{K}}_S^{T,el}$.

Proof. Indeed, for any tame place v, $d(\widetilde{G}_{S\cup\{v\}}^T) \leq d(\widetilde{\mathrm{G}}_S^T) + 1$ with equality if and only if v is ramified in $\widetilde{\mathrm{K}}_{S\cup\{v\}}^{T,el}/\mathrm{K}$. As $\widetilde{V}_{S\cup\{v\}}^T/\mathcal{K}^{\times p} \subset \widetilde{V}_S^T/\mathcal{K}^{\times p} = \{1\}$, the formula (i) of Theorem 3.5 shows that v is ramified in $\widetilde{K}_\Sigma^T/\mathrm{K}$. The relation $d(\widetilde{\mathrm{G}}_{S\cup\Sigma}^T) = d(\widetilde{\mathrm{G}}_S^T) + |\Sigma|$ follows immediately. $\square$

4. Preparatory results

4.1. **A technical result.** We first prove the following:

Proposition 4.1. *Let* S' *and* T *be two finite sets of finite places of* K *with* $S'\cap T = \emptyset$. *There exists a finite set* S *of finite places of* K *such that*

(i) $S \cap T = \emptyset$, $S' \subset S$ *and* $(S - S') \cap \mathrm{Pl}_p = \emptyset$;

(ii) *for any place* $v \in S_p$, *the decomposition group of* v *in* $\widetilde{\mathrm{K}}_S^{T,el}/\mathrm{K}$ *is of maximal rank, i.e.* $1 + [\mathrm{K}_v : \mathbb{Q}_p] + \theta(\mathrm{K}_v)$;

(iii) *For any place* $v \in \mathrm{Pl}_p - (S_p \cup T)$, *the decomposition group of* v *in* $\widetilde{\mathrm{K}}_S^{T,el}/\mathrm{K}$ *is of maximal rank, i.e* 2;

(iv) *any place* $v \in S_0 = S - S_p$ *is ramified in* $\widetilde{\mathrm{K}}_S^{T,el}/\mathrm{K}$;

(v) *for any place* $v \in S \cup (\mathrm{Pl}_p - T)$, *there is a cyclic subgroup* H_v *of* $\widetilde{\mathrm{G}}_S^{T,el}$ *of order* p *such that* $H_v \cap I_v = \{1\}$, *where* I_v *is the inertia group of* v *in* $\widetilde{\mathrm{G}}_S^{T,el}$.

Proof. (i)-(iv). Put $P = \mathrm{Pl}_p - T$. It is sufficient to make sure that

$$\mathcal{J}/(\mathcal{J}^p\mathcal{K}^\times \prod_{v\notin S\cup T} \widetilde{\mathcal{U}}_v \prod_{v\in T} \mathcal{N}_v) \to \mathcal{J}/(\mathcal{J}^p\mathcal{K}^\times \prod_{v\notin S\cup T\cup P} \widetilde{\mathcal{U}}_v \prod_{v\in P} \mathcal{K}_v^\times \prod_{v\in T} \mathcal{N}_v),$$

is of maximal rank, i.e. that

$$\left(\mathcal{J}^p\mathcal{K}^\times \prod_{v\notin S\cup T} \widetilde{\mathcal{U}}_v \prod_{v\in T} \mathcal{N}_v\right) \cap \prod_{v\in P} \mathcal{K}_v^\times = \prod_{v\in S_p} \mathcal{K}_v^{\times p} \prod_{v\in P-S_p} \mathcal{K}_v^{\times p}\widetilde{\mathcal{U}}_v.$$

According to Corollary 3.3 (for T and S'), there is a finite set Σ of tame places such that, for any place v of $S'\cup\Sigma$, we have $\widetilde{V}_{S'\cup\Sigma-\{v\}}^T/\mathcal{K}^{\times p} = \{1\}$. Put $S = S'\cup\Sigma$. The elementary p-extension $\widetilde{\mathrm{K}}_S^{T,el}/\mathrm{K}$ realizes the quotients ${}_p\overline{\mathrm{G}}_v^{ab}$ for any $v \in S$ and is cyclotomically ramified for $v \in \mathrm{Pl}_p - (S_p \cup T)$ with maximal p-rank (thanks to Corollary 3.13, we know that all the tame places of S are ramified in $\widetilde{\mathrm{K}}_S^{T,el}/\mathrm{K}$).

(v) It is clear for any place v of $\mathrm{Pl}_p - T$. For a tame place $v \in S_0$, we have $I_v \simeq \mathbb{Z}/p\mathbb{Z}$. It is thus also clear if $\mathrm{Pl}_p - T \neq \emptyset$. Otherwise, we have to ensure that $d_p\widetilde{\mathrm{G}}_S^{T,el} \geq 2$, which can also be done enlarging S if necessary. $\square$

Remark 4.2. *Condition (ii) can be weakened. We will see later that it is sufficient to make sure that the decomposition group of v in $\mathrm{K}_S^{T,el}$ contains a certain cyclic extension of degree p (see Section 5.3.1)*

4.2. **Gras-Munnier criterium.** In order to obtain the cohomological dimension less than or equal to 2 for the group $\widetilde{\mathrm{G}}_S^T$, we need to add tame places to S. Corollary 3.13 is not sufficient. The extensions ramified at the new places, i.e. at $v \in \Sigma - S$, have to live above K. More precisely, one looks for places v of K for which there exists an extension L/K of degree p, totally ramified at v and totally split at T.

This problem has been considered and solved by Gras and Munnier in [GM] (see also the book of Gras [Gr], Chapter V, Corollary 2.4.4). One can find it partially in the work of Schmidt [Sch2] (when $\mathrm{Cl}_{\mathrm{K}}^T[p] = \{1\}$).

If S and T are two finite sets of places of K with $S \cap T = \emptyset$, let $\mathrm{K}_S^{T,el}$ denote the maximal abelian elementary p-extension of K, unramified outside S and totally split at T. We apply the Gras-Munnier criterium to exhibit places v for which the extensions $\mathrm{K}_{\{v\}}^{T,el}/\mathrm{K}$ have degree p and are totally ramified at v.

Proposition 4.3 (Gras-Munnier, [GM])**.** *Let T be a finite set of places of* K *such that $\mathrm{Cl}_{\mathrm{K}}^T[p] = \{1\}$, where $\mathrm{Cl}_{\mathrm{K}}^T$ stands for the group of T-classes of* K*. Let v be a place of* K *not dividing p. Then $\mathrm{K}_{\{v\}}^{T,el}/\mathrm{K}$ is non trivial (in fact, it is cyclic of order p) if and only if v splits totally in $\mathrm{K}'(\sqrt[p]{E_{\mathrm{K},T}})/\mathrm{K}$, where $\mathrm{K}' = \mathrm{K}(\zeta_p)$ and where $E_{\mathrm{K},T}$ denotes the group of T-units of* K*. In that case the extension $\mathrm{K}_{\{v\}}^{T,el}/\mathrm{K}$ is totally ramified at v.*

4.3. **Schmidt's elements.** Throughout this paragraph, let S, T again be two disjoint sets of places, with $T \subset Pl_p$. Let $\mathbf{T}$ be a finite set of places of K such that

- $T \subset \mathbf{T}$ et $\mathbf{T} \cap \mathrm{Pl}_p = T \cap \mathrm{Pl}_p$;
- $\mathrm{Cl}_{\mathrm{K}}^{\mathbf{T}}[p] = \{1\}$.

For such $\mathbf{T}$, we have

$$\mathrm{K}^\times/(\mathrm{K}^{\times p} E_{\mathrm{K},\mathbf{T}}) \xrightarrow[\approx]{\omega} \bigoplus_{v \notin \mathbf{T}} \mathbb{Z}/p\mathbb{Z} ,$$

the isomorphism being induced by the valuation. This is used to define Schmidt's elements for $\mathbf{T}$.

Definition 4.4 (Schmidt's elements)**.** *For $v \notin \mathbf{T}$, let $s_v \in \mathrm{K}^\times$ be such that $\omega(s_v)$ equals $\overline{1}$ at v and $\overline{0}$ elsewhere: $v(s_v) \equiv 1 \mod p$, $v'(s_v) \equiv 0 \mod p$, for $v' \notin \{v\} \cup \mathbf{T}$.*

The key point is the next proposition:

Proposition 4.5. *Let $S, T, \mathbf{T}$ and Schmidt's elements (s_v) be as above. Let $\mathfrak{q}$ be a place of* K *not dividing p.*

a) The extension $\mathrm{K}_{\{\mathfrak{q}\}}^{\mathbf{T},el}/\mathrm{K}$ is a subextension of $\widetilde{\mathrm{K}}_{S\cup\{\mathfrak{q}\}}^{T,el}/\mathrm{K}$. It is non-trivial if and only if $\mathfrak{q}$ is totally split in $\mathrm{K}'(\sqrt[p]{E_{\mathrm{K},\mathbf{T}}})/\mathrm{K}$. In this case, the extension $\mathrm{K}_{\{\mathfrak{q}\}}^{\mathbf{T},el}/\mathrm{K}$ is totally ramified at $\mathfrak{q}$.

b) Let $\mathfrak{p}$ be a place of K *prime to $\mathfrak{q}$ ($\mathfrak{p}$ may divide p). Then $\mathfrak{p}$ splits totally in $\mathrm{K}_{\{\mathfrak{q}\}}^{\mathbf{T},el}/\mathrm{K}$ if and only if $\mathfrak{q}$ splits totally in $\mathrm{K}'(\sqrt[p]{s_\mathfrak{p}}, \sqrt[p]{E_{\mathrm{K},T}})/\mathrm{K}'(\sqrt[p]{E_{\mathrm{K},T}})$.*

Proof. a) The first point is clear, whereas the second follows from the triviality of $\mathrm{Cl}_{\mathrm{K}}^{\mathbf{T}}[p]$: this is the Gras-Munnier argument.

b) By class field theory, the place $\mathfrak{p}$ splits totally in $\mathrm{K}_{\{\mathfrak{q}\}}^{\mathbf{T},el}/\mathrm{K}$ if and only if the morphism

$$\mathcal{J}/\mathcal{J}^p\mathcal{K}^\times \prod_{v\in\mathbf{T}}\mathcal{K}_v^\times \prod_{v\notin\mathbf{T}\cup\{\mathfrak{q}\}}\mathcal{U}_v \longrightarrow\!\!\!\!\!\rightarrow \mathcal{J}/\mathcal{J}^p\mathcal{K}^\times \prod_{v\in\mathbf{T}\cup\{\mathfrak{p}\}}\mathcal{K}_v^\times \prod_{v\notin\mathbf{T}\cup\{\mathfrak{q}\}}\mathcal{U}_v$$

is injective, that is, if and only if

$$\mathcal{K}_{\mathfrak{p}}^\times/(\mathcal{J}^p\mathcal{K}^\times \prod_{v\in\mathbf{T}}\mathcal{K}_v^\times \prod_{v\notin\mathbf{T}\cup\{\mathfrak{q}\}}\mathcal{U}_v)\cap\mathcal{K}_{\mathfrak{p}}^\times=\{1\}.$$

We remark that $\mathcal{U}_{\mathfrak{p}}$ is in the denominator. Thus the triviality of this quotient is equivalent to the existence of an element $\alpha\in\mathrm{K}^\times$ such that

(i) $\mathfrak{p}(\alpha)\equiv 1 \mod p$;
(ii) for any $v\notin\mathbf{T}\cup\{\mathfrak{p}\}$, $v(\alpha)\equiv 0 \mod p$;
(iii) $\alpha\in(\mathrm{K}_{\mathfrak{q}}^\times)^p$.

The element $s_{\mathfrak{p}}$ satisfies these properties as soon as $\mathfrak{q}$ splits totally in the extension $\mathrm{K}'(\sqrt[p]{s_{\mathfrak{p}}},\sqrt[p]{E_{\mathrm{K},T}})/\mathrm{K}'(\sqrt[p]{E_{\mathrm{K},T}})$.

Conversely, the existence of such α implies that $\omega(\alpha/s_{\mathfrak{q}})=0$, $\alpha\in s_{\mathfrak{q}}(\mathrm{K}^\times)^pE_{\mathrm{K},\mathbf{T}}$ and $\mathfrak{q}$ is totally split in $\mathrm{K}'(\sqrt[p]{s_{\mathfrak{p}}},\sqrt[p]{E_{\mathrm{K},T}})/\mathrm{K}'(\sqrt[p]{E_{\mathrm{K},T}})$. ☐

Corollary 4.6. *Suppose the conditions of Proposition 4.5 hold and let $\mathfrak{q}$ be a place satisfying (a).*

Put $\Gamma_{\mathfrak{q}}=\mathrm{Gal}(\mathrm{K}_{\{\mathfrak{q}\}}^{\mathbf{T},el}/\mathrm{K})$. Then $\Gamma_{\mathfrak{q}}$ is a quotient of $\widetilde{\mathrm{G}}_{S\cup\{\mathfrak{q}\}}^{T,el}$ and the morphism

$$H^1(\widetilde{\mathrm{G}}_S^{T,el})\oplus H^1(\Gamma_{\mathfrak{q}})\longrightarrow H^1(\widetilde{\mathrm{G}}_{S\cup\{\mathfrak{q}\}}^{T,el})$$

induced by the inflation map is an isomorphism.

Proof. By class field theory, one knows that $d(\widetilde{\mathrm{G}}_{S\cup\{\mathfrak{q}\}}^{T,el})\leq d(\widetilde{\mathrm{G}}_S^{T,el})+1$. On the other hand, the condition on $\mathfrak{q}$ implies that the extension $\mathrm{K}_{\{\mathfrak{q}\}}^{\mathbf{T},el}/\mathrm{K}$ is cyclic of degree p and is ramified at $\mathfrak{q}$. Thus $\mathrm{K}_{\{\mathfrak{q}\}}^{\mathbf{T},el}\cap\widetilde{\mathrm{K}}_S^{T,el}=\mathrm{K}$ and $\widetilde{\mathrm{K}}_{S\cup\{\mathfrak{q}\}}^{T,el}=\mathrm{K}_{\{\mathfrak{q}\}}^{\mathbf{T},el}\widetilde{\mathrm{K}}_S^{T,el}$. The results follows immediately. ☐

5. Proof of Theorem 1.1

The proof relies strongly on the work of Schmidt in [Sch2]. Put $\mathrm{K}'=\mathrm{K}(\zeta_p)$.

5.1. **On Schmidt's conditions.** In this paragraph again, let S be a finite set of finite places whose tame places $\mathfrak{p}$ satisfy $N\mathfrak{p}=1 \mod p$, let $T\subset Pl_p$, and let $\mathbf{T}$ be a finite set of places of K such that

- $T\subset\mathbf{T}$ and $\mathbf{T}\cap\mathrm{Pl}_p=T\cap\mathrm{Pl}_p$;
- $\mathrm{Cl}_{\mathrm{K}}^{\mathbf{T}}[p]=\{1\}$.

Definition 5.1. *Let $S=\{\mathfrak{p}_1,\cdots,\mathfrak{p}_m\}$, T and $\mathbf{T}$ three finite sets of places of* K *satisfying the previous conditions. Let Σ be a finite set of places of* K *with $\Sigma\cap(\mathbf{T}\cup S\cup\mathrm{Pl}_p)=\emptyset$.*

We say that the set $\Sigma=\{\mathfrak{q}_1,\cdots,\mathfrak{q}_n\}$ satisfies Schmidt's conditions with respect to the sets S and $\mathbf{T}$ (and T) if, for any $\mathfrak{q}\in\Sigma$ and any $\mathfrak{Q}|\mathfrak{q}$, $\mathfrak{Q}$ prime of K' above $\mathfrak{q}$, we have:

(i) *for any* $a = 1, \dots, n$, $\mathrm{Frob}_{\mathfrak{Q}_a} \notin I_{\mathfrak{p}_a}$, *where* $I_{\mathfrak{p}_a}$ *is the inertia group* $\mathfrak{p}_a$ *in* $\widetilde{\mathrm{K}}_S^{T,el}/\mathrm{K}$.

(ii) *for* $a \neq b$, $\mathfrak{Q}_a$ *splits in* $\mathrm{K}'(\sqrt[p]{s_{\mathfrak{p}_b}})/\mathrm{K}'$;

(iii) *for* $a = 1, \dots, n$, $\mathfrak{Q}_a$ *decomposes totally in* $\mathrm{K}'(\sqrt[p]{E_{\mathrm{K},\mathbf{T}}})/\mathrm{K}$;

(iv) *for* $a = 1, \dots, n$, *the place* $\mathfrak{Q}_a$ *is inert in* $\mathrm{K}'(\sqrt[p]{s_{\mathfrak{p}_a}})/\mathrm{K}'$;

(v) *for* $a < b$, $\mathfrak{Q}_b$ *is totally split in* $\mathrm{K}_{\{\mathfrak{q}_a\}}^{T,el}/\mathrm{K}$;

(vi) *for* $a < b$, $\mathfrak{Q}_b$ *is totally split in* $\mathrm{K}'(\sqrt[p]{s_{\mathfrak{q}_a}})/\mathrm{K}$.

Remark 5.2. *The considered tame places* $v \in S$ *are such that* K_v *contains the* p^{th} *roots of unity.*

The next proposition, which can (mostly) be found in [Sch3], is of crucial importance:

Proposition 5.3. *Let* $S, T, \mathbf{T}$ *be as above.*

(a) *Let* $\Sigma = \{\mathfrak{q}_1, \cdots, \mathfrak{q}_n\}$ *(possibly empty) be a set satisfying Schmidt's conditions with respect to* S *and* $\mathbf{T}$. *Then the fields* $\mathrm{K}'(\sqrt[p]{E_{\mathrm{K},\mathbf{T}}})$, $\mathrm{K}'(\sqrt[p]{s_{\mathfrak{p}_1}}), \cdots$, $\mathrm{K}'(\sqrt[p]{s_{\mathfrak{p}_m}})$, $\widetilde{\mathrm{K}}_S^{T,el}\mathrm{K}'$ *and* $\mathrm{K}'(\sqrt[p]{s_{\mathfrak{q}_1}}), \cdots, \mathrm{K}'(\sqrt[p]{s_{\mathfrak{q}_n}})$, *are linearly disjoint over* K'.

(b) *There exists a set* Σ *of cardinality* $|S| = m$ *satisfying Schmidt's conditions with respect to* S *and* $\mathbf{T}$.

Proof. See the proof of [Sch3], Theorem 6.1. Let us briefly prove the two points.
(a) By construction and Kummer theory, the fields $\mathrm{K}'(\sqrt[p]{E_{\mathrm{K},\mathbf{T}}})$, $\mathrm{K}'(\sqrt[p]{s_{\mathfrak{p}_1}}), \cdots$, $\mathrm{K}'(\sqrt[p]{s_{\mathfrak{p}_m}})$ and $\mathrm{K}'(\sqrt[p]{s_{\mathfrak{q}_1}}), \cdots, \mathrm{K}'(\sqrt[p]{s_{\mathfrak{q}_n}})$ are linearly disjoint over K'. Let L denote their compositum. Because μ_p is not contained in K, the Galois group $\mathrm{Gal}(k(\mu_p)/k)$ acts non trivially by conjugation on $\mathrm{Gal}(\mathrm{L}/\mathrm{K}')$ and this action shows that for any subfield L' of L strictly containing K', L'/K is not an abelian extension, so that $\mathrm{L}' \cap \widetilde{\mathrm{K}}_S^{T,el} = \mathrm{K}'$. Therefore (a) holds.
(b) We take $\mathfrak{Q}_1, \dots, \mathfrak{Q}_n$ recursively using Cebotarev density theorem. The assertions $(ii) - (iv)$ and (vi) are made possible because of the fact that the extensions as linearly disjoint. For (i) one needs in addition Proposition 4.1 (v) and for the point (v) one needs (the proof of) Corollary 4.6. ☐

5.2. **The choice of tame places.** Let $T \subset \mathrm{Pl}_p$ and S' two disjoints finite sets of K as in Theorem 1.1.

- We first consider a set S_{old} satisfying Proposition 4.1 for S' and T. In particular, we have $\widetilde{V}_{S_{old}}^T/\mathcal{K}^{\times p} = \{1\}$.
- Then we choose a set T' of tame places of K, such that $\mathrm{Cl}_{\mathrm{K}}^{\mathbf{T}}[p] = 0$, where $\mathbf{T} = T \cup T'$.
- Let now S_{new} be a set of places of K with cardinality $m = |S_{old}|$ satisfying Schmidt's conditions with respect to S_{old} et $\mathbf{T}$. Let S be the resulting set $S = S_{old} \cup S_{new}$.

Thanks to (iii) of Definition 5.1 and Corollary 4.6, the inflation map induces an isomorphism

$$H^1(\widetilde{\mathrm{G}}_S^{T,el}) \simeq H^1(\widetilde{\mathrm{G}}_{S_{old}}^{T,el}) \oplus \bigoplus_{\mathfrak{q} \in S_{new}} H^1(\Gamma_{\mathfrak{q}}).$$

We are going to compute the cup-product

$$\begin{array}{ccc} H^1(\widetilde{\mathrm{G}}_S^T) \times H^1(\widetilde{\mathrm{G}}_S^T) & \stackrel{\cup}{\longrightarrow} & H^2(\widetilde{\mathrm{G}}_S^T) \\ (x,y) & \mapsto & x \cup y \end{array}$$

5.3. **Computation of cup-products.** Let $V = \bigoplus_{\mathfrak{q}\in S_{new}} H^1(\Gamma_{\mathfrak{q}})$.

Let $(\mathfrak{p}_i)$ denote the primes of $S_{old} \cup (\mathrm{Pl}_p - S_p)$ and $(\mathfrak{q}_j)$ those of S_{new}. For $a = 1, \cdots, m$, let η_a be a generator of $H^1(\Gamma_{\mathfrak{q}_a})$ seen as an element of $H^1(\widetilde{\mathrm{G}}_S^{T,el})$.

5.3.1. *Local cup-products.*

In $\overline{\mathrm{G}}_v$. Let v be a finite place of K such that $\zeta_p \in \mathrm{K}_v$. In that case, the cup-product $\cup : H^1(\overline{\mathrm{G}}_v) \times H^1(\overline{\mathrm{G}}_v) \longrightarrow H^2(\overline{\mathrm{G}}_v) \simeq \mathbb{Z}/p\mathbb{Z}$ is a non-degenerate bilinear form.

Put $H^1_{nr} := Inf(H^1(\mathrm{G}_v^{nr}))$ and let φ be a generator of H^1_{nr}.

Let $\psi \in H^1(\overline{\mathrm{G}}_v)$. We recall (see [NSW], Proposition 7.2.13) that $\psi \cup \varphi = \psi(\sigma_\varepsilon)$, where $\mathrm{K}_v(\sqrt[p]{\varepsilon})/\mathrm{K}_v$ is the unique unramified extension of degree p of K_v and where σ_ε is the element of ${}_p\overline{\mathrm{G}}_v^{ab}$ associated via class field theory to ε. In particular, as we can ensure that ε is a unit, σ_ε is an element of $\overline{I_v}$.

We see immediately that $H^1_{nr} \subset (H^1_{nr})^\perp$. If in addition $v \nmid p$, we have $(H^1_{nr})^\perp = H^1_{nr}$ by dimension. Thus, for $v \nmid p$, if $\chi \in H^1_{nr}$ is an unramified non-trivial character of ${}_p\overline{\mathrm{G}}_v$ and if $\psi \in H^1(\overline{\mathrm{G}}_v)$, then $\chi \cup \psi \in H^2(\overline{\mathrm{G}}_v)$ is non-zero if and only if $\psi(\overline{I_v}) \neq 0$.

In G_v^{cr}. Let $v|p$. The cup product $\cup : H^1(\mathrm{G}_v^{cr}) \times H^1(\mathrm{G}_v^{cr}) \longrightarrow H^2(\mathrm{G}_v^{cr})$ can be easily described thanks to the fact that $\mathrm{G}_v^{cr} \simeq \mathbb{Z}_p^2$: if χ is unramified (and non trivial) and if $\psi \in H^1(\mathrm{G}_v^{cr})$, then $\chi \cup \psi \in H^2(\mathrm{G}_v^{cr})$ is non-zero if and only if $\psi(I_v) \neq 0$, where I_v denotes the inertia group of v in G_v^{cr}.

5.3.2. *Global cup-products.* Denote by Θ the monomorphism (recall that $\widetilde{V}_S^T/\mathcal{K}^{\times p} = \{1\}$):

$$H^2(\widetilde{\mathrm{G}}_S^T) \overset{\Theta}{\hookrightarrow} \bigoplus_{\mathfrak{p}\in S_{old}} H^2(\overline{\mathrm{G}}_{\mathfrak{p}}) \oplus \bigoplus_{\mathfrak{p}\in \mathrm{Pl}_p-(S_p\cup T)} H^2(\mathrm{G}^{cr}_{\mathfrak{p}}) \oplus \bigoplus_{\mathfrak{q}\in S_{new}} H^2(\overline{\mathrm{G}}_{\mathfrak{q}}).$$

We study the cup-product $H^1(\widetilde{\mathrm{G}}_S^T) \cup H^1(\widetilde{\mathrm{G}}_S^T)$ from the following triangle:

$$\begin{tikzcd} H^1(\widetilde{\mathrm{G}}_S^T) \times H^1(\widetilde{\mathrm{G}}_S^T) \arrow[d, "\cup"] \arrow[dr, dotted] & \\ H^2(\widetilde{\mathrm{G}}_S^T) \arrow[r, hook, "\Theta=\oplus_v\Theta_v"] & \displaystyle\bigoplus_{\mathfrak{p}\in S_{old}} H^2(\overline{\mathrm{G}}_{\mathfrak{p}}) \oplus \bigoplus_{\mathfrak{p}\in \mathrm{Pl}_p-(S_p\cup T)} H^2(\mathrm{G}^{cr}_{\mathfrak{p}}) \oplus \bigoplus_{\mathfrak{q}\in S_{new}} H^2(\overline{\mathrm{G}}_{\mathfrak{q}}) \end{tikzcd}$$

As p is odd, for any character $\chi \in H^1(\widetilde{\mathrm{G}}_S^T)$ and any place v, $\Theta_v(\chi \cup \chi) = 0$.

As the places $\mathfrak{p}_i$ are unramified in the extensions $\widetilde{\mathrm{K}}_{\{\mathfrak{q}\}}^{\mathbf{T},el}/\mathrm{K}$, it follows from the local computations that $\Theta_{\mathfrak{p}_i}(\eta_a \cup \eta_b) = 0$ for any a, b. Moreover $\Theta_{\mathfrak{q}_k}(\eta_a \cup \eta_b) = 0$ as soon as $a \neq k$ and $b \neq k$ for the same reason. For $a < b$, by Definition 5.1 (v), $\Theta_{\mathfrak{q}_a}(\eta_a \cup \eta_b) = 0$ and $\Theta_{\mathfrak{q}_a}(\eta_a \cup \eta_b) = 0$ thanks to (vi) together with Proposition 4.5. Finally, for any a, b, $\Theta(\eta_a \cup \eta_b) = 0$ and therefore, by injectivity of Θ, $\eta_a \cup \eta_b = 0$.

Thus we have shown that $V \cup V = 0$.

Let χ be a character of $H^1(\widetilde{\mathrm{G}}_{S_{old}}^{T,el})$. As χ is unramified at the places $\mathfrak{q}$ of S_{new}, it follows that $\Theta_{\mathfrak{q}_b}(\chi\cup\eta_a)=0$ for $a\neq b$. As η_a is totally ramified at $\mathfrak{q}_a$, $\Theta_{\mathfrak{q}_a}(\chi\cup\eta_a)=0$ if and only if $\chi(\mathrm{Frob}_{\mathfrak{q}_a})=0$.

Finally $\Theta_{\mathfrak{p}_i}(\chi\cup\eta_a)=0$ for $i\neq a$ because of the points (ii) and (iii) of Definition 5.1 associated to Proposition 4.5.

5.4. **End of proof.**

5.4.1. *A reduction.* Keeping the notation of Section 5.3, we prove first our main theorem with the following assumptions.

Proposition 5.4. *Suppose that for any $v\in S_p$, $\theta(\mathrm{K}_v)=1$. Then the cup product $H^1(\widetilde{\mathrm{G}}_S^T)\times V\xrightarrow{\cup}H^2(\widetilde{\mathrm{G}}_S^T)$ is surjective and Θ is an isomorphism.*

Proof. For m characters $\gamma_i\in H^1(\widetilde{\mathrm{G}}_S^T)$, $i=1,\cdots,m$, consider the $m\times m$ matrices with coefficients in $\mathbb{F}_p$:

$$A(\gamma_i)=\big(\Theta_{\mathfrak{p}_j}(\gamma_i\cup\eta_i)\big)_{i,j}\,,$$

and

$$B(\gamma_i)=\big(\Theta_{\mathfrak{q}_j}(\gamma_i\cup\eta_i)\big)_{i,j}\,.$$

For $i=1,\cdots,m$, one chooses characters $\chi_i\in H^1(\widetilde{\mathrm{G}}_{S_{old}}^{T,el})$ such that :

(a) $\chi_i(\mathrm{Frob}_{\mathfrak{q}_i})=0$;
(b) $\chi_i(I_{\mathfrak{p}_i})\neq 0$ if $\mathfrak{p}_i$ is prime to p or in Pl_p-S_p;
(c) for $\mathfrak{p}_i|p$ with $\mathfrak{p}_i\in S_p$, $\chi_i(\sigma_{\varepsilon_i})\neq 0$ (cf. Section 5.3.1).

Such χ_i's exist because $\sigma_{\varepsilon_i}\in I_{\mathfrak{p}_i}$ and $\mathrm{Frob}_{\mathfrak{q}_i}\notin I_{\mathfrak{p}_i}$ ((i) of Definition 5.1).

For $i=1,\dots,m$, we choose characters $\psi_i\in H^1(\widetilde{\mathrm{G}}_{S_{old}}^{T,el})$ such that $\psi_i(\mathrm{Frob}_{\mathfrak{q}_i})\neq 0$.

By Section 5.3.2, the matrix $B(\chi_i)$ and $B(\psi_i)$ are diagonal. The condition $\chi_i(\mathrm{Frob}_{\mathfrak{q}_i})=0$ implies that the matrix $B(\chi_i)$ is zero. The condition $\psi_i(\mathrm{Frob}_{\mathfrak{q}_i})\neq 0$ implies that the diagonal matrix $B(\psi_i)$ is invertible (diagonal elements are not zero).

The matrix $A(\chi_i)$ is diagonal: this is the last point of Section 5.3.2. On the diagonal of $A(\chi_i)$, the elements are non-zero (Conditions (b) or (c) on the χ_i's).

Thus the matrix $\begin{pmatrix}\Theta(\chi_i)_i\\ \Theta(\psi_i)_i\end{pmatrix}=\begin{pmatrix}A(\chi_i) & B(\chi_i)\\ A(\psi_i) & B(\psi_i)\end{pmatrix}$ is invertible. Therefore the morphism Θ is an isomorphism and the cup product is surjective. $\square$

5.4.2. *The general case.* Put $\mathrm{Pl}'_p=\{v\in\mathrm{Pl}_p,\ \theta(\mathrm{K}_v)=1\}$ and $S'=(S\cap\mathrm{Pl}'_p)\cup S_0$.

Proposition 5.5. *Suppose:*

(i) $\widetilde{V}_{S'}^T/\mathcal{K}^{\times p}=\{1\}$;
(ii) $H^1(\widetilde{\mathrm{G}}_{S'}^T)=U\oplus V$;
(ii) $V\cup V=0\in H^2(\widetilde{\mathrm{G}}_{S'}^T)$;
(iv) $U\cup V=H^2(\widetilde{\mathrm{G}}_{S'}^T)$.

Then $\widetilde{\mathrm{G}}_S^T$ has cohomological dimension at most 2 and $\chi(\widetilde{\mathrm{G}}_S^T)=r_1+r_2-\sum_{v\in S_p}[\mathrm{K}_v:\mathbb{Q}_p]$.

Proof. As $\widetilde{V}_S^T \subset \widetilde{V}_{S'}^T$, we have $\widetilde{V}_S^T = \mathcal{K}^{\times p}$. Thus

$$\Theta : H^2(\widetilde{\mathrm{G}}_S^T) \longrightarrow \bigoplus_{v\in S} H^2(\overline{\mathrm{G}}_v) \bigoplus \bigoplus_{v\in \mathrm{Pl}_p-(S_p\cup T)} H^2(\mathrm{G}_v^{cr})$$

is injective.

As for $v \in S - S'$, $H^2(\overline{\mathrm{G}}_v) = 0$, we obtain the following commutative diagram

$$\begin{tikzcd} H^2(\widetilde{\mathrm{G}}_{S'}^T) \arrow[r, hook, two heads] \arrow[d, "Inf"'] & \bigoplus_{v\in S'} H^2(\overline{\mathrm{G}}_v) \bigoplus \bigoplus_{v\in \mathrm{Pl}_p-(S'_p\cup T)} H^2(\mathrm{G}_v^{cr}) \arrow[d, two heads] \\ H^2(\widetilde{\mathrm{G}}_S^T) \arrow[r, hook] & \bigoplus_{v\in S} H^2(\overline{\mathrm{G}}_v) \bigoplus \bigoplus_{v\in \mathrm{Pl}_p-(S_p\cup T)} H^2(\mathrm{G}_v^{cr}) \end{tikzcd}$$

One deduces:

$$H^2(\widetilde{\mathrm{G}}_{S'}^T) \twoheadrightarrow H^2(\widetilde{\mathrm{G}}_S^T) \simeq \bigoplus_{v\in S'} H^2(\overline{\mathrm{G}}_v) \bigoplus \bigoplus_{v\in \mathrm{Pl}_p-(S_p\cup T)} H^2(\mathrm{G}_v^{cr}).$$

On the other hand, the inflation map $H^1(\widetilde{\mathrm{G}}_{S'}^T) \hookrightarrow H^1(\widetilde{\mathrm{G}}_S^T)$ induces the decomposition $H^1(\widetilde{\mathrm{G}}_S^T) = U \oplus V \oplus U'$, where U' is any complementary vector space of $U \oplus V$. As the cup-product commutes with the inflation map, the conditions of Theorem 2.6 are all satisfied. $\square$

From Section 5.3 and Proposition 5.4, we see that the condition of Proposition 5.5 are satisfied for S', taking $U = H^1(\mathrm{G}_{S'_{old}}^{T,el})$ and $V = \bigoplus_{\mathfrak{q}\in S'_{new}} H^1(\Gamma_{\mathfrak{q}})$. The last proposition concludes the proof.

References

[As] J. Assim, Analogues étales de la p-tour des corps de classes, *J. Théor. Nombres Bordeaux* **15** (2003), no. 3, 651–663.

[Gi] R. Gillard, Transformation de Mellin-Leopoldt des fonctions elliptiques, *J. Number Theory* **25** (1987), no. 3, 379–393.

[Gr] G. Gras, Class field theory. From theory to practice, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[GM] G. Gras et A. Munnier, Extensions cycliques T-toalement ramifiées, Publications Mathématiques de Besançon, 1999.

[Ja] J.-F. Jaulent, Sur l'indépendance ℓ-adique de nombres algébriques, *J. Number Theory* **20** (1985), no. 2, 149–158.

[JM1] J.-F. Jaulent et C. Maire, Radical hilbertien et tour localement cyclotomique, *Japan. J. Math.* **28** (2002), no. 2, 203–213.

[JM2] J.-F. Jaulent et C. Maire, Sur les invariants d'Iwasawa des tours cyclotomiques, *Canad. Math. Bull.* **46** (2003), no. 2, 178–190.

[JM3] J.-F. Jaulent et C. Maire, A propos de la tour localement cyclotomique d'un corps de nombres, *Abh. Math. Sem. Hamburg* **70**, 2000, 239-250.

[JSa] J.-F. Jaulent et O. Sauzet, Pro-ℓ-extensions de corps de nombres $\mathfrak{l}$-rationnels, *J. Number Theory* **65** (1999), 240-267.

[JSo] J.-F. Jaulent et F. Soriano, Sur les tours localement cyclotomiques, *Arch. Math.* **73** (1999), no. 2, 132–140.

[Ko] H. Koch, Galoissche Theorie der p-Erweiterungen, Springer-Verlag, Berlin-New York; VEB Deutscher Verlag der Wissenschaften, Berlin, 1970.

[La] J. Labute, Mild pro-p-groups and Galois groups of p-extensions of $\mathbb{Q}$, *J. Reine Angew. Math.* **596** (2006), 155–182.

[Ma] C. Maire, Sur la structure galoisienne de certaines pro-p-extensions de corps de nombres, *Math. Zeit.*, **267**(2011), 887-913.

[NSW] J. Neukirch, A. Schmidt, K. Wingberg, Cohomology of Number Fields, GMW 323, Springer 2008.

[PA] PARI/GP, version `2.5.3`, Bordeaux, 2012, `http://pari.math.u-bordeaux.fr/`.

[Sa] L. Salle, Sur les pro-p-extensions à ramification restreinte au-dessus de la $\mathbb{Z}_p$-extension cyclotomique d'un corps de nombres, *J. Théor. Nombres Bordeaux* **20** (2008), no. 2, 485–523.

[Sch1] A. Schmidt, Rings of integers of type $K(\pi, 1)$. Doc. Math. **12** (2007), 441–471.

[Sch2] A. Schmidt, On the $K(\pi, 1)$-property for rings of integers in the mixed case. Algebraic number theory and related topics 2007, 91–100, RIMS Kôkyûroku Bessatsu, B12, Res. Inst. Math. Sci. (RIMS), Kyoto, 2009.

[Sch3] A. Schmidt, Über pro-p-fundamentalgruppen markierter arithmetischer kurven, *J. Reine Angew. Math.* **640** (2010), 203–235.

[Schn] L. Schneps, On the μ-invariant of p-adic L-functions attached to elliptic curves with complex multiplication, *J. Number Theory* **25** (1987), no. 1, 20–33.

[Se] J.-P. Serre, Cohomologie Galoisienne (cinquième édition), Lecture Notes in Mathematics 5, Springer-Verlag, 1994.

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