Notions of solution for conservation laws with fractional diffusion

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Plan of the talk

1. **Fractional convection-diffusion equations**
2. **Classical, weak and entropy solutions**
   - Definitions of weak and entropy solutions
   - Some well-posedness and regularity results
   - Ill-posedness: construction of a non-entropy weak solution
3. **Kinetic formulation for fractional conservation laws**
   - Kinetic formulation, the local case
   - Kinetic formulation, adaptation to the fractional case
   - A glimpse into the proof
4. **Renormalized formulation for fractional diffusion**
   - Renormalized solutions, the local case
   - Renormalized solutions, adaptation to the fractional case

PROBLEM CONSIDERED

Problem considered

We look at the “fractal conservation laws”

$$\partial_t u + \mathrm{div}_x f(u) + \mathcal{L}_\lambda[u] = 0, \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{1}$$

$$u(0,\cdot) = u_0, \quad \text{on } \mathbb{R}^N, \tag{2}$$

$\mathcal{L}_\lambda$ denotes the fractional power $(-\Delta)^{\lambda/2}$ of $-\Delta$.
This is a non-local, pseudodifferential operator of order $\lambda, 0 < \lambda < 2$.

Motivations :

- Gaz detonation (Clavin-Denet-He’01), phenomenological
- “Anomalous diffusion” phenomena :
  Lévi processes as a generalization of $\mathcal{L}_\lambda$
- A “scalar quasi-geostrophic equation”: $\lambda = 1$
- ...

The two reference (limit) case, deeply studied :

- $\lambda = 2$: the parabolic case, similar to the heat equation
- $\lambda = 0$: the pure hyperbolic case (scalar conservation law)

The behaviour of generic solutions is very different in the two cases.

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What is the fractional laplacian ?

If φ is regular (e.g., for a function φ from the Schwartz class $\mathcal{S}(\mathbb{R})$), $\mathcal{L}_\lambda[\varphi]$ can be defined through the Fourier transform:

$$\mathcal{F}(\mathcal{L}_\lambda[\varphi])(\xi) := |\xi|^\lambda \mathcal{F}(\varphi)(\xi). \tag{3}$$

In absence of regularity, a more general definition is provided by the *Lévi-Khinchine formula*: (case $0 < \lambda < 1$: the integral is convergent)

$$const\, \mathcal{L}_\lambda[\varphi](x) := \; v.p. \int_{\mathbb{R}} \frac{\varphi(x+z) - \varphi(x)}{|z|^{N+\lambda}}\, dz. \tag{4}$$

Hint : The kernel $\dfrac{1}{|z|^{N+\lambda}}$ being singular at the origin, split (4) into regular ("order zero") and singular ("order λ") parts:

$$\mathcal{L}_\lambda[\varphi] = -G_\lambda\Big(\int_{\{|z|>r\}} \frac{\varphi(\cdot+z)-\varphi(\cdot)}{|z|^{N+\lambda}}\, dz + v.p. \int_{\{|z|<r\}} \frac{\varphi(\cdot+z)-\varphi(\cdot)}{|z|^{N+\lambda}}\, dz\Big)$$

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Heuristics: competition of regularizing and de-regularizing effects

Heuristically, the knowledge of the purely hyperbolic ($\lambda = 0$) and of the parabolic ($\lambda = 2$) cases permits to set up a few conjectures :

- in the case $1 < \lambda < 2$, the fractional diffusion operator $\mathcal{L}[u]$ is the leading term. In particular:
  - smooth data give rise to globally defined smooth solutions
  - non-smooth data undergo an instanteneous regularizing effect
  - there is well-posedness in the framework of weak solutions.
- in the case $0 < \lambda < 1$, the fractional diffusion term $\mathcal{L}[u]$ is dominated by the term $\mathrm{div}_x f(u)$. In particular:
  - one expects that even for very smooth initial data, there is no globally defined in time classical solution
  - the notion of a weak (distributional) solution permits to get existence for rather general data...
  - ...but it may lead to non-uniqueness
  - a "good" notion of solution should be inspired by the kinetic solutions/entropy solutions coming from conservation laws.
- Case $\lambda = 1$. Hmmmm... no *a priori* conjectures ! For some applications, one needs techniques that allow for a wide range of values of λ, including $\lambda = 1$...

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An overview of the talk

- Highlight the issues essential for the convection-dominated range $0 < \lambda < 1$ (but that remain relevant for the full range $0 \leq \lambda \leq 2$)
- Discuss difficulties and advantages of the non-local case , compared to the local cases ($\lambda = 0$ or $\lambda = 2$):
  NB It can be advantageous (technically) to see the classical local case $\lambda = 2$ as $\lambda = 2^-$, i.e., as the limit of the local case (Alibaud, Cifani, Jakobsen'14)
- Point out drawbacks of classical and weak solutions ($\lambda < 1$)
- Focus on three relevant notions of solution ($0 < \lambda < 2$):
  - entropy (Kruzhkov) solutions , L^∞ setting
  - kinetic solutions , L^1 setting
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Classical, Weak and Entropy solutions

Notions of solution: weak solutions

Definition (Weak solution)

Let $u_0 \in L^\infty(\mathbb{R}^N)$. A function $u \in L^\infty((0,\infty)\times\mathbb{R}^N)$ is said to be a weak solution to (1),(2) if for all $\varphi \in \mathcal{D}([0,\infty)\times\mathbb{R}^N)$,

$$\int_0^\infty \int_{\mathbb{R}^N} (u\partial_t\varphi + f(u)\cdot\nabla_x\varphi - u\mathcal{L}_\lambda[\varphi]) + \int_{\mathbb{R}^N} u_0\varphi(0) = 0.$$

Important remark: at least for regular enough u and v there holds the integration-by-parts formula

$$\int \mathcal{L}_\lambda[u]v = \int u\mathcal{L}_\lambda[v] = const \iint (u(x)-u(y))(v(x)-v(y))\,\frac{dxdy}{|x-y|^{N+\lambda}}.$$

Therefore the definition of a weak solution just says

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$$\partial_t \eta(u) + \mathrm{div}_x q(u) + \eta'(u)\, \mathcal{R}^r_\lambda[u] + \mathcal{S}^r_\lambda[\eta(u)] \le 0 \quad \text{in } \mathcal{D}'.$$

Here η is an "entropy", q is the associated "entropy flux"; these notions are inherited from the Kruzhkov theory of conservation laws.

The definition of Alibaud is based upon the fractional Kato inequality:

$$\forall r > 0 \quad \eta'(u)\, \mathcal{R}^r_\lambda[u] + \mathcal{S}^r_\lambda[\eta(u)] \le \eta'(u)\, \mathcal{L}_\lambda[u].$$

To be compared with the Kato inequality used in the Kruzhkov theory:

$$-\varepsilon\Delta\eta(u) \le -\varepsilon\Delta\eta(u) + \varepsilon\eta''(u)|\nabla u|^2 = \eta'(u)\,(-\varepsilon\Delta u)\,.$$

NB smaller is the parameter r, less information is lost.

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WELL-POSEDNESS RESULTS

Well (and ill)-posedness results

- for the case $1/2 < \lambda < 2$, an H^1 solution exists globally and is unique for small H^1 data (Biler-Funaki-Woyczynski'98)
- for $1 < \lambda < 2$, there exists a unique weak solution for L^∞ data, and $u(t, \cdot)$ falls within C^∞ for $t > 0$ (Droniou-Gallouët-Vovelle'02)
- for $0 < \lambda < 2$, there exists a unique entropy solution (Alibaud'07).

For $0 < \lambda < 1$ and the Burgers flux $f(u) = \frac{u^2}{2}$ in dimension one:

- assume the initial datum u_0 presents an initial discontinuity (say, at zero) with $u_0(0-) > u_0(0+)$ and belongs to some class $\mathfrak{C}$ $\Longrightarrow$ the discontinuity is persistent, at least for small times
- specially selected smooth initial data in $\mathfrak{C}$ $\Longrightarrow$ the unique entropy solution, which becomes discontinuous (but not instantly)
- small Lipschitz data in the class $\mathfrak{C}$ $\Longrightarrow$ global Lipschitz solutions

(Alibaud-Droniou-Vovelle'07; the main tool : characteristics)

For the same fractional Burgers equation in the "hyperbolic regime",

- Non-entropy weak solutions can be constructed
- Consequently, uniqueness of a weak solution may fail

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CONSTRUCTION OF A "WRONG" WEAK SOLUTION

Ideas of the construction

Goal: construct u a stationary, discontinuous, weak solution to the fractional Burgers equation that violates the entropy condition (Alibaud, A.'10)

- We try to mimic the simplest "wrong" weak solution of the Burgers conservation law $\partial_t u + \left(\frac{u^2}{2}\right) = 0$. This is the discontinuous stationary solution
$$u(t,x) := \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$
- One simple reason why this is not an entropy solution is that it fails to satisfy the Oleĭnik inequality $\partial_x u(t,x) \leq \frac{const}{t}$ in $\mathcal{D}'$.
- We prove that the Oleĭnik inequality still holds for entropy solutions of the fractional Burgers equation
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Ideas of the construction

Goal: construct u a stationary, discontinuous, weak solution to the fractional Burgers equation that violates the entropy condition (Alibaud, A.'10)

- We try to mimic the simplest "wrong" weak solution of the Burgers conservation law $\partial_t u + \left(\frac{u^2}{2}\right) = 0$. This is the discontinuous stationary solution $u(t,x) := \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$
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Ideas of the proof

- Work on the space of odd functions $H^1_{odd}(\mathbb{R}_*)$ with jump at $x = 0$.
- Construct solutions in the space H^1_{odd} of the regularized stationary problem

$$\varepsilon(u - \Delta u) + \left(\frac{u^2}{2}\right)_x + \mathcal{L}[u] = 0, \quad u(0\pm) = \pm 1.$$

(they are $\mathcal{D}'(\mathbb{R}_*)$ solutions; in $\mathcal{D}'(\mathbb{R})$, singular term $-2\varepsilon(\delta_0)_x$ appears!)
Technique : mainly the Schauder fixed-point theorem.

- Pass to the limit, as $\varepsilon \downarrow 0$. The things to be cared of:
  - Compactness (in $H^{\lambda/2}(\mathbb{R}^\pm)$-weak and for the a.e. convergence): this comes from the uniform in ε estimate for $H^{\lambda/2}(\mathbb{R}^\pm)$ energy
  - Passage to the limit in the weak formulation: straightforward
  - Guarantee that the discontinuity of u^ε at $x = 0$ persists at the limit

The last item is challenging. We have two proofs .

– a first one , with an explicit construction of barriers and adapted entropies are used to get “odd-functions” comparison principle.

– a second proof , with a passage to the limit in the traces of u^ε: use the Green-Gauss formula and the weak formulation of the PDE at hand.

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Kinetic solutions

The pure hyperbolic case

Given a function $u(t,x)$, one introduces the auxiliary quantity

$$\chi(t,x;\xi) = \chi(\xi, u(t,x)) = \begin{cases} 1, & 0 < \xi < u \\ -1, & u < \xi < 0 \\ 0, & \text{otherwise} \end{cases}$$

NB for all Lipschitz $S(.)$ there holds $S(u) = \int_{\mathbb{R}} S'(\xi)\chi(\xi,u)\,d\xi$.

Represent the nonlinearities from entropy inequalities in this way ; cancel $S'(u) \Rightarrow$ kinetic formulation for scalar conservation law $u_t + \operatorname{div} f(u) = 0$:

$$\partial_t \chi(\xi,u) + f'(\xi) \cdot \nabla_x \chi(\xi,u) = \partial_\xi m$$

where $m = m(t,x;\xi)$ is some finite nonnegative measure responsible for the dissipation of entropy.

Outcome: Full well-posedness theory for $u_t + \operatorname{div} f(u) = 0$, L^1 data.
In $L^1 \cap L^\infty$: kinetic solutions equivalent to Kruzhkov entropy solutions
Proofs: arguments different from Kruzhkov (no variables' doubling).
Lions, Perthame, Tadmor'94; book Perthame'02.

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The degenerate parabolic case

Extension to local degenerate convection-diffusion equation :

$$u_t + \operatorname{div} f(u) - \Delta A(u) = 0$$

(and even to anisotropic diffusion case: Chen, Perthame'03).
The kinetic formulation takes the form

$$\partial_t \chi(\xi, u) + f'(\xi) \cdot \nabla_x \chi(\xi, u) - A'(\xi)\Delta[\chi(\xi, u)] = \partial_\xi (m+n)$$

where m, n are finite nonnegative measures. Moreover,
the dissipation measure n of the local diffusion operator is given by

$$n(t, x; \xi) \ := \ \delta_0(u(t, x) - \xi) \, A'(\xi) |\nabla u(t, x)|^2.$$

NB It is essential that n is prescribed by an explicit formula
(while m need not be precisely determined)

Outcome: Full well-posedness for $u_t + \operatorname{div} f(u) - \Delta A(u) = 0$, L^1 data.
Connection to renormalized solutions in isotropic or anisotropic case
(e.g. Ouédraogo, Maliki'09)

Fractional kinetic dissipation measure

The definition is almost contained in Karlsen, Ulusoy'11:
in this paper, the dissipation measure for fractional diffusion was written for smooth entropies (and with $\mathcal{R}^r_\lambda + \mathcal{S}^r_\lambda$ splitting...) as

$$\int_{\mathbb{R}^N} S'(u(t,x))\,(u(t,x+z)-u(t,x))\;\frac{const}{|z|^{N+2\lambda}}\,dz.$$

Our observation stems from the elementary Taylor's identity:

$$\forall a,b\;\; S'(a)(b-a) = S(b)-S(a) - \int_{\mathbb{R}} S''(\xi)|b-\xi|\mathbb{1}_{\mathrm{conv}\{a,b\}}(\xi)\,d\xi.$$

Using singular (Kruzhkov) entropies ($S''(\xi)=2\delta_0(\xi-k)$), we guess the dissipation measure suitable for the fractional Laplacian:

$$n_\lambda(t,x,\xi) := \int_{\mathbb{R}^N} |u(t,x+z)-\xi|\mathbb{1}_{\mathrm{conv}\{u(t,x),u(t,x+z)\}}(\xi)\,\frac{const}{|z|^{N+2\lambda}}\,dz. \quad (5)$$

The kinetic formulation with fractional laplacian takes the form

$$\partial_t\chi(\xi,u) + f'(\xi)\cdot\nabla_x\chi(\xi,u) + \mathcal{L}_\lambda[\chi(\xi,u)] = \partial_\xi(m+n_\lambda)$$

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Proof by following Chen-Perthame '03

NB Dissipation measure n_λ is less singular than in the local case.

Proof: main issue is uniqueness + L^1 contraction.

As usual, the proof is in two "steps":
formal proof based on the "algebra of $\{-1, 0, 1\}$-valued functions":

$$|\chi(\xi,u)-\chi(\xi,v)| = |\chi(\xi,u)-\chi(\xi,v)|^2 = \chi(\xi,u)+\chi(\xi,v)-2\chi(\xi,u)\chi(\xi,v),$$

rigorous one with convolution regularization of the formal calculation.

The goal is to get to inequality $\partial_t|\chi(\xi,u)-\chi(\xi,v)| \le 0$ in $\mathcal{D}'_{(t,x;\xi)}$; integrating in $\xi \in \mathbb{R}$ yields

$$\partial_t\|u-v\|_{L^1(\mathbb{R}^N)} = \int \partial_t|\chi(\xi,u)-\chi(\xi,v)|\,d\xi \le 0 \quad \text{in } \mathcal{D}'_{(t,x)},$$

and L^1 contraction is justified.

Existence: entropy solutions for $L^1 \cap L^\infty$ data are kinetic solutions; extension to L^1 data by density as soon as L^1-contraction is justified.

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Use of the representation formula for n_λ

The result follows from the tedious case-by-case observation :

Lemma

There holds

$$\forall a,b,c,d \in \mathbb{R} \qquad F(a,b,c,d) \le G(a,b,c,d), \quad (6)$$

$$F(a,b,c,d) := sign(a-b)sign(c-d) \int_{\mathbb{R}} 1\!\!1_{conv\{a,b\}}(\xi) 1\!\!1_{conv\{c,d\}}(\xi)\, d\xi$$

$$G(a,b,c,d) := \int_{\mathbb{R}} \Big(|b-\xi|\delta(\xi-c) 1\!\!1_{conv\{a,b\}}(\xi) + |d-\xi|\delta(\xi-a) 1\!\!1_{conv\{c,d\}}(\xi) \Big)\, d\xi.$$

Definition (5) of dissipation measure n_λ + (formal) calculations say:

$$\text{inequality (6)} \;\Leftrightarrow\; \partial_t |\chi(\xi,u) - \chi(\xi,v)| \le 0.$$

With a mollification step, rigorous proof is achieved.

NB No need to split $\mathcal{L}_\lambda = \mathcal{R}^r_\lambda + \mathcal{S}^r_\lambda$!

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RENORMALIZED SOLUTIONS

Elliptic problems beyond the variational setting. "Closure solutions".

Consider for instance the elliptic problem

$$u - \Delta u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega;$$

the classical setting is the variational one: $u \in H_0^1(\Omega)$, with $f \in H^{-1}(\Omega)$.

But one also has the property $\|u - \widehat{u}\|_{L^1(\Omega)} \le \|f - \widehat{f}\|_{L^1(\Omega)}$
for solutions $u, \hat{u}$ corresponding to data $f, \hat{f} \in L^1 \cap H^{-1}$.
Since every function $f \in L^1$ can be approximated in $\|\cdot\|_{L^1}$ by $L^1 \cap H^{-1}$ functions f_n, then solutions u_n exist, and $(u_n)_n$ is a Cauchy sequence.

From the abstract point of view, one has the solution operator $L^1 \cap H^{-1} \to L^1 \cap H_0^1$ and its closure: $L^1 \to L^1$ (we get "closure solutions").
NB: the nonlinear semigroup theory then yields "mild solutions" of $u_t = \Delta u$.
"Closure" and "mild" solutions may fall out of the energy space !

Then one faces the following question:

in which sense "closure solutions" satisfy the original equation ?

NB: kinetic solutions provide a characterization of "closure solutions"...
renormalized solutions provide another, "more nonlinear" characterization.

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renormalized solutions provide another, "more nonlinear" characterization.

Elliptic problems beyond the variational setting. “Closure solutions”.

Consider for instance the elliptic problem

$$u - \Delta u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega;$$

the classical setting is the variational one: $u \in H_0^1(\Omega)$, with $f \in H^{-1}(\Omega)$.

But one also has the property $\|u - \widehat{u}\|_{L^1(\Omega)} \leq \|f - \widehat{f}\|_{L^1(\Omega)}$
for solutions $u, \hat{u}$ corresponding to data $f, \hat{f} \in L^1 \cap H^{-1}$.
Since every function $f \in L^1$ can be approximated in $\|\cdot\|_{L^1}$ by $L^1 \cap H^{-1}$ functions f_n, then solutions u_n exist, and $(u_n)_n$ is a Cauchy sequence.

From the abstract point of view, one has the solution operator
$L^1 \cap H^{-1} \to L^1 \cap H_0^1$ and its closure: $L^1 \to L^1$ (we get “closure solutions”).
NB: the nonlinear semigroup theory then yields “mild solutions” of $u_t = \Delta u$.
“Closure” and “mild” solutions may fall out of the energy space !

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Truncations. Integrability constraints. Renormalized solutions.

Consider the functions $T_k(\cdot)$ and $\varphi_k(\cdot)$ defined, for $k>0$, by

$$T_k : r \mapsto \operatorname{sign} r \, \min\{k,|r|\} \quad \text{and} \quad \varphi_k : r \mapsto T_k(r+1)-T_k(r);$$

$T_k(\cdot)$ is the truncation function at level $k>0$.

• If we formally take $T_k(u)$ and $\varphi_k(u)$ for the test functions in the equation $u-\Delta u=f$, we find estimates "for finite u" , "for infinite u"

$$\|T_k(u)\|^2_{H^1_0}=\int_{[|u|<k]}|\nabla u|^2\leq k\int|f|, \qquad \int_{[k<|u|<k+1]}|\nabla u|^2\leq\int_{[|u|>k]}|f|\to_{k\to\infty}0.$$

• multiplying by $H(u)\phi$ with compactly supported H we get

$$\int(u-f)H(u)\phi+H(u)\nabla u\cdot\nabla\phi+H'(u)|\nabla u|^2\phi=0 \ \text{ for } \phi\in H^1_0\cap L^\infty.$$

NB $\forall k$, $T_k(u)\in H^1_0$ & $H(.)$ compactly supported $\Rightarrow$ integrals converge

In general, saying that u satisfies "complementary PDEs" + "finite u and infinite u estimates" yields a weaker notion of solution (without asking $u\in H^1_0$). This is called renormalized solution.

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How to renormalize the fractional laplacian

Difficulty: non-locality $\Rightarrow$ no chain rule like $\nabla u 1\!\!1_{|u|<k} = \nabla T_k(u)$

Result: With appropriate notation, definitions and re-interpretation of truncation terms, one finds the same well-posedness results as for the local case: existence, uniqueness, L^1 contraction for renormalized solutions of $u + \mathcal{L}[u] = f$ for all L^1 data (Alibaud, A., Bendahmane'10)

Notation: write $T_k u$ for $T_k(u)$, etc.

- Let $d\mu(z)$ is the measure with the density $g(z) := G_s|z|^{-(N+s)}$, i.e.,

$$\mathcal{L}[\phi] = \int_{\mathbb{R}^N} (\phi(x+z) - \phi(x))\, d\mu(z).$$

More general measures μ can be considered (Lévy diffusions).

- Let $d\pi(x,y)$ denote the measure $\frac{1}{2}g(x-y)\, dxdy$ on $\mathbb{R}^{2N}$.

Recall the following representation of the quadratic form $(\mathcal{L}u, v)_{L^2(\mathbb{R}^N)}$: For all $u, v \in \mathcal{D}(\mathbb{R}^N)$ (and more, by density)

$$\int_{\mathbb{R}^N} (\mathcal{L}u)\, v = \iint_{\mathbb{R}^{2N}} (u(x)-u(y))\, (v(x)-v(y))\, d\pi(x,y).$$

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Definition of renormalized solutions

Definition

Let $f \in L^1_{loc}(\mathbb{R}^N)$. A measurable function $u : \mathbb{R}^N \longrightarrow \mathbb{R}$ is called renormalized solution of the problem $u + \mathcal{L}u = f$ if $u \in L^1_{loc}(\mathbb{R}^N)$ and

(i) for all $k > 0$, $\displaystyle\iint_{\mathbb{R}^{2N}} (u(x) - u(y))\,(T_k u(x) - T_k u(y))\, d\pi(x,y) < +\infty$;

$$\lim_{k\to+\infty} \iint_{\mathbb{R}^{2N}} (u(x) - u(y))\,(\varphi_k u(x) - \varphi_k u(y))\, d\pi(x,y) = 0;$$

(ii) for all compactly supported renormalization function $H \in W^{1,\infty}(\mathbb{R})$, for all test function $\phi \in \mathcal{D}(\mathbb{R}^N)$

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NB: local product-and-chain-rule: $\nabla(H(u)\phi) = H(u)\nabla\phi + \phi H'(u)\nabla u$;
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"Phase plane" splitting and rewriting the constraint

Set $H_\mu := \{\, v \mid \delta_{x,y} v \in L^2(\mathbb{R}^{2N}, d\pi) \};$ recall that $\delta_{x,y} v = v(x) - v(y)$.

The quotient space $H_\mu / \{v \equiv const\}$ is a Hilbert space under the scalar product $(v, w) \mapsto \iint_{R^{2N}} (\delta_{x,y} v)(\delta_{x,y} w)\, d\pi(x, y)$.

A close examination shows that the "estimates for finite u" and "estimates for infinite u" in the Definition imply the properties

$$\|T_k u\|_{H_\mu} \le k\|f\|_{L^1} \quad \text{and} \quad \lim_{k\to+\infty} \iint_{[(u(x),u(y))\in A_k]} |u(x) - u(y)|\, d\pi(x, y) = 0,$$

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Further, denote the region $\{\, (u, v) \in \mathbb{R}^2 \mid |u| \le k,\ |v| \le k \,\}$ by B_k ; finally, the "remainder region" $C_k = \mathbb{R}^2 \setminus (A_k \cup B_k)$ can be neglected, as $k \to \infty$.
So: due to the "estimates for finite/infinite u", for every truncation level k,
– we control $\delta_{x,y} u$ in L^1 in the region $\{(x, y) \mid (u(x), u(y)) \in A_k\}$;
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These ideas permit to prove existence and uniqueness as in the local case.
NB Convergence for finite differences easier to prove than for gradients !

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THANK YOU — MERCI !